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Selection properties of texture structures

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ABSTRACT

The aim of this paper is to begin with study of selection principles in texture and ditopological texture spaces and to establish relationships between these selection principles and other covering properties of texture or ditopological spaces. We also investigate the behavior of the properties under consideration under standard operations with ditopological spaces.

1. Introduction

Three classical selection principles are defined in a general form as follows.

Let $\mathcal{A}$ and $\mathcal{B}$ be sets consist of families of subsets of an infinite set X. Then:

$\mathsf{S}_1(\mathcal{A},\mathcal{B})$: For each sequence $(A_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$ there is a sequence $(b_n : n \in \mathbb{N})$ such that for each n, $b_n \in A_n$, and $\{b_n : n \in \mathbb{N}\}$ is an element of $\mathcal{B}$.

$\mathsf{S}_{fin}(\mathcal{A},\mathcal{B})$: For each sequence $(A_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$ there is a sequence $(B_n : n \in \mathbb{N})$ of finite sets such that for each n, $B_n \subseteq A_n$, and $\bigcup_{n\in\mathbb{N}} B_n \in \mathcal{B}$.

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$\mathsf{U}_{fin}(\mathcal{A}, \mathcal{B})$: For each sequence $(A_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$ there is a sequence $(B_n : n \in \mathbb{N})$ such that for each n, B_n is a finite subset of A_n and $\{\cup B_n : n \in \mathbb{N}\} \in \mathcal{B}$.

If $\mathcal{O}$ denotes the family of all open covers of a topological space X, then X has the *Rothberger property* (resp. the *Menger property*) if X satisfies $\mathsf{S}_1(\mathcal{O}, \mathcal{O})$ (resp. $\mathsf{S}_{fin}(\mathcal{O}, \mathcal{O})$). Denote by $\mathcal{O}_D$ the family of open sets of X such that $\cup\mathcal{O}_D$ is dense in X. If X satisfies $\mathsf{S}_1(\mathcal{O}, \mathcal{O}_D)$ (resp. $\mathsf{S}_{fin}(\mathcal{O}, \mathcal{O}_D)$), then we say that X has the *weak Rothberger property* (resp. the *weak Menger property*).

There are infinite games for two players, ONE and TWO, associated to these selection properties. The game $\mathsf{G}_{fin}(\mathcal{A}, \mathcal{B})$ is defined in this way: ONE and TWO play a round for each $n \in \mathbb{N}$. In the n-th round ONE chooses $A_n \in \mathcal{A}_n$, and TWO responds by a finite set $B_n \subseteq A_n$. TWO wins a play $A_1, B_1; A_2, B_2; \cdots$ if $\bigcup_{n\in\mathbb{N}} B_n \in \mathcal{B}$; otherwise ONE wins. Other games are defined similarly.

For more information about selection principles and games in topological spaces see the survey papers [7,10,11], and in bitopological spaces the papers [8,9].

These selection properties and associated games will be considered here in two special structures defined below.

Texture space: ([1]) Let S be a set. We work within a subset $\mathcal{S}$ of the power set $\mathcal{P}(S)$ called a *texturing*. A *texturing* of S is a point-separating, complete, completely distributive lattice $(\mathcal{S}, \subseteq) \subseteq \mathcal{P}(S)$, which contains S and $\emptyset$, and for which arbitrary meets coincide with intersections, and finite joins with unions. If $\mathcal{S}$ is a texturing of S, the pair $(S, \mathcal{S})$ is called a *texture*. Throughout the paper we denote by $\bigcap$ and $\bigvee$ meets and joins in a texture $(S, \mathcal{S})$.

For $s \in S$ the sets

$$P_s = \bigcap\{A \in \mathcal{S} \mid s \in A\} \text{ and } Q_s = \bigvee\{A \in \mathcal{S} \mid s \notin A\}$$

are called respectively, the *p-sets* and *q-sets* of $(S, \mathcal{S})$. These sets are used in the definition of many textural concepts.

In a texture, arbitrary joins need not coincide with unions, and clearly, this will be so if and only if $\mathcal{S}$ is closed under arbitrary unions, or equivalently if $P_s \not\subseteq Q_s$ for all $s \in S$. In this case $(S, \mathcal{S})$ is said to be *plain*.

Complementation: ([1]) A mapping $\sigma : \mathcal{S} \to \mathcal{S}$ satisfying $\sigma(\sigma(A)) = A$, $\forall A \in \mathcal{S}$ and $A \subseteq B \Longrightarrow \sigma(B) \subseteq \sigma(A)$, $\forall A, B \in \mathcal{S}$ is called a *complementation* on $(S, \mathcal{S})$ and $(S, \mathcal{S}, \sigma)$ is then said to be a *complemented texture*.

If $\mathcal{F}$ is a subfamily of $\mathcal{S}$, then $\sigma(\mathcal{F})$ denotes the set $\{\sigma(F) : F \in \mathcal{F}\}$.

Examples:

(1) For any set X, $(X, \mathcal{P}(X), \pi_X)$ is the complemented *discrete texture* representing the usual set structure of X. Here the complementation $\pi_X(Y) = X \setminus Y$, $Y \subseteq X$, is the usual set complement. Clearly, $P_x = \{x\}$ and $Q_x = X \setminus \{x\}$ for all $x \in X$.

(2) For $\mathbb{I} = [0, 1]$ define $\mathcal{J} = \{[0, t] \mid t \in [0, 1]\} \cup \{[0, t) \mid t \in [0, 1]\}$. $(\mathbb{I}, \mathcal{J}, \iota)$ is a complemented texture, which we will refer to as the *unit interval texture*. Here $P_t = [0, t]$ and $Q_t = [0, t)$ for all $t \in I$.

(3) The texture $(\mathbb{L}, \mathcal{L})$ is defined by $\mathbb{L} = (0, 1]$ and $\mathcal{L} = \{(0, r] \mid r \in [0, 1]\}$. For $r \in \mathbb{L}$ $P_r = (0, r] = Q_r$.

Ditopology: A *dichotomous topology* on $(S, \mathcal{S})$ or *ditopology* for short, is a pair (τ, κ) of generally unrelated subsets τ, κ of $\mathcal{S}$ satisfying

(τ_1) $S, \emptyset \in \tau$,

(τ_2) $G_1, G_2 \in \tau \Longrightarrow G_1 \cap G_2 \in \tau$,

(τ_3) $G_i \in \tau$, $i \in I \Longrightarrow \bigvee_i G_i \in \tau$,
(κ_1) $S, \emptyset \in \kappa$,
(κ_2) $K_1, K_2 \in \kappa \Longrightarrow K_1 \cup K_2 \in \kappa$,
(κ_3) $K_i \in \kappa$, $i \in I \Longrightarrow \bigcap K_i \in \kappa$.

The elements of τ are called open and those of κ closed. We refer to τ as the *topology* and κ as the *cotopology* of (τ, κ).

If (τ, κ) is a ditopology on a complemented texture $(S, \mathcal{S}, \sigma)$, then we say that (τ, κ) *is complemented* if the equality $\kappa = \sigma[\tau]$ is satisfied. In this article, a complemented ditopological texture space is denoted by $(S, \mathcal{S}, \sigma, \tau, \kappa)$.

Dicompactness: ([1]) Let (τ, κ) be a ditopology on $(S, \mathcal{S})$ and take $A \in \mathcal{S}$. The family $\{G_i \mid i \in I\}$ is called an *open cover* of A if $G_i \in \tau$ for all $i \in I$ and $A \subseteq \bigvee_{i \in I} G_i$. A closed cocover can be defined dually, i.e. the family $\{F_i \mid i \in I\}$ is called a *closed cocover* of A if $F_i \in \kappa$ for all $i \in I$ and $\bigcap_{i \in I} F_i \subseteq A$.

Let (τ, κ) be a ditopology on the texture $(S, \mathcal{S})$ and $A \in \mathcal{S}$.

(1) A is called *compact* if whenever $G_i \in \tau$, $i \in I$ is an open cover of A, then there is a finite subset J of I with $A \subseteq \bigcup_{j \in J} G_j$. The ditopological texture space $(S, \mathcal{S}, \tau, \kappa)$ is called compact if S is compact.
(2) A is called *cocompact* if $F_i \in \kappa$, $i \in I$ is a closed cocover of A then there is a finite subset J of I with $\bigcap_{j \in J} F_j \subseteq A$. The ditopological texture space $(S, \mathcal{S}, \tau, \kappa)$ is called cocompact if $\emptyset$ is cocompact.

In a compact topological space every closed subset is compact. This is not true for compact ditopological spaces. This leads to the following concepts.

(1) (τ, κ) is called *stable* if every $K \in \kappa$ with $K \neq S$ is compact.
(2) (τ, κ) is called *costable* if every $G \in \tau$ with $G \neq \emptyset$ is cocompact.

A ditopological texture space $(S, \mathcal{S}, \tau, \kappa)$ is called *dicompact* if it is compact, cocompact, stable and costable.

2. Selection properties of texture spaces

In this section we define and study the Menger selection property of texture spaces. Game-theoretic and Ramsey-theoretic characterizations of this property are given.

Let $(S, \mathcal{S})$ be a texture. A subset $\mathcal{C}$ of $\mathcal{S}$ is said to be a *cover* of a set $A \subseteq S$ if $A \subseteq \bigvee \mathcal{C}$; if $\bigvee \mathcal{C} = S$, then $\mathcal{C}$ is said to be a cover of S. By $\mathbb{C}$ (or $\mathbb{C}_S$ when it is necessary) we denote the family of all covers of S.

Definition 2.1. A texture space $(S, \mathcal{S})$ is said to be *Menger* (*Rothberger*) if S satisfies the selection property $\mathsf{S}_{fin}(\mathbb{C}, \mathbb{C})$ ($\mathsf{S}_1(\mathbb{C}, \mathbb{C})$).

Observe that textures having the Menger property satisfy: each cover of S has a countable subcover. This property will be called the *Lindelöf property*. For this reason we assume that all covers of a texture are countable.

2.1. Games and selection properties

A classical result of Hurewicz [6] states that a Lindelöf topological space has the Menger property $\mathsf{S}_{fin}(\mathcal{O}, \mathcal{O})$ if and only if ONE does not have a winning strategy in the game $\mathsf{G}_{fin}(\mathcal{O}, \mathcal{O})$. We have the following similar result for texture spaces.

Theorem 2.2. *For a Lindelöf texture space $(S, \mathcal{S})$ the following statements are equivalent:*

(1) *S has the Menger property $\mathsf{S}_{fin}(\mathbb{C}, \mathbb{C})$;*
(2) ONE *does not have a winning strategy in the game $\mathsf{G}_{fin}(\mathbb{C}, \mathbb{C})$ on S.*

Proof. (1) $\Rightarrow$ (2): Let φ be a strategy for ONE in the game $\mathsf{G}_{fin}(\mathbb{C}, \mathbb{C})$ and let ONE's first move be $\mathcal{C}_1 = \varphi(\emptyset)$ – a cover of S. One may assume that $\mathcal{C}_1 = \{C_1 \subseteq C_2 \subseteq \cdots\}$ (otherwise we consider the cover $\mathcal{D}_1 = \{D_n : n \in \mathbb{N}\}$ defined by $D_1 = C_1$, $D_k = \bigvee_{i=1}^{k} C_i = \bigcup_{i=1}^{k} C_i$, $k \geq 2$). Recursively define the family $\{C_q : q \text{ is a finite sequence in } \mathbb{N}\}$ such that for each $q = (n_1, n_2, \cdots, n_k) \in \mathbb{N}^{<\omega}$:

(*a*) $\varphi(C_(n_1), C_(n_1, n_2), \cdots, C_(n_1, n_2, \cdots, n_k)) = (C_{q\frown n} : n \in \mathbb{N}) = (C_{(n_1,n_2,\cdots,n_k,n)} : n \in \mathbb{N})$ is a cover of S;
(*b*) $C_q \subseteq C_{q\frown n}$ for each $n \in \mathbb{N}$;
(*c*) $m < n \Rightarrow C_{q\frown m} \subseteq C_{q\frown n}$.

For each $n, m \in \mathbb{N}$ define now

$$D_{n,m} = \begin{cases} C_{(m)} & \text{if } n = 1, \\ (\bigcap\{C_{q\frown m} : q \in \mathbb{N}^{n-1}\}) \bigcap D_{n-1,m} & \text{if } n \geq 2. \end{cases}$$

We have:

1^0. For each n, $m_1 < m_2 \Rightarrow D_{n,m_1} \subseteq D_{n,m_2}$.

For $n = 1$ this follows from the fact $C_{(m_1)} \subseteq C_{(m_2)}$. Suppose that the claim is true for $2, \cdots, n$. Then $D_{n+1,m_1} = (\bigcap\{C_{q\frown m_1} : q \in \mathbb{N}^{n-1}\}) \bigcap D_{n,m_1} \subseteq (\bigcap\{C_{q\frown m_2} : q \in \mathbb{N}^{n-1}\}) \bigcap D_{n,m_2} = D_{n+1,m_2}$ because of (c) above and the assumption $D_{n,m_1} \subseteq D_{n,m_2}$. By induction we have that 1^0 is true.

2^0. For each n and each $(i_1, i_2, \cdots, i_n)$ with $m \leq \max\{i_1, i_2, \cdots, i_n\}$, $D_{n,m} \subseteq C_{(i_1,i_2,\cdots,i_n)}$.

The proof is by induction on n. For $n = 1$ it follows from $D_{1,m} = C_m \subseteq C_{i_1}$. Let $n \geq 2$. Assume that the assertion is true for $2, \cdots, n$. Let $(i_1, \cdots, i_n, i_{n+1}) \in \mathbb{N}^{n+1}$. If $i_{n+1} < m$, then $m \leq \max\{i_1, \cdots, i_n\}$ and thus $D_{n+1,m} \subseteq D_{n,m} \subseteq C_{(i_1,\cdots;i_n)} \subseteq C_{i_1,\cdots,i_n,i_{n+1}}$. If $i_{n+1} \geq m$, then $D_{n+1,m} \subseteq C_{i_1,\cdots,i_n,m} \subseteq C_{i_1,\cdots,i_n,i_{n+1}}$.

For each $n \in \mathbb{N}$ set $\mathcal{D}_n = \{D_{n,m} : m \in \mathbb{N}\}$. By induction on n, using the fact that each $\mathcal{D}_n$ is an increasing chain, we easily prove that each $\mathcal{D}_n$ is a cover of S. Apply now (1) to the sequence $(\mathcal{D}_n : n \in \mathbb{N})$ and use again that $\mathcal{D}_n$'s are increasing chains to find a function $f : \mathbb{N} \to \mathbb{N}$ such that $\{D_{f(n)} : n \in \mathbb{N}\}$ is a cover of S. Then

$$D_{f(1)}, D_{f(1),f(2)}, \cdots$$

is a sequence of moves of TWO. By 2^0 we have

$$S = D_{f(1)} \bigvee D_{f(1),f(2)} \bigvee \cdots \bigvee D_{f(1),\cdots,f(n)} \bigvee \cdots,$$

i.e. φ is not a winning strategy for ONE.

(2) $\Rightarrow$ (1) It is evident. □

We recall that in [5] it was shown that $(S, \mathcal{S})$ is a texture if and only if $(S, \mathcal{S}^c)$ is a T_0 topological space with completely distributive lattice of open sets, and that the join $\bigvee_{i \in I} A_i$ of $\{A_i : i \in I\} \subseteq \mathcal{S}$ in $(S, \mathcal{S})$ is given by $\bigvee_{i \in I} A_i = Cl_{\mathcal{S}^c}\left(\bigcup_{i \in I} A_i\right)$. So, we have that a texture $(S, \mathcal{S})$ has the Menger property (the Rothberger property) implies that the topological space $(S, \mathcal{S}^c)$ has the weak Menger property (the weak Rothberger property).

For a texture space $(S, \mathcal{S})$ the symbol $\mathbb{C}_\Omega$ denotes the collection of all covers $\mathcal{C}$ of S with the property:

For each k and each partition $\mathcal{C} = \mathcal{C}_1 \cup \cdots \mathcal{C}_k$ there is an $i \leq k$ with $\mathcal{C}_i \in \mathbb{C}$.

Definition 2.3. A texture space $(S, \mathcal{S})$ is said to have the *dual-Menger property* (resp. *dual-Rothberger property*) if for each sequence $(\mathcal{K}_n : n \in \mathbb{N})$ such that for each n, $\mathcal{K}_n \subseteq \mathcal{S}$ and $\bigcap_{n\in\mathbb{N}} \mathcal{K}_n = \emptyset$, there is a sequence $(\mathcal{F}_n : n \in \mathbb{N})$ (resp. a sequence $(K_n : n \in \mathbb{N})$) satisfying: (1) for each n, $\mathcal{F}_n$ is a finite subset of $\mathcal{K}_n$ (resp. for each n, $K_n \in \mathcal{K}_n$), and (2) $\bigcap_{n\in\mathbb{N}} \bigcap\{F : F \in \mathcal{F}_n\} = \emptyset$ (resp. $\bigcap_{n\in\mathbb{N}} K_n = \emptyset$).

Proposition 2.4. *For a texture $(S, \mathcal{S})$ the following statements are equivalent:*

(1) $(S, \mathcal{S})$ *has the dual-Menger (Rothberger) property;*
(2) $(S, \mathcal{S}^c)$ *has the Menger (Rothberger) property.*

Proof. We prove only the Rothberger case.

(1) $\Rightarrow$ (2): Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of open covers of $(S, \mathcal{S}^c)$. Then $(\mathcal{K}_n : n \in \mathbb{N})$, where $\mathcal{K}_n = \{S \setminus U : U \in \mathcal{U}_n\}$, is a sequence of families of sets in $\mathcal{S}$ satisfying for each n, $\bigcap\{K : K \in \mathcal{K}_n\} = \emptyset$. By (1) there are elements $K_n \in \mathcal{K}_n$, $n \in \mathbb{N}$, with $\bigcap\{K_n : n \in \mathbb{N}\} = \emptyset$. Taking for each n, $U_n \in \mathcal{U}_n$ with $K_n = S \setminus U_n$ we obtain a sequence $(U_n : n \in \mathbb{N})$ witnessing that (2) holds.

(2) $\Rightarrow$ (1): Let $(\mathcal{K}_n : n \in \mathbb{N})$ be a sequence of families of sets in $\mathcal{S}$ such that for each n, $\bigcap \mathcal{K}_n = \emptyset$. For each n, set $\mathcal{U}_n = \{S \setminus K : K \in \mathcal{K}_n\}$. One obtains a sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of $\mathcal{S}^c$-open covers of $(S, \mathcal{S}^c)$. Apply (2) and pick an element U_n from each $\mathcal{U}_n$ satisfying $\bigcup_{n\in\mathbb{N}} U_n = S$. Then for each n, $K_n = U_n^c \in \mathcal{K}_n$ and $\bigcap_{n\in\mathbb{N}} K_n = \emptyset$, i.e. (1) is true. □

2.2. Ramsey theoretic approach

Recall the following notion in Ramsey theory, called the *Baumgartner–Taylor partition relation.* For each positive integer k,

$$\mathcal{A} \to \lceil \mathcal{B} \rceil^2_k$$

denotes the following statement:

For each A in $\mathcal{A}$ and for each function $f : [A]^2 \to \{1, \cdots, k\}$ there are a set $B \in \mathcal{B}$ with $B \subseteq A$, a $j \in \{1, \cdots, k\}$, and a partition $B = \bigcup_{n\in\mathbb{N}} B_n$ of B into pairwise disjoint finite sets such that for each $\{a, b\} \in [B]^2$ for which a and b are not from the same B_n, we have $f(\{a, b\}) = j$.

Several covering properties of topological spaces are characterized by the Baumgartner–Taylor partition relation (see [7,10,11]). We will see now the situation in texture spaces.

Call a texture space $(S, \mathcal{S})$ *ω-Lindelöf* if each $\mathcal{C} \in \mathbb{C}_\Omega$ has a countable $\mathcal{C}' \subseteq \mathcal{C}$ with $\mathcal{C}' \in \mathbb{C}_\Omega$.

Theorem 2.5. *Let $(S, \mathcal{S})$ be an ω-Lindelöf texture space. Then the following are equivalent:*

(1) ONE *has no winning strategy in the game* $\mathsf{G}_{fin}(\mathbb{C}_\Omega, \mathbb{C})$*;*
(2) *For each $k \in \mathbb{N}$ the partition relation $\mathbb{C}_\Omega \to \lceil \mathbb{C} \rceil^2_k$ holds.*

Proof. (1) $\Rightarrow$ (2) Let $k \in \mathbb{N}$ and $\mathcal{C} \in \mathbb{C}_\Omega$ be given; one may assume that $\mathcal{C}$ is countable. Let $\mathcal{C} = \{C_1, C_2, \cdots\}$. Fix a function $f : [\mathcal{C}]^2 \to \{1, \cdots, k\}$.

Set $\mathcal{D}_j = \{C_i : i > 1 \text{ and } f(\{C_1, C_i\}) = j\}$, $j = 1, 2, \cdots, k$. We get a partition of $\mathcal{C} \setminus \{C_1\}$ into k many pieces $\mathcal{D}_1, \mathcal{D}_2, \cdots, \mathcal{D}_k$. There is a j for which $\mathcal{D}_j$ is in $\mathbb{C}_\Omega$. Fix such a j and set $i_1 = j$, $\mathcal{C}_1 = \mathcal{D}_j$. Let now $\mathcal{D}_j = \{C_i : i > 2 \text{ and } f(\{C_2, C_i\}) = j\}$. We get a partition of $\mathcal{C} \setminus \{C_1, C_2\}$ into finitely many pieces. One of these pieces, say $\mathcal{D}_j$, is in $\mathbb{C}_\Omega$. This j we denote by i_2 and put $\mathcal{C}_2 = \mathcal{D}_j$. In a similar way we select $\mathcal{C}_n$ and i_n, $n \geq 3$, such that for each n

- $\mathcal{C}_n \in \mathbb{C}_\Omega$;
- $\mathcal{C}_{n+1} \subseteq \mathcal{C}_n$;
- $\mathcal{C}_{n+1} = \{C_i \in \mathcal{C}_n : i > n + 1 \text{ and } f(\{C_{n+1}, C_i\}) = i_{n+1}\}$.

For each $j \in \{1, \cdots, k\}$ define $\mathcal{T}_j = \{C_n : i_n = j\}$. We have that for each n, $\mathcal{C}_n \cap \mathcal{T}_1, \cdots, \mathcal{C}_n \cap \mathcal{T}_k$ is a partition of $\mathcal{C}_n$ into k many pieces. Thus there is a j_n with $\mathcal{C}_n \cap \mathcal{T}_{j_n} \in \mathbb{C}_\Omega$. Since for each n we have $\mathcal{C}_n \supset \mathcal{C}_{n+1}$, one may assume that the same j_n works for all $\mathcal{C}_n$'s; denote this j_n by j^*.

Now define the following strategy φ for ONE in the game $\mathsf{G}_{fin}(\mathbb{C}_\Omega, \mathbb{C})$. In the first round ONE plays $\varphi(\emptyset) = \mathcal{C}_1 \cap \mathcal{T}_{j^*}$. If the respond of TWO is a finite set $\mathcal{P}_1 \subseteq \varphi(\emptyset)$, put $n_1 = 1 + \max\{n : C_n \in \mathcal{P}_1\}$. Let ONE play $\varphi(\mathcal{P}_1) = \mathcal{C}_{n_1} \cap \mathcal{T}_{j^*}$. If TWO's respond is the finite set $\mathcal{P}_2 \subseteq \varphi(\mathcal{P}_1)$, let $n_2 = 1 + \max\{n : C_n \in \mathcal{P}_2\}$; clearly, $n_2 > n_1$. Then ONE plays $\varphi(\mathcal{P}_1, \mathcal{P}_2) = \mathcal{C}_{n_2} \cap \mathcal{T}_{j^*}$.

And so ONE has no winning strategy in $\mathsf{G}_{fin}(\mathbb{C}_\Omega, \mathbb{C})$, hence there is a play $\varphi(\emptyset), \mathcal{P}_1; \varphi(\mathcal{P}_1), \mathcal{P}_2; \varphi(\mathcal{P}_1, \mathcal{P}_2), \mathcal{P}_3; \cdots$ in which ONE used φ but lost. Since TWO won we have $\bigvee_{n\in\mathbb{N}} \mathcal{P}_n \in \mathbb{C}$. It is evident (by the definition of φ) that $p \neq q$ implies $n_p \neq n_q$, and $f(\{G, H\}) = j^*$ for all G and H from distinct $\mathcal{P}_j$'s.

(2) ⇒ (1) Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of elements of $\mathbb{C}_\Omega$. Again we suppose that each $\mathcal{U}_n$ is countable: $\mathcal{U}_n = \{U_{n,m} : m \in \mathbb{N}\}$. Let $\mathcal{V} = \{U_{1,n} \cap U_{n,m} : n, m \in \mathbb{N}\}$. As S is completely distributive lattice, $\mathcal{V} \in \mathbb{C}_\Omega$.

Define a function $f : [\mathcal{V}]^2 \to \{1, 2\}$ by

$$f(\{U_{1,n_1} \cap U_{n_1,m}, U_{1,n_2} \cap U_{n_2,k}\}) = \begin{cases} 1 & \text{if } n_1 = n_2, \\ 2 & \text{otherwise.} \end{cases}$$

By (2) there are a color j, $\mathcal{W} \subseteq \mathcal{V}$ with $\mathcal{W} \in \mathbb{C}$, and a partition $\mathcal{W} = \bigcup_{p\in\mathbb{N}} \mathcal{W}_p$ such that for G and H from distinct $\mathcal{W}_p$'s, $f(\{G, H\}) = j$. We consider two possible cases $j = 1$ and $j = 2$.

$j = 1$: In this case there is $n \in \mathbb{N}$ such that $W \subseteq U_{1,n}$ for each $W \in \mathcal{W}$. This means that $\mathcal{W}$ does not belong to $\mathbb{C}$, so that this contradiction shows that the case $j = 1$ is not possible. So, we have

$j = 2$: For each $p > 1$ let

$$\mathcal{F}_p = \{U_{p,m} : U_{p,m} \text{ is the second coordinate of some } W \in \mathcal{W}\}.$$

Clearly, for each p, $\mathcal{F}_p \subseteq \mathcal{U}_p$ is finite. It is also easy to see that $\mathcal{F} = \bigvee_{p\in\mathbb{N}} \mathcal{F}_p \in \mathbb{C}$. If we put $\mathcal{F}_p = \emptyset$ if $\mathcal{F}_p$ is not defined, we obtain a sequence $(\mathcal{F}_n : n \in \mathbb{N})$ which witnesses for $(\mathcal{U}_n : n \in \mathbb{N})$ that (1) holds. □

3. Selection properties of ditopological texture spaces

In this section we discuss several basic facts about selection properties of ditopological texture spaces.

Definition 3.1. Let $(S, \mathcal{S}, \tau, \kappa)$ be a ditopological texture space and A a subset of S.

(1) A is said to have the *Menger property* if for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of open covers of A there is a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ of finite sets such that for each $n \in \mathbb{N}$, $\mathcal{V}_n \subseteq \mathcal{U}_n$ and $A \subseteq \bigvee_{n\in\mathbb{N}} \bigcup \mathcal{V}_n$. We say that $(S, \mathcal{S}, \tau, \kappa)$ is Menger if the set S is Menger. (This property is denoted by $\mathsf{S}_{fin}(\theta_S, \theta_S)$, where θ_S is the family of open covers of S.)

(2) A is said to have the *co-Menger property* if for each sequence $(\mathcal{F}_n : n \in \mathbb{N})$ of closed cocovers of A there is a sequence $(\mathcal{K}_n : n \in \mathbb{N})$ of finite sets such that for each $n \in \mathbb{N}$, $\mathcal{K}_n \subseteq \mathcal{F}_n$ and $\bigcap_{n\in\mathbb{N}} \bigcap \mathcal{K}_n \subseteq A$. We say that $(S, \mathcal{S}, \tau, \kappa)$ is co-Menger if $\emptyset$ is co-Menger. (We will denote this property by $\mathsf{S}_{cfin}(\Phi_S, \Phi_S)$, where Φ_S is the family of closed cocovers of $\emptyset$.)

Definition 3.2. Let $(S, \mathcal{S}, \tau, \kappa)$ be a ditopological texture space and A a subset of S.

(1) A is said to have the *Rothberger property* if for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of open covers of A there is a sequence $(U_n : n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, $U_n \in \mathcal{U}_n$ and $A \subseteq \bigvee_{n\in\mathbb{N}} U_n$. We say that $(S, \mathcal{S}, \tau, \kappa)$ is Rothberger if the set S is Rothberger. (This property is denoted by $\mathsf{S}_1(\theta_S, \theta_S)$.)
(2) A is said to have the *co-Rothberger property* if for each sequence $(\mathcal{F}_n : n \in \mathbb{N})$ of closed cocovers of A there is a sequence $(F_n : n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, $F_n \in \mathcal{F}_n$ and $\bigcap_{n\in\mathbb{N}} F_n \subseteq A$. We say that $(S, \mathcal{S}, \tau, \kappa)$ is co-Rothberger if $\emptyset$ is co-Rothberger. (We will denote this property by $\mathsf{S}_{c1}(\Phi_S, \Phi_S)$.)

Call a ditopological space $(S, \mathcal{S}, \tau, \kappa)$ *σ-compact* (*σ-cocompact*) if there is a sequence $(A_n : n \in \mathbb{N})$ of compact (cocompact) subsets of S such that $\bigvee_{n\in\mathbb{N}} A_n = S$ ($\bigcap_{n\in\mathbb{N}} A_n = \emptyset$).

Proposition 3.3. *Let $(S, \mathcal{S}, \tau, \kappa)$ be a ditopological texture space.*

a) If $(S, \mathcal{S}, \tau, \kappa)$ is σ-compact, then $(S, \mathcal{S}, \tau, \kappa)$ has the Menger property.
b) If $(S, \mathcal{S}, \tau, \kappa)$ is σ-cocompact, then $(S, \mathcal{S}, \tau, \kappa)$ has the co-Menger property.

Proof. (a) Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of open covers of S. Since S is σ-compact it can be represented in the form $S = \bigvee_{k\in\mathbb{N}} A_k$, where each A_k is compact. For each $k \in \mathbb{N}$ there is a finite subset $\mathcal{V}_k \subseteq \mathcal{U}_k$ such that $A_k \subseteq \bigcup \mathcal{V}_k$. Then the sequence $(\mathcal{V}_n : n \in \mathbb{N})$ shows that S is Menger.

(b) Let $(\mathcal{F}_n : n \in \mathbb{N})$ be a sequence of closed cocovers of $\emptyset$. We have $\emptyset = \bigcap_{k\in\mathbb{N}} A_k$, where each A_k is cocompact. For each $k \in \mathbb{N}$ there is a finite subset $\mathcal{K}_k \subseteq \mathcal{F}_k$ such that $\bigcap \mathcal{K}_k \subseteq A_k$. Then $\bigcap_{n\in\mathbb{N}} \bigcap \mathcal{K}_n \subseteq \bigcap_{n\in\mathbb{N}} A_n = \emptyset$, which means that S is co-Menger. □

Example 3.4. There is a ditopological texture space which is Rothberger (hence Menger), but not compact.

Let $(\mathbb{R}, \mathcal{R}, \tau_\mathbb{R}, \kappa_\mathbb{R})$ be the real line with the texture $\mathcal{R} = \{(-\infty, r] : r \in \mathbb{R}\} \cup \{(-\infty, r) : r \in \mathbb{R}\} \cup \{\mathbb{R}, \emptyset\}$, topology $\tau_\mathbb{R} = \{(-\infty, r) : r \in \mathbb{R}\} \cup \{\mathbb{R}, \emptyset\}$ and cotopology $\kappa_\mathbb{R} = \{(-\infty, r] : r \in \mathbb{R}\} \cup \{\mathbb{R}, \emptyset\}$. This ditopological texture space is neither compact (because the open cover $\mathcal{U} = \{(-\infty, n) : n \in \mathbb{N}\}$ does not contain a finite subcover) nor cocompact (because its closed cocover $\{(-\infty, n] : n \in \mathbb{N}\}$ does not contain a finite cocover). But $(\mathbb{R}, \mathcal{R}, \tau_\mathbb{R}, \kappa_\mathbb{R})$ is Rothberger and co-Rothberger. Let us prove that this space is Rothberger. Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of open covers of $\mathbb{R}$. Write $\mathbb{R} = \cup\{(-\infty, n) : n \in \mathbb{N}\}$. For each n, $\mathcal{U}_n$ is an open cover of $\mathbb{R}$, hence there is some $r_n \in \mathbb{R}$ such that $(-\infty, n) \subseteq (-\infty, r_n) \in \mathcal{U}_n$. Then the collection $\{(-\infty, r_n) : n \in \mathbb{N}\}$ shows that $(\mathbb{R}, \mathcal{R}, \tau_\mathbb{R}, \kappa_\mathbb{R})$ is Rothberger.

For complemented ditopological texture spaces we have:

Proposition 3.5. *Let $(S, \mathcal{S}, \sigma)$ be a texture with the complementation σ and let (τ, κ) be a complemented ditopology on $(S, \mathcal{S}, \sigma)$. Then $S \in \mathsf{S}_{fin}(\theta_S, \theta_S)$ if and only if $\emptyset \in \mathsf{S}_{cfin}(\Phi_S, \Phi_S)$.*

Proof. Let $S \in \mathsf{S}_{fin}(\theta_S, \theta_S)$ and let $(\mathcal{F}_n : n \in \mathbb{N})$ be a sequence of closed cocovers of $\emptyset$. Then $(\sigma(\mathcal{F}_n) : n \in \mathbb{N})$ is a sequence of open covers of S. Since $S \in \mathsf{S}_{fin}(\theta_S, \theta_S)$, there is a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ of finite sets such that for each n, $\mathcal{V}_n \subseteq \sigma(\mathcal{F}_n)$ and $\bigvee_{n\in\mathbb{N}} \bigvee \mathcal{V}_n = S$. We have $(\sigma(\mathcal{V}_n) : n \in \mathbb{N})$ is a sequence of finite sets, and also

$$\emptyset = \sigma(S) = \sigma\left(\bigvee_{n\in\mathbb{N}} \bigvee \mathcal{V}_n\right) = \bigcap_{n\in\mathbb{N}} \bigcap \sigma(\mathcal{V}_n).$$

Hence $\emptyset \in \mathsf{S}_{cfin}(\Phi_S, \Phi_S)$.

The proof of $\emptyset \in \mathsf{S}_{cfin}(\Phi_S, \Phi_S)$ implies $S \in \mathsf{S}_{fin}(\theta_S, \theta_S)$ is the dual of the above and is omitted. □

Proposition 3.6. *Let $(S, \mathcal{S}, \sigma)$ be a texture with complementation σ and let (τ, κ) be a complemented ditopology on $(S, \mathcal{S}, \sigma)$. Then for $K \in \kappa$ with $K \neq S$, K is Menger if and only if G is co-Menger for $G \in \tau$ and $G \neq \emptyset$.*

Proof. ($\Longrightarrow$) Take $G \in \tau$ with $G \neq \emptyset$. Let $(\mathcal{F}_n : n \in \mathbb{N})$ be a sequence of closed cocovers of G. Set $K = \sigma(G)$ and we obtain $K \in \kappa$ and $K \neq S$. Since K is Menger, for the sequence $(\sigma(\mathcal{F}_n) : n \in \mathbb{N})$ of open covers of K there is a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ of finite sets such that for each $n \in \mathbb{N}$, $\mathcal{V}_n \subseteq \sigma(\mathcal{F}_n)$ and $\bigvee_{n\in\mathbb{N}} \bigcup \mathcal{V}_n$ is an open cover of K. Thus $(\sigma(\mathcal{V}_n) : n \in \mathbb{N})$ is a sequence of finite sets such that for each $n \in \mathbb{N}$, $\sigma(\mathcal{V}_n) \subseteq \mathcal{F}_n$ and $\sigma(\bigvee_{n\in\mathbb{N}} \bigcup \mathcal{V}_n) = \bigcap_{n\in\mathbb{N}} \bigcap \sigma(\mathcal{V}_n) \subseteq G$ which gives G is co-Menger.

The proof of G is co-Menger with $G \in \tau$ and $G \neq \emptyset$ implies K is Menger with $K \in \kappa$ with $K \neq S$ is dual to this and is omitted. □

4. Operations

Recall [2] that for a texture space $(S, \mathcal{S})$ and a set $A \in \mathcal{S}$ the texturing $\mathcal{S}_A := \{A \cap K : K \in \mathcal{S}\}$ of A is called the *induced texture* on A, and $(A, \mathcal{S}_A)$ is called a *principal subtexture* of $(S, \mathcal{S})$.

Proposition 4.1. *Let $(S, \mathcal{S}, \sigma, \tau, \kappa)$ be a complemented ditopological texture space. If S is Menger and $A \in \kappa$, then $(A, \mathcal{S}_A, \tau_A, \kappa_A)$ is also Menger.*

Proof. Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of open covers of A. Then for each n, $\mathcal{V}_n = \mathcal{U}_n \cup \{\sigma(A)\}$ is an open cover of S. By assumption there are finite families $\mathcal{W}_n \subseteq \mathcal{V}_n$, $n \in \mathbb{N}$, such that $S = \bigvee_{n\in\mathbb{N}} \bigcup \mathcal{W}_n$. Set $\mathcal{H}_n = \mathcal{W}_n \setminus \{\sigma(A)\}$, $n \in \mathbb{N}$. Then for each $n \in \mathbb{N}$, $\mathcal{H}_n$ is a finite subset of $\mathcal{U}_n$ and $A \subseteq \bigvee_{n\in\mathbb{N}} \bigcup \mathcal{H}_n$, i.e. A is Menger. □

Remark 4.2. By Proposition 3.6, in a complemented ditopological space $(S, \mathcal{S}, \sigma, \tau, \kappa)$ it holds: if S is co-Menger and $A \in \tau$, then $(A, \mathcal{S}_A, \tau_A, \kappa_A)$ is also co-Menger.

Remark 4.3. The situation is quite different when $A \subseteq S$ does not belong to $\mathcal{S}$ because then $\mathcal{S}_A$ need not be a texture of A. In this case we consider the smallest element $\xi A \in \mathcal{S}$ containing A, a surjection $\rho : \xi A \to A$ and $\mathcal{S}_A := \{B \subseteq A : \rho^{\leftarrow}(B) \in \mathcal{S}_{\xi A}\}$. If $\mathcal{S}_A$ is a texture of A, then $(A, \mathcal{S}_A)$ is called a subtexture of $(S, \mathcal{S})$ under ρ. Then Mengerness of A is actually Mengerness of $(A, \mathcal{S}_A)$.

If $(S, \mathcal{S})$ and $(T, \mathcal{T})$ are textures, the product texturing $\mathcal{S} \bigotimes \mathcal{T}$ of $S \times T$ consists of arbitrary intersections of sets of the form $(A \times T) \cup (S \times B)$, $A \in \mathcal{S}$, $B \in \mathcal{T}$, and $(S \times T; \mathcal{S} \bigotimes \mathcal{T})$ is called the *product* of $(S; \mathcal{S})$ and $(T; \mathcal{T})$. If $(S, \mathcal{S}, \tau_S, \kappa_S)$ and $(T, \mathcal{T}, \tau_T, \kappa_T)$ are ditopological texture spaces, then the topology and cotopology on $S \times T$ are defined in the standard way similar to topological case.

The proof of the following statement is standard and similar to the corresponding proof for topological spaces and so is omitted.

Theorem 4.4. *Let $(S, \mathcal{S}, \tau_S, \kappa_S)$ and $(T, \mathcal{T}, \tau_T, \kappa_T)$ be ditopological spaces. Then:*

(1) *If $(S, \mathcal{S}, \tau_S, \kappa_S)$ is Menger, and $(T, \mathcal{T}, \tau_T, \kappa_T)$ compact, then their product is Menger;*
(2) *If $(S, \mathcal{S}, \tau_S, \kappa_S)$ is co-Menger, and $(T, \mathcal{T}, \tau_T, \kappa_T)$ cocompact, then their product is co-Menger.*

The notion of difunction is one of the most important notions in the theory of texture structures. We give here necessary definitions following [3,4]. Let $(S, \mathcal{S})$ and $(T, \mathcal{T})$ be textures. A *relation* from $(S, \mathcal{S})$ to $(T, \mathcal{T})$ is defined as $r \in \mathcal{P}(\mathcal{S}) \bigotimes \mathcal{T}$ such that:

$(r1)$ $r \not\subseteq Q_{(s,t)}$ and $P_{s'} \not\subseteq Q_s$ imply $r \not\subseteq Q_{(s',t)}$;
$(r2)$ if $r \not\subseteq Q_{(s,t)}$, then there is $s' \in S$ such that $P_s \not\subseteq Q_{s'}$ and $r \not\subseteq Q_{(s',t)}$.

A *corelation* from $(S, \mathcal{S})$ to $(T, \mathcal{T})$ is defined as $R \in \mathcal{P}(\mathcal{S}) \bigotimes \mathcal{T}$ such that

$(R1)$ $P_{(s,t)} \not\subseteq R$ and $P_s \not\subseteq Q_{s'}$ imply $P_{(s',t)} \not\subseteq R$;
$(R2)$ if $P_{(s,t)} \not\subseteq R$, then there is $s' \in S$ such that $P_{s'} \not\subseteq Q_s$ and $P_{(s',t)} \not\subseteq R$.

A *direlation* from $(S, \mathcal{S})$ to $(T, \mathcal{T})$ is a pair (r, R) such that r is a relation and R is a corelation.

For $A \in \mathcal{S}$ and $B \in \mathcal{T}$ one defines:

$$r^{\to}A = \bigcap\{Q_t : \forall s,\, r \not\subseteq Q_{(s,t)} \Rightarrow A \subseteq Q_s\},$$
$$R^{\to}A = \bigvee\{P_t : \forall s,\, P_{(s,t)} \not\subseteq R \Rightarrow P_s \subseteq A\}$$
$$r^{\leftarrow}B = \bigvee\{P_s : \forall t,\, r \not\subseteq Q_{(s,t)} \Rightarrow P_t \subseteq B\}$$
$$R^{\leftarrow}B = \bigcap\{Q_s : \forall t,\, P_{(s,t)} \not\subseteq R \Rightarrow B \subseteq Q_t\}.$$

The following definition of difunction is in fact a result in [3, Theorem 2.24].

A direlation (f, F) from $(S, \mathcal{S})$ to $(T, \mathcal{T})$ is a *difunction* if for each $A \in \mathcal{S}$ and each $B \in \mathcal{T}$ the following hold: (a) $f^{\leftarrow}(F^{\to}A) \subseteq A \subseteq F^{\leftarrow}(f^{\to}A)$, and (b) $f^{\to}(F^{\leftarrow}B) \subseteq B \subseteq F^{\to}(f^{\leftarrow}B)$.

A difunction $(f, F) : (S, \mathcal{S}, \tau_S, \kappa_S) \to (T, \mathcal{T}, \tau_T, \kappa_T)$ is: (i) *continuous* if $F^{\leftarrow}G \in \tau_S$ whenever $G \in \tau_T$; (ii) *cocontinuous* if $f^{\leftarrow}K \in \kappa_S$ whenever $K \in \kappa_T$; (iii) *bicontinuous* if it is continuous and cocontinuous.

Theorem 4.5. *Let $(S_1, \mathcal{S}_1, \tau_1, \kappa_1)$ and $(S_2, \mathcal{S}_2, \tau_2, \kappa_2)$ be ditopological texture spaces, and let (f, F) be a continuous difunction between them. If $A \in \mathcal{S}_1$ is Menger, then $f^{\to}(A) \in \mathcal{S}_2$ is also Menger. In particular, if (f, F) is surjective and $(S_1, \mathcal{S}_1, \tau_1, \kappa_1)$ is Menger, then $(S_2, \mathcal{S}_2, \tau_2, \kappa_2)$ is also Menger.*

Proof. Let $(\mathcal{V}_n : n \in \mathbb{N})$ be a sequence of τ_2-open covers of $f^{\to}A$. Then, by [3, Th. 2.24(2a); Cor. 2.12(2)] and continuity of (f, F), for each n we have

$$A \subseteq F^{\leftarrow}(f^{\to}A) \subseteq F^{\leftarrow}\left(\bigvee \mathcal{V}_n\right) = \bigvee F^{\leftarrow}\mathcal{V}_n,$$

so that each $F^{\leftarrow}\mathcal{V}_n$ is a τ_1-open cover of A. As A is Menger, there are for each n finite sets $\mathcal{U}_n \subseteq \mathcal{V}_n$ such that $A \subseteq \bigvee_{n\in\mathbb{N}}(\bigcup F^{\leftarrow}(\mathcal{U}_n))$. Thus, by Theorem 2.24(2b) and Corollary 2.12(2) in [3], we have

$$f^{\to}A \subseteq f^{\to}\left(\bigvee_{n\in\mathbb{N}}\left(\bigcup F^{\leftarrow}\mathcal{U}_n\right)\right) = \bigvee_{n\in\mathbb{N}}\bigcup(f^{\to}F^{\leftarrow}\mathcal{U}_n) \subseteq \bigvee_{n\in\mathbb{N}}\bigcup\mathcal{U}_n,$$

i.e. $f^{\to}A$ is Menger.

For the second part we have only to observe that $F^{\leftarrow}S_2 = f^{\leftarrow}S_2 = S_1$, and that for $A \in \mathcal{S}_1$, $A = S_1$ if and only if $F^{\to}A = S_2$. □

We omit the proof of the following similar statement concerning co-Mengerness.

Theorem 4.6. *Let $(S_1, \mathcal{S}_1, \tau_1, \kappa_1)$ and $(S_2, \mathcal{S}_2, \tau_2, \kappa_2)$ be ditopological texture spaces, and let (f, F) be a continuous difunction between them. If $A \in \mathcal{S}_1$ is co-Menger, then $F^{\to}(A) \in \mathcal{S}_2$ is also co-Menger. In particular, if (f, F) is surjective and $(S_1, \mathcal{S}_1, \tau_1, \kappa_1)$ is co-Menger, then $(S_2, \mathcal{S}_2, \tau_2, \kappa_2)$ is also co-Menger.*

Call a ditopological space $(S, \mathcal{S}, \tau, \kappa)$ *M-stable* if every $K \in \kappa$ with $K \neq S$ is Menger, and *M-costable* if every $G \in \tau$ with $G \neq \emptyset$ is co-Menger.

Theorem 4.7. *Let $(S_1, \mathcal{S}_1, \tau_1, \kappa_1)$ and $(S_2, \mathcal{S}_2, \tau_2, \kappa_2)$ be ditopological texture spaces, and let (f, F) be a bicontinuous surjective difunction between them. If $(S_1, \mathcal{S}_1, \tau_1, \kappa_1)$ is M-stable, then $(S_2, \mathcal{S}_2, \tau_2, \kappa_2)$ is also M-stable.*

Proof. Let $K \in \kappa_2$ be such that $K \neq S_2$. Since (f, F) is cocontinuous we have that $f^{\leftarrow}K \in \kappa_1$. Let us prove that $f^{\leftarrow}K \neq S_1$. Suppose this is not true. Since (f, F) is surjective, we have $f^{\leftarrow}S_2 = S_1$ which, by [3, Lemma 2.28(1c)], implies $f^{\leftarrow}S_2 \subseteq f^{\leftarrow}K$. Since (f, F) is surjective, then Corollary 2.33(1 ii) in [3] implies $S_2 \subseteq K$. This contradiction shows that $f^{\leftarrow}K \neq S_1$.

Since $(S_1, \mathcal{S}_1, \tau_1, \kappa_1)$ is M-stable, the set $f^{\leftarrow}K$ is a Menger set. Continuity of (f, F) and Theorem 4.5 implies that $f^{\to}(f^{\leftarrow}K)$ is a Menger set in $(S_2, \mathcal{S}_2, \tau_2, \kappa_2)$. By [3, Corollary 2.33 (1)] the latter set is equal to K, and this completes the proof of the theorem. □

The proof of the following theorem is omitted because it is dual and parallel to the proof of the previous theorem.

Theorem 4.8. *Let $(S_1, \mathcal{S}_1, \tau_1, \kappa_1)$ and $(S_2, \mathcal{S}_2, \tau_2, \kappa_2)$ be ditopological texture spaces, and let (f, F) be a bicontinuous surjective difunction between them. If $(S_1, \mathcal{S}_1, \tau_1, \kappa_1)$ is M-costable, then $(S_2, \mathcal{S}_2, \tau_2, \kappa_2)$ is also M-costable.*

Call a ditopological texture space $(S, \mathcal{S}, \tau, \kappa)$ *di-Menger* if it is Menger, co-Menger, M-stable and M-costable.

From Theorems 4.5, 4.6, 4.7, 4.8 we have:

Theorem 4.9. *Let (f, F) be a surjective bicontinuous difunction from a di-Menger ditopological texture space $(S_1, \mathcal{S}_1, \tau_1, \kappa_1)$ to a ditopological texture space $(S_2, \mathcal{S}_2, \tau_2, \kappa_2)$. Then $(S_2, \mathcal{S}_2, \tau_2, \kappa_2)$ is also di-Menger.*

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