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Real dicompact textures

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ABSTRACT

A notion of real compactness for completely biregular bi-T_2 ditopological texture spaces is defined and studied under the name real dicompactness. In particular it is shown that real dicompact spaces are nearly plain $*$-spaces, and an important characterization is presented. Finally the connection of this work with topological and bitopological real compactness is discussed in a categorical setting.

1. Introduction

The interplay between various topological properties of a space X and certain types of ideal in the ring $C(X)$ of continuous real-valued functions on X is well known [15], and is intimately involved in the definition and study of real compactness. When seeking to establish a corresponding theory for bitopological spaces, the second author considered in [2, Chapter 3] the set $\mathrm{P}(X)$ of pairwise continuous functions from a bitopological space (X, u, v) in the sense of Kelly [17] to the real bitopological space $(\mathbb{R}, s, t)$, where $s = \{(-\infty, r) \mid r \in \mathbb{R}\} \cup \{\mathbb{R}, \emptyset\}$ and $t = \{(r, \infty) \mid r \in \mathbb{R}\} \cup \{\mathbb{R}, \emptyset\}$. Unlike $C(X)$ this is not a ring, but an additive lattice, and this property of $\mathrm{P}(X)$ was further abstracted to the notion of a T-lattice, that is a distributive lattice A with distinguished element 0 and a suitable family of translations $T_r : A \to A$, $r \in \mathbb{R}$. In the theory developed in [2] the role of the ring ideals is played by the bi-ideals, pairs (L, M) consisting of a lattice ideal L and a dual lattice ideal M satisfying $0 \in L \cap M$. Various notions of regularity for bi-ideals were introduced, including a notion of real bi-ideal, and these were used to define bireal compactness. A characterization of bireal compactness was presented which shows that this notion coincides with the bitopological real compactness considered by Brümmer and Salbany in [10], and several other properties of this class of spaces were investigated.

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The aim of this paper is to carry over the concepts and some of the results given in [2] to the much more general framework of ditopological texture spaces. As in the authors' recent paper [23] on dicompact spaces we work within the same general class of completely biregular bi-T_2 spaces, and this enables us to use many of the notions and results given in that paper. Indeed, the layout of [23] was designed explicitly to provide the foundation necessary for the study of real compactness in a ditopological setting, as well as presenting important results on the more specific class of dicompact spaces. In particular the study in [23] highlights the importance for dicompact bi-T_2 spaces of the notion of a nearly plain texture $(S, \mathcal{S}, \tau, \kappa)$ and its associated plain space $(S_p, \mathcal{S}_p, \tau_p, \kappa_p)$, and also of $*$-spaces, the T-lattices $\mathrm{BA}(S)$ and $\mathrm{BA}(S_p)$, and the notion of a bi-ideal being difixed. All of these are of equal importance in the present study.

The layout of this paper is as follows. The remainder of this introduction is given over to some background material. No attempt is made at completeness, our aim being to give just enough material to enable a casual reader to gain a general idea of the contents of the paper, although an exception is made for the required material on bi-ideals from [2], since this is not currently available as a paper. Section 2 gives the definition of B-real dicompactness for a bigenerating subset B of $\mathrm{BA}(S)$ and various fundamental results are given, including a characterization in terms of powers of the real texture $(\mathbb{R}, \mathcal{R}, \tau_{\mathbb{R}}, \kappa_{\mathbb{R}})$. Finally, Section 3 gives various results of a categorical nature. In particular, the relation between the ditopological theory presented here and the bitopological study in [2] is investigated in some detail.

An earlier version of the main results given in Section 2, and the Hewitt Isomorphism Theorem from Section 3, occur in the PhD thesis of the first author [22], written with the partial support of Grant Number 06 T03 604005 awarded by Hacettepe University.

Ditopological texture spaces

There is now a considerable literature on the theory of ditopological texture spaces, and an adequate introduction to this theory and the motivation for its study may be obtained from [4–8].

Briefly, if S is a set, a *texturing* $\mathcal{S}$ of S is a subset of $\mathcal{P}(S)$ which is a point-separating, complete, completely distributive lattice containing S and $\emptyset$, and for which meet coincides with intersection and finite joins with union. The pair $(S, \mathcal{S})$ is then called a *texture*. We regard a texture $(S, \mathcal{S})$ as a framework in which to do mathematics.

For a texture $(S, \mathcal{S})$, most properties are conveniently defined in terms of the *p-sets* $P_s = \bigcap\{A \in \mathcal{S} \mid s \in A\}$ and the *q-sets*, $Q_s = \bigvee\{A \in \mathcal{S} \mid s \notin A\}$. However, as noted in [3] we may associate with $(S, \mathcal{S})$ the C-space (core-space) [11–13,16,18] $(S, \mathcal{S}^c)$, and then the frequently occurring relationship $P_{s'} \not\subseteq Q_s$, $s, s' \in S$, is equivalent to $s \omega_S s'$, where ω_S is the *interior relation* for $(S, \mathcal{S}^c)$. In this paper we will use whichever notation seems to be the more convenient in each particular instance.

In general a texturing $\mathcal{S}$ need not be closed under the operation of taking the set complement, so in the context of a texture $(S, \mathcal{S})$ the notion of topology is replaced by that of dichotomous topology. A *dichotomous topology*, or *ditopology* for short, on a texture $(S, \mathcal{S})$ is a pair (τ, κ) of subsets of $\mathcal{S}$, where the set of *open sets* τ satisfies

(1) $S, \emptyset \in \tau$,
(2) $G_1, G_2 \in \tau \Rightarrow G_1 \cap G_2 \in \tau$, and
(3) $G_i \in \tau$, $i \in I \Rightarrow \bigvee_i G_i \in \tau$,

and the set of *closed sets* κ satisfies

(1) $S, \emptyset \in \kappa$,
(2) $K_1, K_2 \in \kappa \Rightarrow K_1 \cup K_2 \in \kappa$, and
(3) $K_i \in \kappa$, $i \in I \Rightarrow \bigcap K_i \in \kappa$.

Hence a ditopology is essentially a "topology" for which there is no *a priori* relation between the open and closed sets. As will be clear from the references given above, ditopological texture spaces provide a unified setting for the study of topology, bitopology and fuzzy topology. We will not be concerned with the links with fuzzy topology in this paper, but the relation with bitopology will be considered in some detail in the final section. See also [20] in this context.

We recall the product of textures and of ditopological texture spaces. Let $(S_j, \mathcal{S}_j)$, $j \in J$, be textures and $S = \prod_{j \in J} S_j$. If $A_k \in \mathcal{S}_k$ for some $k \in J$ we write

$$E(k, A_k) = \prod_{j \in J} Y_j \quad \text{where } Y_j = \begin{cases} A_j, & \text{if } j = k, \\ S_j, & \text{otherwise.} \end{cases}$$

Then the *product texturing* $\mathcal{S} = \bigotimes_{j \in J} \mathcal{S}_j$ of S consists of arbitrary intersections of elements of the set

$$\mathcal{E} = \left\{ \bigcup_{j \in J} E(j, A_j) \;\middle|\; A_j \in \mathcal{S}_j \text{ for } j \in J \right\}.$$

Let $(S_j, \mathcal{S}_j)$, $j \in J$ be textures and $(S, \mathcal{S})$ their product. Then for $s = (s_j) \in S$,

$$P_s = \bigcap_{j\in J} E(j, P_{s_j}) = \prod_{j\in J} P_{s_j} \quad \text{and} \quad Q_s = \bigcup_{j\in J} E(j, Q_{s_j}).$$

In case (τ_j, κ_j) is a ditopology on $(S_j, \mathcal{S}_j)$, $j \in J$, the *product ditopology* on the product texture $(S, \mathcal{S})$ has subbase $\{E(j, G) \mid G \in \tau_j,\ j \in J\}$, cosubbase $\gamma = \{E(j, K) \mid K \in \kappa_j,\ j \in J\}$.

We recall from [8, Theorem 4.17] that a ditopological space $(S, \mathcal{S}, \tau, \kappa)$ is bi-T_2 if given $s, s' \in S$ with $Q_s \not\subseteq Q_{s'}$ there exists $H \in \tau, K \in \kappa$ with $H \subseteq K$, $P_s \not\subseteq K$ and $H \not\subseteq Q_{s'}$. This is the form of the Hausdorff property considered in [8], and it arises naturally in various contexts such as that of separated di-uniformities and of dimetrics [19].

Various special classes of textures have been considered. Here we will be concerned primarily with plain textures and nearly plain textures. The texture $(S, \mathcal{S})$ is *plain* if $\mathcal{S}$ is closed under arbitrary unions, equivalently if the corresponding C-space is an Alexandroff-discrete [9] or A-space [12], or if the interior relation ω_S is reflexive. The more general class of nearly plain textures was introduced in [23]. The texture $(S, \mathcal{S})$ is *nearly plain* if given $s \in S$ there exists a point $w \in S$ satisfying $Q_s = Q_w$ and $w\omega_S w$. This "plain" point w is necessarily unique, and setting $\varphi_p(s) = w$ gives a mapping from S to the set S_p of plain points. The texturing $\mathcal{S}$ on S induces a plain texturing $\mathcal{S}_p$ on S_p, and if (τ, κ) is a ditopology on $(S, \mathcal{S})$ we obtain the induced ditopology (τ_p, κ_p) on $(S_p, \mathcal{S}_p)$. The plain ditopological space $(S_p, \mathcal{S}_p, \tau_p, \kappa_p)$ will play an important role in this paper as it does in [23], and the reader is referred to that paper for a detailed discussion of the relation between the spaces $(S, \mathcal{S}, \tau, \kappa)$ and $(S_p, \mathcal{S}_p, \tau_p, \kappa_p)$. We recall from [23] the *joint topology* $\mathcal{J}_{\tau\kappa}$ on S_p, which is defined in terms of its family $\mathcal{J}^c_{\tau\kappa}$ of closed sets by the condition

$$W \in \mathcal{J}^c_{\tau\kappa} \quad \Leftrightarrow \quad \big(s \in S_p,\ G \in \eta^*(s),\ K \in \mu^*(s) \ \Rightarrow\ G \cap W \not\subseteq K\big) \quad \Rightarrow \quad s \in W,$$

where

$$\eta^*(s) = \{A \in \mathcal{S} \mid \exists G_k \in \tau \text{ with } G_k \not\subseteq Q_s,\ 1 \leqslant k \leqslant n \text{ and } G_1 \cap \cdots \cap G_n \subseteq A\},$$

and

$$\mu^*(s) = \{A \in \mathcal{S} \mid \exists F_k \in \kappa \text{ with } P_s \not\subseteq F_k,\ 1 \leqslant k \leqslant n \text{ and } A \subseteq F_1 \cup \cdots \cup F_n\}.$$

Nearly plain textures share with plain textures the property that a difunction [6, Definition 2.22] between them may be represented by an ω-preserving point function between their base sets [23, Theorem 2.10]. Here $\varphi : S \to T$ is *ω-preserving* as a function from $(S, \mathcal{S})$ to $(T, \mathcal{T})$ if $s_1\omega_S s_2 \Rightarrow \varphi(s_1)\omega_T\varphi(s_2)$ (often referred to as "condition (a)" in earlier papers). Note that this condition does not guarantee $\varphi^{-1}[B] \in \mathcal{S}$ for $B \in \mathcal{T}$, so the inverse image $\varphi^{\leftarrow}B$,

$$\varphi^{\leftarrow}B = \bigvee\big\{P_u \bigm| \varphi(u) \in B\big\} = \bigcap\big\{Q_v \bigm| \varphi(v) \notin B\big\},$$

inherited from the inverse image for the corresponding difunction is used in its place. Hence $\varphi : (S, \mathcal{S}, \tau_S, \kappa_S) \to (T, \mathcal{T}, \tau_T, \kappa_T)$ is *bicontinuous* if $G \in \tau_T \Rightarrow \varphi^{\leftarrow}G \in \tau_S$ and $K \in \kappa_T \Rightarrow \varphi^{\leftarrow}K \in \kappa_S$. We note that when these textures are plain we have $\varphi^{\leftarrow}B = \varphi^{-1}[B]$, and we will then use whichever notation seems the most appropriate in a given situation.

It will transpire that, as for dicompact bi-T_2 spaces, real dicompact bi-T_2 spaces are nearly plain, so it suffices to consider ω-preserving point functions in place of difunctions. Such functions are also of interest in a wider context and in particular we recall from [23] the following characterization of complete biregularity [8].

Proposition 1.1. *Let $(S, \mathcal{S}, \tau, \kappa)$ be a ditopological texture space. Then (τ, κ) is completely biregular if and only if the following conditions hold*:

(1) *Given $G \in \tau$, $a \in S$ with $G \not\subseteq Q_a$ there exists an ω-preserving bicontinuous point function $\varphi : (S, \mathcal{S}, \tau, \kappa) \to (\mathbb{R}, \mathcal{R}, \tau_{\mathbb{R}}, \kappa_{\mathbb{R}})$ satisfying $-\mathbf{1} \leqslant \varphi \leqslant \mathbf{1}$, and for which $P_a \subseteq \varphi^{\leftarrow}P_{-1}$ and $\varphi^{\leftarrow}Q_1 \subseteq G$.*
(2) *Given $K \in \kappa$, $a \in S$ with $P_a \not\subseteq K$ there exists an ω-preserving bicontinuous point function $\varphi : (S, \mathcal{S}, \tau, \kappa) \to (\mathbb{R}, \mathcal{R}, \tau_{\mathbb{R}}, \kappa_{\mathbb{R}})$ satisfying $-\mathbf{1} \leqslant \varphi \leqslant \mathbf{1}$, and for which $\varphi^{\leftarrow}Q_1 \subseteq Q_a$ and $K \subseteq \varphi^{\leftarrow}P_{-1}$.*

Our attention will be focused on the set $\mathrm{BA}(S)$ of bicontinuous ω-preserving point functions from $(S, \mathcal{S}, \tau, \kappa)$ to the *real ditopological texture space* $(\mathbb{R}, \mathcal{R}, \tau_{\mathbb{R}}, \kappa_{\mathbb{R}})$ which is defined by $\mathcal{R} = \{(-\infty, r] \mid r \in \mathbb{R}\} \cup \{(-\infty, r) \mid r \in \mathbb{R}\} \cup \{\mathbb{R}, \emptyset\}$, $\tau_{\mathbb{R}} = \{(-\infty, r) \mid r \in \mathbb{R}\} \cup \{\mathbb{R}, \emptyset\}$ and $\kappa_{\mathbb{R}} = \{(-\infty, r] \mid r \in \mathbb{R}\} \cup \{\mathbb{R}, \emptyset\}$. This is a T-lattice under $T_r(\varphi) = \varphi + (-\mathbf{r})$, where $\mathbf{r}$ denotes the constant function with value $r \in \mathbb{R}$. This will be the T-lattice we use in place of $\mathrm{P}(X)$, although we will show that it is equivalent, as far as real dicompactness is concerned, to consider $\mathrm{BA}(S_p)$ instead.

For $\varphi \in \mathrm{BA}(S)$ we recall from [23] the functions $\varphi_*, \varphi^* \in \mathrm{BA}(S)$ given by

$$\varphi_*(s) = \sup\big\{\varphi(v) \bigm| v\omega_S s\big\}, \qquad \varphi^*(s) = \inf\big\{\varphi(u) \bigm| s\omega_S u\big\}, \quad \forall s \in S. \tag{1.1}$$

In general we have $\varphi_* \leqslant \varphi \leqslant \varphi^*$. We note also the equalities

$$(\varphi \vee \psi)_* = \varphi_* \vee \psi_* \quad \text{and} \quad (\varphi \wedge \psi)^* = \varphi^* \wedge \psi^*, \quad \forall \varphi, \psi \in \mathrm{BA}(S), \tag{1.2}$$

the inequalities

$$\varphi^* \vee \psi^* \leqslant (\varphi \vee \psi)^* \quad \text{and} \quad (\varphi \wedge \psi)_* \leqslant \varphi_* \wedge \psi_*, \quad \forall \varphi, \psi \in \mathrm{BA}(S), \tag{1.3}$$

with equality if ψ is constant, and the relation

$$\varphi_* \leqslant \psi \quad \Leftrightarrow \quad \varphi \leqslant \psi^*, \quad \forall \varphi, \psi \in \mathrm{BA}(S) \tag{1.4}$$

for future reference. We note that $(S, \mathcal{S}, \tau, \kappa)$ is called a $*$-*space* [23] if $\varphi_* = \varphi^*$ for all $\varphi \in \mathrm{BA}(S)$. A plain ditopological space is clearly a $*$-space, so for a nearly plain space $(S, \mathcal{S}, \tau, \kappa)$, $(S_p, \mathcal{S}_p, \tau_p, \kappa_p)$ is always a $*$-space.

Finally, we recall the basic constructs **fTex** and **fDitop** from [6,7], respectively.

Bi-ideals in *T*-lattices

We will require a little more of the theory of bi-ideals in T-lattices from [2] than is given in [23].

We recall that a distributive lattice [14] A with a distinguished element 0 is called a T-*lattice* if there exists a mapping $T : \mathbb{R} \times A \to A$ for which the mappings $T_r : A \to A$ given by $T_r(a) = T(r, a)$ $\forall a \in A$, $r \in \mathbb{R}$, satisfy:

(i) $T_r : A \to A$ is a lattice homomorphism for each $r \in \mathbb{R}$.
(ii) $T_r \circ T_s = T_s \circ T_r = T_{r+s}$ for all $r, s \in \mathbb{R}$.
(iii) $T_r(a) = a \Leftrightarrow r = 0$, for all $a \in A$.
(iv) $T_r(a) \leqslant a$ for all $a \in A$ and $r > 0$.

We note in particular that the mapping $r \mapsto T_{-r}(0)$ is an injection of $\mathbb{R}$ into A which takes $0 \in \mathbb{R}$ to the distinguished element $0 \in A$. For this reason, $T_{-r}(0)$ may be denoted by r.

Definition 1.2.

(1) The binary relation ρ on A is a *dispersion* if for all $a, b, a', b' \in A$ it satisfies:
  (i) $a\rho b$, $a \leqslant a'$ and $b \geqslant b' \Rightarrow a'\rho b'$, and
  (ii) $a\rho b$, $a\rho b'$, $a'\rho b$ and $a'\rho b' \Rightarrow (a \wedge a')\rho(b \vee b')$.
(2) A *bi-ideal* in A is a pair (L, M) consisting of a lattice ideal L and a lattice dual ideal M with $0 \in L \cap M$.
(3) If ρ is a dispersion on A, the bi-ideal (L, M) is ρ-*regular* if $(L \times M) \cap \rho = \emptyset$.

Bi-ideals are partially ordered by $(L, M) \preccurlyeq (L', M') \Leftrightarrow L \subseteq L'$ and $M \subseteq M'$. By Zorn's Lemma each ρ-regular bi-ideal has a maximal ρ-regular refinement.

Definition 1.3. On the T-lattice A the dispersions ρ_e and ρ_b are given by

$$\rho_e = \{(a, b) \in A \times A \mid \exists r \in \mathbb{R} \text{ with } b \leqslant r < 0 \text{ or } 0 < r \leqslant a\},$$
$$\rho_b = \{(a, b) \in A \times A \mid \exists r > 0 \text{ with } T_r(a \vee 0) \geqslant b \wedge 0\}.$$

Note that, since $\rho_e \subseteq \rho_b$, a ρ_b-regular bi-ideal is also ρ_e-regular. Also, $a \in L \Rightarrow T_r(a) \notin M$ $\forall r > 0$ and $a \in M \Rightarrow T_r(a) \notin L$ $\forall r < 0$ are both necessary and sufficient conditions for (L, M) to be ρ_b-regular.

For a fixed bi-ideal (L, M) the equivalence relation $\sim$ on A is defined by

$$a \sim b \quad \Leftrightarrow \quad \big(T_r(a) \in L \Leftrightarrow T_r(b) \in L\big) \quad \text{and} \quad \big(T_r(a) \in M \Leftrightarrow T_r(b) \in M\big).$$

The quotient set $A/(L, M)$ is partially ordered by

$$[a] \leqslant [b] \quad \Leftrightarrow \quad \big(T_r(b) \in L \Rightarrow T_r(a) \in L\big) \quad \text{and} \quad \big(T_r(a) \in M \Rightarrow T_r(b) \in M\big),$$

and if (L, M) is ρ_e-regular, $r \mapsto [r]$ is an order preserving injection of $\mathbb{R}$ into $A/(L, M)$. In case the image of $\mathbb{R}$ under this mapping is the whole of $A/(L, M)$ the bi-ideal is called *real*. For a ρ_e regular bi-ideal (L, M), (L, M) is real if and only if given $a \in A$ there exists a unique real number α such that $T_r(a) \in L \cap M \Leftrightarrow r = \alpha$. Each real bi-ideal is maximal ρ_b-regular. It is also *prime*, that is L is a prime ideal and M a prime dual ideal.

If (L, M) is a bi-ideal the bi-ideal (L^+, M^+) is defined by

$$L^+ = \{a \in A \mid T_r(a) \in L \ \forall r > 0\} \quad \text{and} \quad M^+ = \{a \in A \mid T_r(a) \in M \ \forall r < 0\}. \tag{1.5}$$

We note that $(L, M) \preccurlyeq (L^+, M^+)$, and that (L, M) is ρ_e-regular or ρ_b-regular if and only if the same is true of (L^+, M^+). Hence, if (L, M) is real, $(L, M) = (L^+, M^+)$.

The element $[a]$ of $A/(L,M)$ is called *finite* if $[r_1] \leqslant [a] \leqslant [r_2]$ for some $r_1, r_2 \in \mathbb{R}$, it is *infinite* if $[r] \leqslant [a]$ for all $r \in \mathbb{R}$ or $[a] \leqslant [r]$ for all $r \in \mathbb{R}$. The element $a \in A$ is *finite* or *infinite at* (L,M) when $[a]$ has the corresponding property in $A/(L,M)$. Finally, (L,M) is called *finite* if every element of A is finite at (L,M).

According to [2, Lemma 3.1.7, Corollary], if (L,M) is ρ_e-regular then the only infinite elements of $A/(L,M)$ are the greatest and least elements, when these exist. We shall also need the following result:

Proposition 1.4. *(See [2, Proposition 3.1.7, Corollary 2].) Any maximal ρ_b regular refinement of a finite bi-ideal is real.*

Now let B and C be sub-T-lattices of A and suppose that $B \subseteq C$. Then any dispersion ρ on A induces a dispersion on B and C which we continue to denote by ρ.

If (L,M) is a bi-ideal in C then clearly $(L \cap B, M \cap B)$ is a bi-ideal in B. Moreover, if (L,M) is maximal ρ-regular in C, $(L \cap B, M \cap B)$ is maximal ρ-regular in B.

On the other hand suppose that (L,M) is a ρ-regular bi-ideal in B and assume that $\rho_e \subseteq \rho$. Let

$$L_C = \{c \in C \mid \exists b \in L \text{ and } \epsilon > 0 \text{ with } c \wedge \epsilon \leqslant b\},$$
$$M_C = \{c \in C \mid \exists b \in M \text{ and } \epsilon > 0 \text{ with } b \leqslant c \vee (-\epsilon)\}. \tag{1.6}$$

Then (L_C, M_C) is a ρ-regular bi-ideal in C which is contained in every prime ρ-regular bi-ideal in C whose restriction to B is (L,M). Hence if (L,M) is maximal ρ-regular in B there exists at least one maximal ρ-regular bi-ideal in C whose restriction to B is (L,M).

With B, C as above, C is called a (*finite*) *ρ-refinement of B* if every (finite) maximal ρ-regular bi-ideal in B has a unique extension to a (finite) maximal ρ-regular bi-ideal in C.

B is called (*finitely*) *ρ-complete in A* if it has no proper (finite) ρ-refinement in A. A (finitely) ρ-complete (finite) ρ-refinement of B will be called a (*finite*) *ρ-completion of B*.

It is shown in [2, Theorem 3.1.3] that every sub-T-lattice of A has a unique ρ_b-completion and a unique finite ρ_b-completion in A.

An element $a \in A$ is *bounded* if $r_1 \leqslant a \leqslant r_2$ for some $r_1, r_2 \in \mathbb{R}$. The set of bounded elements of A is denoted by A^*. Clearly, A^* is a sub-T-lattice of A. It is shown in [2] that A^* is finitely-ρ_b complete, and that the ρ_b-completion of A^* is A.

Now let B be a subset of A containing 0. We denote by $\langle B\rangle$ the smallest sub-T-lattice of A containing B. Its elements are obtained from those of B by a finite number of applications of the operations $\vee$, $\wedge$ and T_r.

Let $g : B \to \mathbb{R}$ be a function satisfying $g(0) = 0$, and define

$$L^g = \left\{a \in \langle B\rangle \;\middle|\; \exists b_1, \ldots, b_n \in B,\ r > 0 \text{ with } a \wedge r \leqslant \left(\bigvee_{i=1}^{n} T_{g(b_i)} b_i\right) \vee 0\right\},$$
$$M^g = \left\{a \in \langle B\rangle \;\middle|\; \exists b_1, \ldots, b_n \in B,\ r > 0 \text{ with } a \vee -r \geqslant \left(\bigwedge_{i=1}^{n} T_{g(b_i)} b_i\right) \wedge 0\right\}. \tag{1.7}$$

Then (L^g, M^g) is a bi-ideal. If this bi-ideal is ρ_e-regular, g is called a *B-resolution* and (L^g, M^g) the corresponding B-derivative. The set of all B-resolutions is denoted by R_B.

A real bi-ideal (L,M) in $\langle B\rangle$ determines a $\langle B\rangle$-resolution $g : \langle B\rangle \to \mathbb{R}$ by $T_{g(a)}(a) \in L \cap M$, and (L,M) is the $\langle B\rangle$-derivative of g. It is shown in [2, Proposition 3.2.1] that a $\langle B\rangle$-resolution g has a real derivative if and only if belongs to $H_{\langle B\rangle}$, the set of T-lattice homomorphisms of $\langle B\rangle$ to $\mathbb{R}$ considered as a T-lattice under $T_r(x) = x - r$, $r, x \in \mathbb{R}$. Hence, $H_{\langle B\rangle}$ is in one to one correspondence with the real bi-ideals in $\langle B\rangle$.

If $g \in H_{\langle B\rangle}$ we have the equalities

$$L^g = \{a \in \langle B\rangle \mid g(a) \leqslant 0\} \quad \text{and} \quad M^g = \{a \in \langle B\rangle \mid g(a) \geqslant 0\},$$

and we also note that in this case $(L^{g|_B}, M^{g|_B}) = (L^g, M^g)$.

Finally we mention some concepts from [23] specific to the T-lattice $\mathrm{BA}(S)$. For $s \in S$ we define $L(s) = \{\varphi \in \mathrm{BA}(S) \mid \varphi(s) \leqslant 0\}$, $M(s) = \{\varphi \in \mathrm{BA}(S) \mid \varphi(s) \geqslant 0\}$. Then $(L(s), M(s))$ is a real bi-ideal in $\mathrm{BA}(S)$ and a ρ_b regular bi-ideal (L,M) in $\mathrm{BA}(S)$ is called *difixed* if there is a (necessarily unique) element $s \in S_p$ satisfying $(L,M) = (L(s), M(s))$.

A bi-ideal (L,M) in $\mathrm{BA}(S)$ is called a $*$bi-ideal if L and M are closed under the operations $_*$ and *.

There is an important link between real bi-ideals in $\mathrm{BA}(S)$ and difilters [21] in the space $(S, \mathcal{S}, \tau, \kappa)$. In particular, if (L,M) is a ρ_b-regular $*$bi-ideal then $Z_d(L,M)$ defined by $Z_d(L,M) = \mathcal{F}_L \times \mathcal{G}_M$,

$$\mathcal{F}_L = \{A \in \mathcal{S} \mid \exists \varphi \in L,\ r > 0 \text{ with } \varphi^{\leftarrow} P_r \subseteq A\},$$
$$\mathcal{G}_M = \{A \in \mathcal{S} \mid \exists \varphi \in M,\ r > 0 \text{ with } A \subseteq \varphi^{\leftarrow} Q_{-r}\},$$

is a regular difilter. Moreover, if (L,M) is a real $*$bi-ideal, then $Z_d(L,M)$ is diconvergent if and only if (L,M) is difixed.

Naturally, in a $*$-space these properties hold for a bi-ideal (L,M) with the appropriate property.

2. Real dicompactness of ditopological texture spaces

Throughout the remainder of this paper, $(S, \mathcal{S}, \tau, \kappa)$ will denote a completely biregular bi-T_2 ditopological texture space, unless stated otherwise.

We will be interested in subsets of $\mathrm{BA}(S)$ which are "bigenerating" in the sense that they contain enough functions to determine the topology τ and cotopology κ. The following lemma and the definition that follows will make this concept exact.

Lemma 2.1. *For $B \subseteq \mathrm{BA}(S)$ the following are equivalent*:

(1) (i) *the family $\{\varphi^{\leftarrow} Q_r \mid \varphi \in B,\ r \in \mathbb{R}\}$ is a subbase for τ, and*
(ii) *the family $\{\varphi^{\leftarrow} P_r \mid \varphi \in B,\ r \in \mathbb{R}\}$ is a subbase of κ.*

(2) *For each $s \in S^\flat$,*
(i) $\eta^*(s) = \{A \in \mathcal{S} \mid \exists \varphi_1, \ldots, \varphi_n \in B,\ \epsilon_1, \ldots, \epsilon_n > 0$ *with* $T_{\varphi_i(s)}(\varphi_i)^{\leftarrow} Q_{\epsilon_i} \not\subseteq Q_s\ \forall i$ *and* $\bigcap_{i=1}^n T_{\varphi_i(s)}(\varphi_i)^{\leftarrow} Q_{\epsilon_i} \subseteq A\}$, *and*
(ii) $\mu^*(s) = \{A \in \mathcal{S} \mid \exists \varphi_1, \ldots, \varphi_n \in B,\ \epsilon_1, \ldots, \epsilon_n > 0$ *with* $P_s \not\subseteq T_{\varphi_i(s)}(\varphi_i)^{\leftarrow} P_{-\epsilon_i}\ \forall i$ *and* $A \subseteq \bigcup_{i=1}^n T_{\varphi_i(s)}(\varphi_i)^{\leftarrow} P_{-\epsilon_i}\}$.

Proof. Straightforward. □

Definition 2.2. A set $B \subseteq \mathrm{BA}(S)$ which satisfies $\mathbf{0} \in B$ will be called *bigenerating* if it satisfies one, and hence both, of the conditions (1) and (2) in Lemma 2.1.

The set $\mathrm{BA}(S)$ itself is bigenerating. Indeed $\mathbf{0} \in \mathrm{BA}(S)$, so take $G \in \tau$ and $s \in S$ with $G \not\subseteq Q_s$. By Proposition 1.1 there exists $\varphi \in \mathrm{BA}(S)$ with $P_s \subseteq \varphi^{\leftarrow} P_{-1} \subseteq \varphi^{\leftarrow} Q_{-\frac{1}{2}} \subseteq \varphi^{\leftarrow} Q_1 \subseteq G$. This shows that the family $\{\varphi^{\leftarrow} Q_r \mid \varphi \in \mathrm{BA}(S),\ r < 0\}$ is a base for τ, whence Lemma 2.1(1)(i) holds, and (1)(ii) may be proved likewise.

Definition 2.3. For $s \in S$ the mapping $\hat{s} : \mathrm{BA}(S) \to \mathbb{R}$ is defined by $\hat{s}(\varphi) = \varphi(s)$.

Clearly, $\hat{s} \in H_{\mathrm{BA}(S)}$. Take $B \subseteq \mathrm{BA}(S)$ with $\mathbf{0} \in B$. To give a necessary and sufficient condition for B to be bigenerating let us define:

$$L^{\hat{s}|_B}_{\mathrm{BA}(S)} = \left\{\varphi \in \mathrm{BA}(S) \;\middle|\; \exists \varphi_1, \ldots, \varphi_n \in B,\ \delta > 0 \text{ with } \varphi_* \wedge \boldsymbol{\delta} \leqslant \bigvee_{i=1}^n T_{\hat{s}(\varphi_i)}(\varphi_i) \vee \mathbf{0}\right\},$$

$$M^{\hat{s}|_B}_{\mathrm{BA}(S)} = \left\{\varphi \in \mathrm{BA}(S) \;\middle|\; \exists \varphi_1, \ldots, \varphi_n \in B,\ \delta > 0 \text{ with } \varphi^* \vee -\boldsymbol{\delta} \geqslant \bigwedge_{i=1}^n T_{\hat{s}(\varphi_i)}(\varphi_i) \wedge \mathbf{0}\right\}.$$

See (1.1) for the definitions of φ_*, φ^*. Using (1.2) and (1.3) it is easy to see that $(L^{\hat{s}|_B}_{\mathrm{BA}(S)}, M^{\hat{s}|_B}_{\mathrm{BA}(S)})$ is a bi-ideal. Moreover, by (1.6) and (1.7), in a $*$-space this bi-ideal coincides with $((L^{\hat{s}|_B})_{\mathrm{BA}(S)}, (M^{\hat{s}|_B})_{\mathrm{BA}(S)})$.

Proposition 2.4. *The following are equivalent for $\mathbf{0} \in B \subseteq \mathrm{BA}(S)$.*

(1) *B is bigenerating.*
(2) *For $s, t \in S$ with $t \omega_S s$ we have $(L(s), M(t)) \preccurlyeq ([L^{\hat{t}|_B}_{\mathrm{BA}(S)}]^+, [M^{\hat{s}|_B}_{\mathrm{BA}(S)}]^+)$.*

Proof. *Necessity*. To prove the first inclusion take $P_s \not\subseteq Q_t$, $\varphi \in L(s)$ and $\epsilon > 0$. Then $\varphi(s) \in Q_\epsilon$ and so $P_s \subseteq \varphi^{\leftarrow} Q_\epsilon$, whence $\varphi^{\leftarrow} Q_\epsilon \not\subseteq Q_t$. Since B is bigenerating we now have $\varphi_1, \ldots \varphi_n \in B$, $r_1, \ldots, r_n \in \mathbb{R}$ satisfying

$$\bigcap_{i=1}^n \varphi_i^{\leftarrow} Q_{r_i} \subseteq \varphi^{\leftarrow} Q_\epsilon \quad \text{and} \quad \bigcap_{i=1}^n \varphi_i^{\leftarrow} Q_{r_i} \not\subseteq Q_t.$$

From the second relation we have $\varphi_i(t) \in Q_{r_i}$, hence $\varphi_i(t) < r_i$ and we may choose $\delta > 0$ for which $\varphi_i(t) + \delta < r_i$ for $1 \leqslant i \leqslant n$. It is now easy to verify that

$$(\varphi_* - \boldsymbol{\epsilon}) \wedge \boldsymbol{\delta} \leqslant \bigvee_{i=1}^n \bigl(\varphi_i - \varphi_i(t)\bigr) \vee \mathbf{0},$$

whence $T_\epsilon(\varphi)_* \wedge \boldsymbol{\delta} \leqslant \bigvee_{i=1}^n T_{\hat{t}(\varphi_i)}(\varphi_i) \vee \mathbf{0}$. This gives $T_\epsilon(\varphi) \in L^{\hat{t}|_B}_{\mathrm{BA}(S)}$, and since $\epsilon > 0$ is arbitrary, $\varphi \in [L^{\hat{t}|_B}_{\mathrm{BA}(S)}]^+$, as required. The proof of $M(t) \subseteq [M^{\hat{s}|_B}_{\mathrm{BA}(S)}]^+$ is dual to this, and is omitted.

Sufficiency. Since the sets $\varphi^{\leftarrow} Q_r$ form a base for τ, take $s \in S$ with $\varphi^{\leftarrow} Q_r \not\subseteq Q_s$. We must show the existence of $\varphi_1, \ldots, \varphi_n \in B$ and $r_1, \ldots, r_n \in \mathbb{R}$ satisfying

$$P_s \subseteq \bigcap_{i=1}^{n} \varphi_i^{\leftarrow} Q_{r_i} \subseteq \varphi^{\leftarrow} Q_r. \tag{2.1}$$

Choose $t \in S$ with $\varphi^{\leftarrow} Q_r \not\subseteq Q_t$ and $P_t \not\subseteq Q_s$. By hypothesis $L(t) \subseteq [L^{\hat{s}|_B}_{\mathrm{BA}(S)}]^+$, and $\varphi(t) < r$. Choose $\alpha \in \mathbb{R}$ with $\varphi(t) < \alpha < r$. Then $T_\alpha(\varphi) \in L^{\hat{s}|_B}_{\mathrm{BA}(S)}$, and so there exist $\varphi_1, \ldots, \varphi_n \in B$ and $\delta > 0$ with $T_\alpha(\varphi)_* \wedge \boldsymbol{\delta} \leqslant \bigvee_{i=1}^{n} T_{\hat{s}(\varphi_i)}(\varphi_i) \vee \mathbf{0}$. If we choose $\theta \in \mathbb{R}$ satisfying $0 < \theta < \min\{\delta, r - \alpha\}$ it is not difficult to verify (2.1) for these $\varphi_i \in B$ and $r_i = \varphi_i(s) + \theta \in \mathbb{R}$, $i = 1, \ldots, n$.

This verifies that B generates the topology τ, and likewise it generates the cotopology κ. □

Definition 2.5. Let $(S, \mathcal{S}, \tau, \kappa)$ be a ditopological texture space and $B \subseteq \mathrm{BA}(S)$ bigenerating.

(i) $\langle B \rangle^*$ denotes the smallest T-lattice containing B which is also closed under the operations $_*$ and *. That is, the elements of $\langle B \rangle^*$ are obtained from B by a finite number of applications of the operations $\vee$, $\wedge$, T_r, $_*$ and *.
(ii) $(S, \mathcal{S}, \tau, \kappa)$ will be called *B-real dicompact* if every real bi-ideal in $\langle B \rangle^*$ is difixed.

In particular a $\mathrm{BA}(S)$-real dicompact space will be called *real dicompact*.

Example 2.6. Consider the space $(\mathbb{R}, \mathcal{R}, \tau_{\mathbb{R}}, \kappa_{\mathbb{R}})$ and put $B = \{\iota_{\mathbb{R}}, \mathbf{0}\}$, where $\iota_{\mathbb{R}}$ is the identity function on $\mathbb{R}$. Lemma 2.1(1) is trivially satisfied by B, so it is a bigenerating subset of $\mathrm{BA}(\mathbb{R})$. We show that $(\mathbb{R}, \mathcal{R}, \tau_{\mathbb{R}}, \kappa_{\mathbb{R}})$ is B-real dicompact.

We begin by noting that $(\mathbb{R}, \mathcal{R}, \tau_{\mathbb{R}}, \kappa_{\mathbb{R}})$ is plain and hence a $*$-space. Hence $\langle B \rangle^* = \langle B \rangle$, and we take a real bi-ideal (L, M) in $\langle B \rangle$. For some $\alpha \in \mathbb{R}$ we have $T_\alpha(\iota_{\mathbb{R}}) \in L \cap M$. We show that (L, M) is difixed by α. However for $\varphi = \iota_{\mathbb{R}} \in B$ and $\varphi = \mathbf{0} \in B$ we have $T_r(\varphi) \in L \Leftrightarrow \varphi(\alpha) \leqslant r$ and $T_r(\varphi) \in M \Leftrightarrow \varphi(\alpha) \geqslant r$, while a simple induction argument on the form of the elements of $\langle B \rangle$ shows the same to be true for all $\varphi \in \langle B \rangle$, so we deduce $(L, M) = (L(\alpha), M(\alpha))$, as required. This establishes that $(\mathbb{R}, \mathcal{R}, \tau_{\mathbb{R}}, \kappa_{\mathbb{R}})$ is B-real dicompact.

Proposition 2.7. *Let $B, C \subseteq \mathrm{BA}(S)$ be bigenerating and suppose that $\langle B \rangle^* \subseteq \langle C \rangle^*$. Then if $(S, \mathcal{S}, \tau, \kappa)$ is B-real dicompact it is C-real dicompact.*

Proof. Let (L, M) be a real bi-ideal in $\langle C \rangle^*$. Then $(L \cap \langle B \rangle^*, M \cap \langle B \rangle^*)$ is a real bi-ideal in $\langle B \rangle^*$, and hence difixed by some $s \in S_p$. Hence $(L \cap \langle B \rangle^*, M \cap \langle B \rangle^*) = (L(s) \cap \langle B \rangle^*, M(s) \cap \langle B \rangle^*)$, while as $s \omega_S s$, $(L(s), M(s)) = ([L^{\hat{s}|_B}_{\mathrm{BA}(S)}]^+, [M^{\hat{s}|_B}_{\mathrm{BA}(S)}]^+)$ by Proposition 2.4 and the definitions. Set $L^{\hat{s}|_B}_{\langle C \rangle^*} = L^{\hat{s}|_B}_{\mathrm{BA}(S)} \cap \langle C \rangle^*$. We wish to show that $L^{\hat{s}|_B}_{\langle C \rangle^*} \subseteq L$. Suppose on the contrary that there exists $\varphi \in L^{\hat{s}|_B}_{\langle C \rangle^*}$ with $\varphi \notin L$. Then $L' = \{\psi \in \langle C \rangle^* \mid \exists \mu \in L \text{ with } \psi \leqslant \mu \vee \varphi\}$ is an ideal in $\langle C \rangle^*$ properly containing L. Suppose that (L', M) is not ρ_b-regular. Then we have $\psi \in L'$, $\theta \in M$ and $r > 0$ for which $T_r(\psi \vee \mathbf{0}) \geqslant \theta \wedge \mathbf{0}$. Take $\mu \in L$ with $\psi \leqslant \mu \vee \varphi$, $\varphi_1, \ldots, \varphi_n \in B$ and $\delta > 0$ satisfying $\varphi_* \wedge \boldsymbol{\delta} \leqslant \bigvee_{i=1}^{n} T_{\hat{s}(\varphi_i)}(\varphi_i) \vee \mathbf{0}$. Finally, choose $r' \in \mathbb{R}$ with $0 < r' < \min\{r, \delta\}$. Bearing in mind that $\varphi_* \wedge \boldsymbol{\delta} \leqslant \bigvee_{i=1}^{n} T_{\hat{s}(\varphi_i)}(\varphi_i) \vee \mathbf{0}$ is equivalent to $\varphi \wedge \boldsymbol{\delta} \leqslant (\bigvee_{i=1}^{n} T_{\hat{s}(\varphi_i)}(\varphi_i))^* \vee \mathbf{0}$ by (1.4) and (1.3) as $\boldsymbol{\delta}$ is constant, it is then straightforward to verify that

$$\theta \wedge \mathbf{0} \leqslant T_{r'}\left(\mu \vee \left(\bigvee_{i=1}^{n} T_{\hat{s}(\varphi_i)}(\varphi_i)\right)^* \vee \mathbf{0}\right).$$

However, $(\bigvee_{i=1}^{n} T_{\hat{s}(\varphi_i)}(\varphi_i))^* \in L^{\hat{s}|_B}_{\mathrm{BA}(S)} \cap \langle B \rangle^*$, so $\mu \vee (\bigvee_{i=1}^{n} T_{\hat{s}(\varphi_i)}(\varphi_i))^* \in L$ and we have a contradiction to the fact that (L, M) is ρ_b-regular. Hence, (L', M) is ρ_b-regular, and since it contains (L, M) properly this contradicts the fact that (L, M) is maximal ρ_b-regular. Hence $L^{\hat{s}|_B}_{\langle C \rangle^*} \subseteq L$ and so

$$L(s) \cap \langle C \rangle^* = \left[L^{\hat{s}|_B}_{\langle C \rangle^*}\right]^+ \subseteq L^+ = L.$$

Likewise we may show that $M(s) \cap \langle C \rangle^* \subseteq M$ and so (L, M) is difixed by s. Hence, $(S, \mathcal{S}, \tau, \kappa)$ is C-real dicompact. □

Corollary 2.8. *A B-real dicompact space is real dicompact.*

Proof. For a bigenerating set B we have $\langle B \rangle^* \subseteq \mathrm{BA}(S) = \langle \mathrm{BA}(S) \rangle^*$, so the result follows from Proposition 2.7. □

In particular, it follows from Example 2.6 and the above corollary that the real ditopological texture space $(\mathbb{R}, \mathcal{R}, \tau_{\mathbb{R}}, \kappa_{\mathbb{R}})$ is real dicompact.

Proposition 2.9. *If $(S, \mathcal{S}, \tau, \kappa)$ is B-real dicompact then it is a nearly plain $*$-space.*

Proof. By Corollary 2.8 we may assume that $(S, \mathcal{S}, \tau, \kappa)$ is real dicompact. Take $s \in S$. Then $(L(s), M(s))$ is a real bi-ideal in $\mathrm{BA}(S)$ and hence difixed by some $w \in S_p$. Since $L(s) = L(w)$ we have $\varphi(s) = \varphi(w)$ for all $\varphi \in \mathrm{BA}(S)$. Hence, since $\varphi_* \in \mathrm{BA}(S)$ and $w \omega_S w$ we have $\varphi_*(s) = \varphi_*(w) = \varphi(w) = \varphi(s)$. Since s is arbitrary $\varphi_* = \varphi$, and likewise $\varphi^* = \varphi$, so $(S, \mathcal{S}, \tau, \kappa)$ is a $*$-space.

To show that $(S, \mathcal{S})$ is nearly plain it will suffice to show that $Q_s = Q_w$. Suppose that $Q_w \nsubseteq Q_s$. Then since $(S, \mathcal{S}, \tau, \kappa)$ is bi-T_2 we have $H \in \tau$, $K \in \kappa$ with $H \subseteq K$, $P_w \nsubseteq K$ and $H \nsubseteq Q_s$ by [8, Theorem 4.17]. By complete biregularity we have $\varphi, \psi \in \mathrm{BA}(S)$, $r_1, r_2 \in \mathbb{R}$ with $\varphi^{\leftarrow} Q_{r_1} \subseteq H$, $\varphi^{\leftarrow} Q_{r_1} \nsubseteq Q_s$ and $K \subseteq \psi^{\leftarrow} P_{r_2}$, $P_w \nsubseteq \psi^{\leftarrow} P_{r_2}$. Now $\varphi(s) < r_1$, $\psi(w) > r_2$ so we may choose $\delta > 0$ with $\varphi(s) + \delta < r_1$, $\psi(w) - \delta > r_2$, and it is then easy to verify that

$$\big(\psi - \psi(w) + \boldsymbol{\delta}\big) \wedge \boldsymbol{\delta} \leqslant \big(\varphi - \varphi(s)\big)^* \vee \mathbf{0}.$$

Hence, $\delta = (\psi(w) - \psi(w) + \delta) \leqslant (\varphi - \varphi(s))^*(w) \vee 0 = (\varphi - \varphi(s))(w) \vee 0 = 0$ since $\varphi(w) = \varphi(s)$, and we have a contradiction. Hence $Q_w \subseteq Q_s$, and the opposite inclusion is proved likewise. □

Remark 2.10. If for a bigenerating set B we assume only that the real ideals in $\langle B \rangle$ are difixed the latter part of the above proof may easily be modified to show that $(S, \mathcal{S})$ is nearly plain. However, to conclude that $(S, \mathcal{S}, \tau, \kappa)$ is a $*$-space would seem to require the stronger assumption used in the definition of B-real dicompactness, namely that every real bi-ideal in $\langle B \rangle^*$ is difixed.

Naturally, if it is known in advance that $(S, \mathcal{S}, \tau, \kappa)$ is a completely biregular bi-T_2 $*$-space then $\langle B \rangle^* = \langle B \rangle$ for all bigenerating sets B, so the B-real dicompact spaces are characterized by the condition that all the real bi-ideals in $\langle B \rangle$ are difixed. Moreover, since all bi-ideals are then $*$-bi-deals, it is equivalent to require all the real $*$-bi-ideals in $\langle B \rangle$ are difixed.

Proposition 2.11. *Let $B, C \subseteq \mathrm{BA}(S)$ be bigenerating sets with $\langle B \rangle \subseteq \langle C \rangle$ and suppose that $(S, \mathcal{S}, \tau, \kappa)$ is C-real dicompact. Then the following are equivalent*:

(1) *The space $(S, \mathcal{S}, \tau, \kappa)$ is B-real dicompact.*
(2) *The bi-ideal $(L_{\langle C \rangle}, M_{\langle C \rangle})$ is finite for each real bi-ideal (L, M) in $\langle B \rangle$.*
(3) *The T-lattice $\langle C \rangle$ is a finite ρ_b-refinement of $\langle B \rangle$.*

Here, $(L_{\langle C \rangle}, M_{\langle C \rangle})$ is defined as in (1.6).

Proof. (1) ⇒ (2). Let (L, M) be a real bi-ideal in $\langle B \rangle$. Then (L, M) is difixed by some $s \in S_p$, so $(L, M) = (L(s), M(s)) = (L^{\hat{s}|_B}, M^{\hat{s}|_B})$ and we obtain

$$\big(L_{\langle C \rangle}, M_{\langle C \rangle}\big) = \big(\big(L^{\hat{s}|_B}\big)_{\langle C \rangle}, \big(M^{\hat{s}|_B}\big)_{\langle C \rangle}\big). \tag{2.2}$$

Since B is bigenerating we may apply Proposition 2.4 to give $(L(s), M(s)) = ([L^{\hat{s}|_B}_{\mathrm{BA}(S)}]^+, [M^{\hat{s}|_B}_{\mathrm{BA}(S)}]^+)$ since $s \omega_S s$. Hence, bearing in mind that we are dealing with a $*$-space we have $(L(s) \cap \langle C \rangle, M(s) \cap \langle C \rangle) = ([(L^{\hat{s}|_B})_{\langle C \rangle}]^+, [(M^{\hat{s}|_B})_{\langle C \rangle}]^+)$. Comparing this with (2.2) now gives

$$\big(\big[L_{\langle C \rangle}\big]^+, \big[M_{\langle C \rangle}\big]^+\big) = \big(L(s) \cap \langle C \rangle, M(s) \cap \langle C \rangle\big).$$

It follows that $([L_{\langle C \rangle}]^+, [M_{\langle C \rangle}]^+)$ is real in $\langle C \rangle$, and hence in particular finite. Hence $(L_{\langle C \rangle}, M_{\langle C \rangle})$ is finite also.

(2) ⇒ (3). Let (L, M) be a real bi-ideal in $\langle B \rangle$, and (L', M') any maximal ρ_b-regular extension of (L, M) to $\langle C \rangle$. Since $(L_{\langle C \rangle}, M_{\langle C \rangle}) \preccurlyeq (L', M')$, and $(L_{\langle C \rangle}, M_{\langle C \rangle})$ is finite by hypothesis, we have from Proposition 1.4 that (L', M') is real. Since $(S, \mathcal{S}, \tau, \kappa)$ is C-real dicompact, (L', M') is difixed by some $s \in S_p$, and so $(L', M') = (L(s) \cap \langle C \rangle, M(s) \cap \langle C \rangle)$. In particular, $(L, M) = (L(s) \cap \langle B \rangle, M(s) \cap \langle B \rangle)$, so (L, M) is difixed by s.

Now if (L, M) is also difixed by s' then since the points s and s' are plain, the argument used in the proof of [23, Lemma 4.9] gives $(L(s) \cap \langle B \rangle, M(s) \cap \langle B \rangle) = (L(s') \cap \langle B \rangle, M(s') \cap \langle B \rangle) \Rightarrow s \omega_S s'$ and $s' \omega_S s \Rightarrow s = s'$. This means that $(L(s) \cap \langle C \rangle,$ $M(s) \cap \langle C \rangle)$ is the unique maximal ρ_b-regular extension of (L, M) to $\langle C \rangle$, and hence $\langle C \rangle$ is a finite ρ_b-refinement of $\langle B \rangle$.

(3) ⇒ (1). Let (L, M) be a real bi-ideal in $\langle B \rangle$ and (L', M') its unique real extension to $\langle C \rangle$. By hypothesis (L', M') is difixed, and clearly (L, M) is difixed by the same element of S_p. Hence, $(S, \mathcal{S}, \tau, \kappa)$ is B-real dicompact. □

Corollary 2.12. *Let $(S, \mathcal{S}, \tau, \kappa)$ be real dicompact and $B \subseteq \mathrm{BA}(S)$ bigenerating. Then $(S, \mathcal{S}, \tau, \kappa)$ is B-real dicompact if and only if the finite ρ_b-completion of $\langle B \rangle$ is* $\mathrm{BA}(S)$.

Lemma 2.13. *Let $(S, \mathcal{S}, \tau, \kappa)$ be nearly plain, $(S_p, \mathcal{S}_p, \tau_p, \kappa_p)$ the associated plain space and $\varphi_p : S \to S_p$ the bicontinuous surjection. Then*:

(1) *For $\varphi \in \mathrm{BA}(S_p)$ we have $(\varphi \circ \varphi_p)|_{S_p} = \varphi$.*
(2) *For $\varphi \in \mathrm{BA}(S)$ we have $\varphi_* \leqslant (\varphi|_{S_p}) \circ \varphi_p \leqslant \varphi^*$.*

Proof. (1) is trivial since φ_p is the identity on S_p, so we prove (2). If the first inequality is false we have $s \in S$ with $\varphi_*(s) > (\varphi|_{S_p})(\varphi_p(s))$. Let $w = \varphi_p(s) \in S_p$. Then $\varphi_*(s) > \varphi(w)$ so we have $P_s \not\subseteq Q_v$ with $\varphi(v) > \varphi(w)$. However $P_w \not\subseteq Q_w = Q_s$, so $P_w \not\subseteq Q_v$ and we have the contradiction $\varphi(v) \leqslant \varphi(w)$ because φ is ω-preserving. Hence the first inequality is established, and the proof of the second inequality is dual and hence omitted. □

Proposition 2.14. *Let $(S, \mathcal{S}, \tau, \kappa)$ be a completely biregular bi-T_2 space and B a subset of* BA(S).

(1) *Let B be bigenerating and $(S, \mathcal{S}, \tau, \kappa)$, B-real dicompact. Then $(S, \mathcal{S}, \tau, \kappa)$ is a nearly plain $*$-space. Denote by α the mapping $\varphi \mapsto \varphi|_{S_p}$ from* BA(S) *to* BA(S_p). *Then α is a T-lattice isomorphism, $\alpha(B)$ is bigenerating in $(S_p, \mathcal{S}_p, \tau_p, \kappa_p)$ and $(S_p, \mathcal{S}_p, \tau_p, \kappa_p)$ is $\alpha(B)$-real dicompact.*
(2) *Let $(S, \mathcal{S}, \tau, \kappa)$ be a nearly plain $*$-space and define the isomorphism α as in* (1). *Let $\alpha(B)$ be bigenerating and $(S_p, \mathcal{S}_p, \tau_p, \kappa_p)$, $\alpha(B)$-real dicompact. Then $(S, \mathcal{S}, \tau, \kappa)$ is B-real dicompact.*

Proof. (1) Firstly, $(S, \mathcal{S}, \tau, \kappa)$ is a nearly plain $*$-space by Proposition 2.9. The mapping $\alpha : \mathrm{BA}(S) \to \mathrm{BA}(S_p)$ given by $\alpha(\varphi) = \varphi|_{S_p}$ is clearly a T-lattice homomorphism which is onto by Lemma 2.13(1). Since $(S, \mathcal{S}, \tau, \kappa)$ is a $*$-space we have $\varphi = (\varphi|_{S_p}) \circ \varphi_p$ for each $\varphi \in \mathrm{BA}(S)$ by Lemma 2.13(2), so α is injective and hence a T-lattice isomorphism. Moreover, the inverse α^{-1} is given by $\varphi \mapsto \varphi \circ \varphi_p$, $\varphi \in \mathrm{BA}(S_p)$.

It is straightforward to show that $\alpha(B) \subseteq \mathrm{BA}(S_p)$ is bigenerating, either directly or by verifying $\alpha(L(s)) = L_p(s)$, $\alpha(L_{\mathrm{BA}(S)}^{\hat{s}|_B}) = L_{\mathrm{BA}(S_p)}^{\hat{s}|_{\alpha(B)}}$, and corresponding results for the dual ideals, for all $s \in S_p$ and using Proposition 2.4. The space $(S_p, \mathcal{S}_p, \tau_p, \kappa_p)$, being plain, is a $*$-space so by Remark 2.10 we must show every real bi-ideal (L, M) in $\langle \alpha(B) \rangle$ is difixed. However, $(\alpha^{-1}L, \alpha^{-1}M)$ is a real bi-ideal in $\langle B \rangle$ and so difixed by some $s \in S_p$, again by Remark 2.10, and it is easy to verify that (L, M) is also difixed by s. Hence, $(S_p, \mathcal{S}_p, \tau_p, \kappa_p)$ is $\alpha(B)$-real dicompact.

(2) Under the given hypothesis the properties of α stated above still hold. In this case for $s \in S$ we may verify $\alpha(L(s)) = L_p(\varphi_p(s))$, $\alpha(L_{\mathrm{BA}(S)}^{\hat{s}|_B}) = L_{\mathrm{BA}(S_p)}^{\widehat{\varphi_p(s)}|_{\alpha(B)}}$, etc., and the proof then follows similarly to that of (1). □

We now present an important characterization of B-real dicompact spaces. In view of Proposition 2.14 there will be no loss of generality in considering a nearly plain $*$-space $(S, \mathcal{S}, \tau, \kappa)$, and in restricting our attention to the plain space $(S_p, \mathcal{S}_p, \tau_p, \kappa_p)$ and the T-lattice $\mathrm{BA}(S_p)$. Throughout the discussion below, therefore, B will be a bigenerating subset of $\mathrm{BA}(S_p)$ and we seek a necessary and sufficient condition for $(S_p, \mathcal{S}_p, \tau_p, \kappa_p)$ to be B-real dicompact.

Denote by $(\mathbb{R}^{\langle B \rangle}, \mathcal{R}^{\langle B \rangle}, \tau_{\mathbb{R}}^{\langle B \rangle}, \kappa_{\mathbb{R}}^{\langle B \rangle})$ the product of $\langle B \rangle$ copies of the ditopological texture space $(\mathbb{R}, \mathcal{R}, \tau_{\mathbb{R}}, \kappa_{\mathbb{R}})$. Since a product of plain textures is plain, this is a plain texture and hence its restriction to the subset $H_{\langle B \rangle}$ of $\mathbb{R}^{\langle B \rangle}$ is also a plain ditopological texture space. We denote this space by $(H_{\langle B \rangle}, \mathcal{H}_{\langle B \rangle}, \tau_{\langle B \rangle}, \kappa_{\langle B \rangle})$, and define the mapping

$$\xi_{\langle B \rangle} : S_p \to H_{\langle B \rangle}$$

by $\xi_{\langle B \rangle}(s) = \hat{s}|_{\langle B \rangle}$ for all $s \in S_p$. Note that in $(H_{\langle B \rangle}, \mathcal{H}_{\langle B \rangle}, \tau_{\langle B \rangle}, \kappa_{\langle B \rangle})$ we have

$$P_h = H_{\langle B \rangle} \cap \bigcap_{\varphi \in \langle B \rangle} E(\varphi, P_{h(\varphi)}), \qquad Q_h = H_{\langle B \rangle} \cap \left(\bigcup_{\varphi \in \langle B \rangle} E(\varphi, Q_{h(\varphi)}) \right), \tag{2.3}$$

for $h \in H_{\langle B \rangle}$ by [6, Proposition 1.3] and the fact that the texture is plain, and the p-sets and q-sets in the subspace $\xi_{\langle B \rangle}(S_p)$ are given by the same formulae with $H_{\langle B \rangle}$ replaced by $\xi_{\langle B \rangle}(S_p)$. We have:

Lemma 2.15. *For $s_1, s_2 \in S_p$ the following are equivalent.*

(1) $s_1 \omega_p s_2$.
(2) $\varphi(s_1) \leqslant \varphi(s_2)$ $\forall \varphi \in \langle B \rangle$.
(3) $\xi_{\langle B \rangle}(s_1) \omega_{\xi_{\langle B \rangle}(S_p)} \xi_{\langle B \rangle}(s_2)$.

Proof. (1) ⇒ (2) is immediate since the elements of $\langle B \rangle$ are ω-preserving, and (2) ⇔ (3) follows from the formulae (2.3). It remains to prove (2) ⇒ (1). Suppose $\varphi(s_1) \leqslant \varphi(s_2)$ $\forall \varphi \in \langle B \rangle$ but that $P_{s_2} \subseteq Q_{s_1}$. Then $P_{s_2} \not\subseteq Q_{s_2}$ gives $Q_{s_1} \not\subseteq Q_{s_2}$. Since $(S_p, \mathcal{S}_p, \tau_p, \kappa_p)$ is bi-T_2 we have $H \in \tau_p$, $K \in \kappa_p$ with $P_{s_2} \subseteq H \subseteq K$ and $P_{s_1} \not\subseteq K$, so since B is bigenerating we have $\varphi_i, \psi_j \in B$, $\epsilon_i, \delta_j > 0$ with

$$\bigcap_{i=1}^{n} T_{\varphi(s_2)}(\varphi_i)^{\leftarrow} Q_{\epsilon_i} \subseteq H \subseteq K \subseteq \bigcup_{j=1}^{m} T_{\psi_j(s_1)}(\psi_j)^{\leftarrow} P_{-\delta_j}.$$

However this implies $s_2 \in T_{\psi_j(s_1)}(\psi_j)^{\leftarrow} P_{-\delta_j}$ for some j, and hence we obtain the contradiction $0 \leqslant \psi_j(s_2) - \psi_j(s_1) < -\delta_j$ since $\psi_j \in B \subseteq \langle B \rangle$. □

Corollary 2.16. *$\xi_{\langle B\rangle}$ is a **fTex** isomorphism between $(S_p, \mathcal{S}_p)$ and the texture induced by $(H_{\langle B\rangle}, \mathcal{H}_{\langle B\rangle})$ on $\xi_{\langle B\rangle}(S_p)$.*

Proof. By Lemma 2.15, $\xi_{\langle B\rangle}$ is a bijection between S_p and the subset $\xi_{\langle B\rangle}(S_p)$ of $H_{\langle B\rangle}$ since in a plain texture $(T, \mathcal{T})$, $t_1 = t_2 \Leftrightarrow P_{t_1} \subseteq P_{t_2}$ and $P_{t_2} \subseteq P_{t_1} \Leftrightarrow t_1 \omega_T t_2$ and $t_2 \omega_T t_1$. Moreover, the same lemma shows that both $\xi_{\langle B\rangle}$ and its inverse are ω-preserving. For plain textures this is sufficient to ensure that $\xi_{\langle B\rangle}$ is a **fTex** isomorphism [6]. □

The following proposition considerably strengthens the above result.

Proposition 2.17.

(1) *$\xi_{\langle B\rangle}$ is an **fDitop** isomorphism between $(S_p, \mathcal{S}_p, \tau_p, \kappa_p)$ and its image in $(H_{\langle B\rangle}, \mathcal{H}_{\langle B\rangle}, \tau_{\langle B\rangle}, \kappa_{\langle B\rangle})$. Moreover, $\xi_{\langle B\rangle}(S_p)$ is a jointly dense subset of $H_{\langle B\rangle}$.*
(2) *$H_{\langle B\rangle}$ is jointly closed in $(\mathbb{R}^{\langle B\rangle}, \mathcal{R}^{\langle B\rangle}, \tau_{\mathbb{R}}^{\langle B\rangle}, \kappa_{\mathbb{R}}^{\langle B\rangle})$.*

Proof. (1) The bicontinuity of $\xi_{\langle B\rangle}$ follows from the evident equality

$$\xi_{\langle B\rangle}^{\leftarrow}\big(\xi_{\langle B\rangle}(S_p) \cap E(\varphi, A)\big) = \varphi^{\leftarrow}A, \quad \forall A \in \mathcal{R} \text{ and } \forall \varphi \in \langle B\rangle,$$

and the fact that $\{\xi_{\langle B\rangle}(S_p) \cap E(\varphi, G) \mid \varphi \in \langle B\rangle,\ G \in \tau_{\mathbb{R}}\}$ is a subbase for the topology and $\{\xi_{\langle B\rangle}(S_p) \cap E(\varphi, K) \mid \varphi \in \langle B\rangle, K \in \kappa_{\mathbb{R}}\}$ a subbase for the cotopology on $\xi_{\langle B\rangle}(S_p)$. Likewise, the bicontinuity of the inverse mapping $\xi_{\langle B\rangle}^{-1}$ follows from

$$\big(\xi_{\langle B\rangle}^{-1}\big)^{\leftarrow}\varphi^{\leftarrow}A = \xi_{\langle B\rangle}(S_p) \cap E(\varphi, A), \quad \forall A \in \mathcal{R} \text{ and } \forall \varphi \in \langle B\rangle$$

and Lemma 2.1(1).

Finally, we must show that if $\xi_{\langle B\rangle}(S_p) \subseteq W \in \mathcal{J}^c_{\tau_{\langle B\rangle}\kappa_{\langle B\rangle}}$ then $W = H_{\langle B\rangle}$. Suppose on the contrary that there exists such a set W with $W \neq H_{\langle B\rangle}$, and take $h \in H_{\langle B\rangle}$ with $h \notin W$. Then we have $G \in \eta^*(h)$, $K \in \mu^*(h)$ satisfying $G \cap W \subseteq K$, and hence

$$G \cap \xi_{\langle B\rangle}(S_p) \subseteq K. \tag{2.4}$$

By the definition of the product ditopology we have $\varphi_i \in \langle B\rangle$, $r_i \in \mathbb{R}$, $i = 1, \dots, n$ with $P_h \subseteq H_{\langle B\rangle} \cap \bigcap_{i=1}^n E(\varphi_i, Q_{r_i}) \subseteq G$ and $\psi_j \in \langle B\rangle$, $k_j \in \mathbb{R}$, $j = 1, \dots, m$ with $K \subseteq H_{\langle B\rangle} \cap (\bigcup_{j=1}^m E(\psi_j, P_{k_j})) \subseteq Q_h$. This gives $h(\varphi_i) < r_i$, $h(\psi_j) > k_j$ and so we may choose $\epsilon > 0$ with $h(\varphi_i) + 2\epsilon < r_i$, $h(\psi_j) - 2\epsilon > k_j$ for all i and j. Let $r_i' = h(\varphi_i) + \epsilon$, $k_j' = h(\psi_j) - \epsilon$ and consider the real bi-ideal (L^h, M^h), where $L^h = \{\varphi \in \langle B\rangle \mid h(\varphi) \leqslant 0\}$, $M^h = \{\psi \in \langle B\rangle \mid h(\psi) \geqslant 0\}$ since $h \in H_{\langle B\rangle}$. We may apply [23, Proposition 3.5(2)] to deduce that $Z_d(L^h, M^h)$ is a regular difilter since we are working in the plain texture $(S_p, \mathcal{S}_p, \tau_p, \kappa_p)$. However, we clearly have $\varphi_i - r_i' \in L^h$, $i = 1, \dots, n$ and $\psi_j - k_j' \in M^h$, $j = 1, \dots, m$, so

$$\bigcap_{i=1}^n \big(\varphi_i - r_i'\big)^{\leftarrow} P_\epsilon \not\subseteq \bigcup_{j=1}^m \big(\psi_j - k_j'\big)^{\leftarrow} Q_{-\epsilon},$$

and taking $s \in \bigcap_{i=1}^n (\varphi_i - r_i')^{\leftarrow} P_\epsilon$, $s \notin \bigcup_{j=1}^m (\psi_j - k_j')^{\leftarrow} Q_{-\epsilon}$ leads easily to $\xi_{\langle B\rangle}(s) \in G$, $\xi_{\langle B\rangle}(s) \notin K$, which contradicts (2.4).

(2) By the discussion in [23], a subbase for the joint topology on the plain space $(\mathbb{R}^{\langle B\rangle}, \mathcal{R}^{\langle B\rangle}, \tau_{\mathbb{R}}^{\langle B\rangle}, \kappa_{\mathbb{R}}^{\langle B\rangle})$ is

$$\big\{E(\varphi, Q_r) \bigm| \varphi \in \langle B\rangle,\ r \in \mathbb{R}\big\} \cup \big\{\mathbb{R}^{\langle B\rangle} \setminus E(\psi, P_k) \bigm| \psi \in \langle B\rangle,\ k \in \mathbb{R}\big\}. \tag{2.5}$$

Take g in the closure of $H_{\langle B\rangle}$ for this topology. We must show that $g \in H_{\langle B\rangle}$.

If $g(\mathbf{0}) < 0$ then $E(\mathbf{0}, Q_0)$ is a nhd. of g for the joint topology, so we may find $h \in E(\mathbf{0}, Q_0) \cap H_{\langle B\rangle}$. However, $h(\mathbf{0}) = 0$ for $h \in H_{\langle B\rangle}$, which gives the contradiction $h \notin E(\mathbf{0}, Q_0)$. Likewise, $g(\mathbf{0}) > 0$ leads to a contradiction and we have established $g(\mathbf{0}) = 0$.

Let us verify that

$$g(\varphi \vee \psi) = g(\varphi) \vee g(\psi)$$

for all $\varphi, \psi \in \langle B\rangle$. Suppose that $g(\varphi \vee \psi) > g(\varphi) \vee g(\psi)$ and let $\epsilon > 0$ satisfy $g(\varphi \vee \psi) - \epsilon = g(\varphi) \vee g(\psi) + \epsilon$. Now

$$E\big(\varphi, Q_{g(\varphi)+\epsilon}\big) \cap E\big(\psi, Q_{g(\psi)+\epsilon}\big) \cap \big(\mathbb{R}^{\langle B\rangle} \setminus E\big(\varphi \vee \psi, P_{g(\varphi\vee\psi)-\epsilon}\big)\big)$$

is a nhd. of g in the joint topology on $\mathbb{R}^{\langle B\rangle}$, and so meets $H_{\langle B\rangle}$ in some element h. Hence $h(\varphi) < g(\varphi) + \epsilon$, $h(\psi) < g(\psi) + \epsilon$ and $h(\varphi \vee \psi) > g(\varphi \vee \psi) - \epsilon$, so

$$g(\varphi \vee \psi) - \epsilon < h(\varphi \vee \psi) = h(\varphi) \vee h(\psi) < g(\varphi) \vee g(\psi) + \epsilon,$$

which is a contradiction. Likewise, $g(\varphi \vee \psi) < g(\varphi) \vee g(\psi)$ leads to a contradiction, and we have shown that $g(\varphi \vee \psi) = g(\varphi) \vee g(\psi)$. The equalities $g(\varphi \wedge \psi) = g(\varphi) \wedge g(\psi)$ and $g(\varphi - \mathbf{r}) = g(\varphi) - r$, $r \in \mathbb{R}$ may be proved likewise, and the details are left to the interested reader. We deduce that $g \in H_{\langle B\rangle}$, so $H_{\langle B\rangle}$ is closed in $\mathbb{R}^{\langle B\rangle}$ for the joint topology, as required. □

Corollary 2.18.

(1) *For a real dicompact ditopological texture space* $(S, \mathcal{S}, \tau, \kappa)$, $(S_p, \mathcal{S}_p, \tau_p, \kappa_p)$ *can be embedded as a jointly closed subspace of a product of copies of the space* $(\mathbb{R}, \mathcal{R}, \tau_{\mathbb{R}}, \kappa_{\mathbb{R}})$.
(2) *The joint topology on* S_p *of a real dicompact ditopological texture space* $(S, \mathcal{S}, \tau, \kappa)$ *is real compact.*

Proof. (1) If $(S, \mathcal{S}, \tau, \kappa)$ is real dicompact then by Proposition 2.14(1), $(S_p, \mathcal{S}_p, \tau_p, \kappa_p)$ is also. Hence, by Proposition 2.17(2) it will be sufficient to show that whenever $(S_p, \mathcal{S}_p, \tau_p, \kappa_p)$ is B-real dicompact we have $\xi_{\langle B\rangle}(S_p) = H_{\langle B\rangle}$. For $h \in H_{\langle B\rangle}$ the bi-ideal (L^h, M^h) is real in $\langle B\rangle$, and we have the equalities $L^h = \{\varphi \in \langle B\rangle \mid h(\varphi) \leqslant 0\}$, $M^h = \{\varphi \in \langle B\rangle \mid h(\varphi) \geqslant 0\}$. By hypothesis (L^h, M^h) is difixed by some $s \in S_p$, whence $(L^h, M^h) = (L(s) \cap \langle B\rangle, M(s) \cap \langle B\rangle)$ and we easily deduce that $h = \hat{s}|_{\langle B\rangle} \in \xi_{\langle B\rangle}(S_p)$.

(2) By (2.5) it is easy to see that the joint topology of $(\mathbb{R}^{\langle B\rangle}, \mathcal{R}^{\langle B\rangle}, \tau_{\mathbb{R}}^{\langle B\rangle}, \kappa_{\mathbb{R}}^{\langle B\rangle})$ coincides with the product topology of the spaces $\mathbb{R}$ with their standard topology. It follows that the joint topological space of $(S, \mathcal{S}, \tau, \kappa)$ on S_p may be embedded as a closed subspace of a product of copies of the space $\mathbb{R}$ under its usual topology, and hence is real compact by [15]. □

We now consider the converse of Corollary 2.18(1).

Theorem 2.19. *A completely biregular bi-*T_2 *nearly plain* $*$*-space* $(S, \mathcal{S}, \tau, \kappa)$ *is real dicompact if and only if the space* $(S_p, \mathcal{S}_p, \tau_p, \kappa_p)$ *can be embedded as a jointly closed subspace of a product of the spaces* $(\mathbb{R}, \mathcal{R}, \tau_{\mathbb{R}}, \kappa_{\mathbb{R}})$.

Proof. Necessity is just Corollary 2.18(1), so we prove sufficiency. To simplify the notation we assume without loss of generality that $(S_p, \mathcal{S}_p, \tau_p, \kappa_p)$ is a jointly closed subspace of some product $(\mathbb{R}^J, \mathcal{R}^J, \tau_{\mathbb{R}}^J, \kappa_{\mathbb{R}}^J)$ of copies of $(\mathbb{R}, \mathcal{R}, \tau_{\mathbb{R}}, \kappa_{\mathbb{R}})$. Denoting the projection mappings by π_j it is clear that $B = \{\pi_j|_{S_p} \mid j \in J\} \cup \{\mathbf{0}\}$ is a bigenerating subset of $\mathrm{BA}(S_p)$ and so by Proposition 2.14(2) and Corollary 2.8 it will be sufficient to show that $(S_p, \mathcal{S}_p, \tau_p, \kappa_p)$ is B-real dicompact.

Let (L, M) be a real bi-ideal in $\langle B\rangle$, and let $h \in H_{\langle B\rangle}$ be the $\langle B\rangle$-resolution of (L, M). For $j \in J$ let $s_j = h(\pi_j|_{S_p})$ and consider $s = (s_j) \in \mathbb{R}^J$. We first show that $s \in S_p$. Suppose this is not so. Then, since S_p is jointly closed in $\mathbb{R}^{\langle B\rangle}$ we have a joint nhd

$$\bigcap_{\alpha=1}^{m} E(j_\alpha, Q_{r_\alpha}) \cap \bigcap_{\beta=1}^{n} \big(\mathbb{R}^{\langle B\rangle} \setminus E(j_\beta, P_{k_\beta})\big) \tag{2.6}$$

of s which does not meet S_p. Now $s_{j_\alpha} < r_\alpha$, $s_{j_\beta} > k_\beta$ so we may choose $\epsilon > 0$ for which $s_{j_\alpha} + \epsilon < r_\alpha$, $s_{j_\beta} - \epsilon > k_\beta$ for all $1 \leqslant \alpha \leqslant m$ and $1 \leqslant \beta \leqslant n$. By definition $L = L^h = \{\varphi \in \langle B\rangle \mid h(\varphi) \leqslant 0\}$, and $h(\pi_{j_\alpha}|_{S_p} - s_{j_\alpha}) = h(\pi_{j_\alpha}|_{S_p}) - s_{j_\alpha} = 0$, so $\bigvee\{\pi_{j_\alpha}|_{S_p} - s_{j_\alpha} \mid 1 \leqslant \alpha \leqslant m\} \vee \mathbf{0} \in L$. Likewise, $\bigwedge\{\pi_{j_\beta}|_{S_p} - s_{j_\beta} \mid 1 \leqslant \beta \leqslant n\} \wedge \mathbf{0} \in M$. On the other hand, since (L, M) is ρ_b-regular we have $T_\epsilon(\bigvee\{\pi_{j_\alpha}|_{S_p} - s_{j_\alpha} \mid 1 \leqslant \alpha \leqslant m\} \vee \mathbf{0}) \notin M$, so

$$\bigwedge\{\pi_{j_\beta}|_{S_p} - s_{j_\beta} \mid 1 \leqslant \beta \leqslant n\} \wedge \mathbf{0} \nleq T_\epsilon\Big(\bigvee\{\pi_{j_\alpha}|_{S_p} - s_{j_\alpha} \;\big|\; 1 \leqslant \alpha \leqslant m\} \vee \mathbf{0}\Big).$$

We now have $t \in S_p$ for which $(\pi_{j_\beta}(t) - s_{j_\beta}) \wedge 0 > (\pi_{j_\alpha} - s_{j_\alpha}) \vee 0 - \epsilon$ for all α and β. However, this implies that t is in the nhd. (2.6) and this contradiction shows that $s \in S_p$.

For $\varphi \in B$ it is immediate that $\varphi(s) = h(\varphi)$, and a simple induction argument of the form of the elements in $\langle B\rangle$ shows that $\varphi(s) = h(\varphi)$ for all $\varphi \in \langle B\rangle$. Hence

$$\big(L(s) \cap \langle B\rangle, M(s) \cap \langle B\rangle\big) = \big(L^h, M^h\big) = (L, M),$$

and so (L, M) is difixed by $s \in S_p$ as required. □

An important special choice for B is $\mathrm{BA}^*(S)$, the set of bounded elements of $\mathrm{BA}(S)$. It is easy to verify that this is a bigenerating sub-T-lattice of $\mathrm{BA}(S)$. In parallel with the topological and bitopological cases we refer to a space $(S, \mathcal{S}, \tau, \kappa)$ with $\mathrm{BA}^*(S) = \mathrm{BA}(S)$ as *pseudo dicompact*.

Proposition 2.20. *The following are equivalent for* $(S, \mathcal{S}, \tau, \kappa)$.

(i) *Pseudo dicompact and real dicompact.*
(ii) $\mathrm{BA}^*(S)$*-real dicompact.*
(iii) B*-real dicompact for all bigenerating* $B \subseteq \mathrm{BA}^*(S)$.
(iv) *Dicompact.*

Proof. (i) ⇒ (ii). Immediate from the definitions.

(ii) ⇒ (iii). This follows from Proposition 2.11 since in $\mathrm{BA}^*(S)$ all ρ_e-regular bi-ideals are finite.

(iii) ⇒ (iv). Let $B \subseteq \mathrm{BA}^*(S)$ be a bigenerating sub-T-lattice of $\mathrm{BA}^*(S)$. By the proof of Corollary 2.18(1) we may embed $(S_p, \mathcal{S}_p, \tau_p, \kappa_p)$ as a jointly closed subspace of $(\mathbb{R}^B, \mathcal{R}^B, \tau_{\mathbb{R}}^B, \kappa_{\mathbb{R}}^B)$. Moreover, each $\varphi \in B$ is bounded and if we take $\mathbf{a}_\varphi \leqslant \varphi \leqslant \mathbf{b}_\varphi$ it is not difficult to see that we may replace the φ^{th} factor in this product with $(\mathbb{I}_{a_\varphi,b_\varphi}, \mathcal{I}_{a_\varphi,b_\varphi}, \tau_{a_\varphi,b_\varphi}, \kappa_{a_\varphi,b_\varphi})$ (see [8, Notes 5.4(4)]). These are dicompact plain spaces and hence their product is a dicompact plain space. By [23, Proposition 4.6(2)] the joint topology of this space is compact, and hence the jointly closed subspace corresponding to $(S_p, \mathcal{S}_p, \tau_p, \kappa_p)$ is also compact. This proves that the joint topology of $(S_p, \mathcal{S}_p, \tau_p, \kappa_p)$ is compact. Hence $(S_p, \mathcal{S}_p, \tau_p, \kappa_p)$ is dicompact by [23, Proposition 4.6(2)], and so $(S, \mathcal{S}, \tau, \kappa)$ is dicompact since these spaces are isomorphic in **dfDitop**.

(iv) ⇒ (i) This is just (1) ⇒ (2) in [23, Theorem 4.2]. □

Corollary 2.21. *A completely biregular bi-T_2 nearly plain $*$-space $(S, \mathcal{S}, \tau, \kappa)$ is dicompact if and only if $(S_p, \mathcal{S}_p, \tau_p, \kappa_p)$ may be embedded as a jointly closed subspace of a product of spaces of the form $(\mathbb{I}_{a,b}, \mathcal{I}_{a,b}, \tau_{a,b}, \kappa_{a,b})$.*

3. Categorical results

We begin by recalling the construct **ifNpDitop** of nearly plain ditopological spaces and ω-preserving mappings, the adjoint functor $\mathfrak{J}$: **ifNpDitop** → **Top** given by

$$\mathfrak{J}\big((S, \mathcal{S}, \tau_S, \kappa_S) \xrightarrow{\varphi} (T, \mathcal{T}, \tau_T, \kappa_T)\big) = (S_p, \mathcal{J}_{\tau_S\kappa_S}) \xrightarrow{\varphi|_{S_p}} (T_p, \mathcal{J}_{\tau_T\kappa_T}),$$

and its co-adjoint

$$\mathfrak{T}\big((X, \mathcal{T}) \xrightarrow{\varphi} (Y, \mathcal{V})\big) = \big(X, \mathcal{P}(X), \mathcal{T}, \mathcal{T}^c\big) \xrightarrow{\varphi} \big(Y, \mathcal{P}(Y), \mathcal{V}, \mathcal{V}^c\big)$$

from **Top** to **ifNpDitop** [23, Theorem 4.4, Corollary 4.5]. By Corollary 2.18(2) we see that $\mathfrak{J}$ may be restricted to a functor $\mathfrak{J}$: **ifRdiComp₂** → **RComp**, where **ifRdiComp₂** denotes the construct of real dicompact bi-T_2 spaces and **RComp** the construct of (Hausdorff) real compact topological spaces.

We wish to show that if $(X, \mathcal{T})$ is real compact then $\mathfrak{T}(X, \mathcal{T})$ is real dicompact. Clearly $\mathrm{BA}(X) = C(X)$, and denoting this set by B the characterization theorem for real compact spaces [15, 11.12] says that X may be embedded as a closed subspace of $\mathbb{R}^B$ under the mapping $\xi_B : X \to \mathbb{R}^B$, where the spaces $\mathbb{R}$ have their usual topology. We have already noted that the product topology on $\mathbb{R}^B$ coincides with the joint topology of the ditopological texture space $(\mathbb{R}^B, \mathcal{R}^B, \tau_{\mathbb{R}}^B, \kappa_{\mathbb{R}}^B)$, so to establish the real dicompactness of $(X, \mathcal{P}(X), \mathcal{T}, \mathcal{T}^c)$ by Theorem 2.19 it will suffice to show that for $Y \in \mathcal{P}(X)$ there exists a set $Y_1 \in \mathcal{R}^B$ satisfying $Y_1 \cap \xi_B(X) = \xi_B(Y)$. However

$$Y_1 = \bigcup_{y \in Y}\Big(\bigcap_{\varphi \in B} E(\varphi, P_{\varphi(y)})\Big)$$

is easily seen to have the required properties and we have established:

Theorem 3.1. *The functor $\mathfrak{J}$:* **ifRdiComp₂** → **RComp** *is an adjoint and $\mathfrak{T}$:* **RComp** → **ifRdiComp₂** *its co-adjoint. In particular, $\mathfrak{T}$ embeds* **RComp** *as a coreflective subcategory of* **ifRdiComp₂**.

Next we extend [23, Theorem 4.8] to the real dicompact case. The names used for the categories are an obvious modification of the ones used for the dicompact case. The proof follows the same lines as the proof of [23, Theorem 4.8], and is omitted.

Theorem 3.2. *The functor $\mathfrak{B}$:* **ifRdiComp₂**$/\sim_p$ → **Tlat_DT** *is faithful and creates isomorphisms.*

This leads at once to analogues of the Hewitt Isomorphism Theorem [15].

Corollary 3.3 *(Hewitt Isomorphism Theorem). Let $(S_k, \mathcal{S}_k, \tau_k, \kappa_k)$, $k = 1, 2$, be real dicompact bi-T_2 spaces. Then these spaces are isomorphic in* **dfRdiComp₂** *(resp., in* **ifRdiComp₂**$/\sim_p$*) if and only if the T-lattices* $\mathrm{BDF}(S_k)$ *(resp.,* $\mathrm{BA}(S_k)$*), $k = 1, 2$, are isomorphic.*

As mentioned in the introduction, the theory of T-lattices and bi-ideals was first conceived in a bitopological context. We conclude this paper by looking in greater detail at the relationship between the ditopological and bitopological cases insofar as real compactness is concerned.

We begin by defining a functor $\mathfrak{U}$ from **ifNpDitop** to **Bitop** by

$$\mathfrak{U}\big((S, \mathcal{S}, \tau_S, \kappa_S) \xrightarrow{\varphi} (T, \mathcal{T}, \tau_T, \kappa_T)\big) = \big((S_p, (\tau_S)_p, (\kappa_S)_p^c\big) \xrightarrow{\varphi|_{S_p}} \big(T_p, (\tau_T)_p, (\kappa_T)_p^c\big)\big).$$

It is trivial to verify that this is indeed a functor and we omit the details. Clearly, it generalizes the functor with the same name from **fPDitop** to **Bitop** given in [7]. To define a suitable functor in the opposite direction we restrict ourselves to weakly pairwise T_0 bitopological spaces (X,u,v), and consider the smallest subset $\mathcal{K}_{uv}$ of $\mathcal{P}(X)$ which contains $u \cup v^c$ and is closed under arbitrary intersections and unions. Clearly the elements of $\mathcal{K}_{uv}$ have the form

$$A = \bigcap_{j\in J} A_j, \quad \text{where } A_j = U_j \cup \bigcup_{i\in I_j} \bigl\{ \bigl(V_i^j\bigr)^c \bigm| V_i^j \in v \bigr\},\ U_j \in u,\ j \in J. \tag{3.1}$$

To show that $\mathcal{K}_{uv}$ is a (plain) texturing of X it remains only to verify that it separates points. For $x_1, x_2 \in X$ with $x_1 \neq x_2$ we have $x_1 \notin \overline{x_2}^u \cap \overline{x_2}^v$ or $x_2 \notin \overline{x_1}^u \cap \overline{x_1}^v$ by the weak pairwise T_0 axiom, whence x_1, x_2 are separated by a set in $u \cup v^c \subseteq \mathcal{K}_{uv}$, as required. We consider the plain ditopological texture space $(X, \mathcal{K}_{uv}, u, v^c)$ and define $\mathfrak{K}$ by

$$\mathfrak{K}\bigl((X,u_X,v_X) \xrightarrow{\varphi} (Y,u_Y,v_Y)\bigr) = \bigl((X,\mathcal{K}_{u_X v_X},u_X,v_X^c) \xrightarrow{\varphi} \bigl(Y,\mathcal{K}_{u_Y v_Y},u_Y,v_Y^c\bigr)\bigr).$$

Now when $(X,u_X,v_X) \xrightarrow{\varphi} (Y,u_Y,v_Y)$ is pairwise continuous, $(X,\mathcal{K}_{u_X v_X},u_X,v_X^c) \xrightarrow{\varphi} (Y,\mathcal{K}_{u_Y v_Y},u_Y,v_Y^c)$ is ω-preserving and bicontinuous. Indeed, take $x_1 \omega_X x_2$. Now from (3.1) we clearly have, since $(X,\mathcal{K}_{u_X v_X})$ is plain,

$$\begin{aligned} x_1\omega_X x_2 \quad &\Leftrightarrow \quad x_1 \in P_{x_2} \\ &\Leftrightarrow \quad (x_2 \in U \in u_X \Rightarrow x_1 \in U) \wedge (x_1 \in V \in v_X \Rightarrow x_2 \in V). \end{aligned} \tag{3.2}$$

Hence, $\varphi(x_2) \in U \in u_Y \Rightarrow x_2 \in \varphi^{-1}[U] \in u_X \Rightarrow x_1 \in \varphi^{-1}[U] \Rightarrow \varphi(x_1) \in U$, and likewise $\varphi(x_1) \in V \in v_Y \Rightarrow \varphi(x_2) \in V$, whence $\varphi(x_1)\omega_Y\varphi(x_2)$. This shows φ is ω-preserving, and bicontinuity is an immediate consequence of the pairwise continuity of $(X,u_X,v_X) \xrightarrow{\varphi} (Y,u_Y,v_Y)$ and the fact that for $A \in \mathcal{K}_{u_Y v_Y}$ we have $\varphi^{\leftarrow}A = \varphi^{-1}[A]$ since the textures are plain. Hence $\mathfrak{K}$ is a functor.

We consider the preservation of certain properties under the functors $\mathfrak{U}$ and $\mathfrak{K}$.

Lemma 3.4. *With the notation as above*:

(1) *If (X,u,v) is weakly pairwise T_2 then $\mathfrak{K}(X,u,v)$ is bi-T_2.*
(2) *If $(S,\mathcal{S},\tau,\kappa)$ is bi-T_2 then $\mathfrak{U}(S,\mathcal{S},\tau,\kappa)$ is weakly pairwise T_2.*
(3) *If $(S,\mathcal{S},\tau,\kappa)$ is completely biregular then $\mathfrak{U}(S,\mathcal{S},\tau,\kappa)$ is pairwise completely regular.*
(4) *If (X,u,v) is pairwise completely regular then $\mathfrak{K}(X,u,v)$ is completely biregular.*

Proof. (1) Take $x_1, x_2 \in X$ with $Q_{x_1} \not\subseteq Q_{x_2}$ in $(X,\mathcal{K}_{uv})$. Now $P_{x_1} \not\subseteq P_{x_2}$, that is $x_1 \notin P_{x_2}$ and from (3.2) we deduce that $x_1 \notin \overline{x_2}^u$ or $x_2 \notin \overline{x_1}^v$. Since (X,u,v) is weakly pairwise T_2 it is pairwise R_1 and hence in both cases we have $U \in u$, $V \in v$ with $x_1 \in U$, $x_2 \in V$ and $U \cap V = \emptyset$. Now $U \not\subseteq Q_{x_1}$, $U \subseteq K = X \setminus V \in v^c$ and $P_{x_2} \not\subseteq K$ so $\mathfrak{K}(X,u,v)$ is bi-T_2.

(2) Left to the interested reader.

(3) First we note that $\mathfrak{U}(\mathbb{R},\mathcal{R},\tau_{\mathbb{R}},\kappa_{\mathbb{R}}) = (\mathbb{R},s,t)$ and that the elements of $\mathrm{BA}(S)$ are precisely the **ifNpDitop** morphisms from $(S,\mathcal{S},\tau,\kappa)$ to $(\mathbb{R},\mathcal{R},\tau_{\mathbb{R}},\kappa_{\mathbb{R}})$.

$$\begin{array}{ccc} (S,\mathcal{S},\tau,\kappa) & \xrightarrow{\ \mathfrak{U}\ } & (S_p,\tau_p,\kappa_p^c) \\ \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \varphi|_{S_p}} \\ (\mathbb{R},\mathcal{R},\tau_{\mathbb{R}},\kappa_{\mathbb{R}}) & \xrightarrow{\ \mathfrak{U}\ } & (\mathbb{R},s,t) \end{array}$$

It will clearly suffice to show that $B = \{\varphi|_{S_p} \mid \varphi \in \mathrm{BA}(S)\}$ is a bigenerating subset of $\mathrm{P}(S_p)$. Certainly $B \subseteq \mathrm{P}(S_p)$ since $\mathfrak{U}$ is a functor. Let $F \subseteq S_p$ be τ_p-closed and take $u \in S_p$ with $u \notin F$. Then $u \in H = S_p \setminus F \in \tau_p$ so we have $H_1 \in \tau$ with $H = H_1 \cap S_p$, and clearly $H_1 \not\subseteq Q_u$ in $(S,\mathcal{S},\tau,\kappa)$. Since $(S,\mathcal{S},\tau,\kappa)$ is completely biregular, by Proposition 1.1(1) we have $\varphi \in \mathrm{BA}(S)$ satisfying $-\mathbf{1} \leqslant \varphi \leqslant \mathbf{1}$, $P_u \subseteq \varphi^{\leftarrow}P_{-1}$ and $\varphi^{\leftarrow}Q_1 \subseteq H_1$. Now $\varphi|_{S_p} \in B$, $\varphi|_{S_p}(u) = -1$ and $\varphi|_{S_p}[F] \subseteq \{1\}$, so the functions in B generate the topology τ_p on S_p. Likewise they generate the topology κ_p^c, so B is bigenerating and in particular (S_p,τ_p,κ_p^c) is pairwise completely regular.

(4) On noting that $\mathfrak{K}(\mathbb{R},s,t) = (\mathbb{R},\mathcal{R},\tau_{\mathbb{R}},\kappa_{\mathbb{R}})$ we obtain the diagram below.

$$\begin{array}{ccc} (X,u,v) & \xrightarrow{\ \mathfrak{K}\ } & (X,\mathcal{K}_{uv},u,v^c) \\ \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \varphi} \\ (\mathbb{R},s,t) & \xrightarrow{\ \mathfrak{K}\ } & (\mathbb{R},\mathcal{R},\tau_{\mathbb{R}},\kappa_{\mathbb{R}}) \end{array}$$

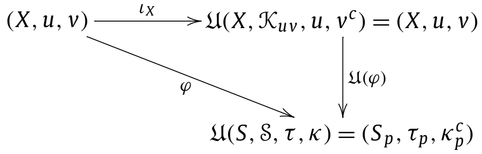
Clearly $\mathfrak{K}$ is full, so $\mathrm{P}(X) = \mathrm{BA}(X)$. The remainder of the proof is straightforward, and is omitted. □

The above lemma shows that with respect to these functors our basic assumption that $(S,\mathcal{S},\tau,\kappa) \in \mathrm{Ob}$ **ifNpDitop** is a completely biregular bi-T_2 space is fully consistent with the assumption in [2, Chapter 3] that (X,u,v) is pairwise completely regular and weakly pairwise Hausdorff. Moreover, the additional requirement that $(S,\mathcal{S},\tau,\kappa)$ should be a $*$-space will not cause a problem as the image of (X,u,v) under $\mathfrak{K}$ is plain and hence also a $*$-space.

For the purposes of this paper we denote the construct of pairwise completely regular weakly pairwise Hausdorff bitopological spaces and pairwise continuous functions by $\mathbf{pCReg_{w2}}$, and the category of completely biregular bi-T_2 nearly plain ($*$-) spaces and ω-preserving bicontinuous functions by $\mathbf{ifNpCbiR_2}$ (respectively, $\mathbf{ifNpCbiR_2^*}$).

Theorem 3.5. *$\mathfrak{U} : \mathbf{ifNpCbiR_2^*} \to \mathbf{pCReg_{w2}}$ is an adjoint functor and $\mathfrak{K} : \mathbf{pCReg_{w2}} \to \mathbf{ifNpCbiR_2^*}$ a co-adjoint of $\mathfrak{U}$.*

Proof. Take $(X, u, v) \in \mathrm{Ob}\,\mathbf{pCReg_{w2}}$. We show that $(\iota_X, (X, \mathcal{K}_{uv}, u, v^c))$ is a $\mathfrak{U}$-universal arrow with domain (X, u, v). It is clearly a $\mathfrak{U}$-structured arrow, so take $(S, \mathcal{S}, \tau, \kappa) \in \mathrm{Ob}\,\mathbf{ifNpCbiR_2^*}$ and $\varphi \in \mathbf{pCReg_{w2}}((X, u, v), (S_p, \tau_p, (\kappa_p)^c))$. We verify first that $\varphi : (X, \mathcal{K}_{uv}, u, v^c) \to (S, \mathcal{S}, \tau, \kappa)$ is an **ifNpTex** morphism, whence it is the unique such morphism for which the following diagram in commutative.



Suppose on the contrary that φ is not ω-preserving. Then we have $x_1, x_2 \in X$ with $x_1 \omega_X x_2$ and $P_{\varphi(x_2)} \subseteq Q_{\varphi(x_1)}$. Since $\varphi(x_2) \in S_p$ we have $P_{\varphi(x_2)} \not\subseteq Q_{\varphi(x_2)}$, whence $Q_{\varphi(x_1)} \not\subseteq Q_{\varphi(x_2)}$. Since $(S, \mathcal{S}, \tau, \kappa)$ is bi-T_2 we have $H \in \tau$, $K \in \kappa$ with $H \subseteq K$, $P_{\varphi(x_1)} \not\subseteq K$ and $H \not\subseteq Q_{\varphi(x_2)}$. Now $U = \varphi^{-1}[H \cap S_p] \in u$ by pairwise continuity, and $x_2 \in U$ so by (3.2) we have $x_1 \in U$ and hence the contradiction $\varphi(x_1) \in H \subseteq K$.

On the other hand, for $A \in \mathcal{S}$ we have $\varphi^{-1}[A \cap S_p] = \varphi^{\leftarrow} A$ since $(X, \mathcal{K}_{uv})$ is plain and φ maps into S_p. Hence $\varphi \in \mathbf{ifNpCbiR_2^*}((X, \mathcal{K}_{uv}, u, v^c), (S, \mathcal{S}, \tau, \kappa))$, and since $(X, \mathcal{K}_{uv}, u, v^c) = \mathfrak{K}(X, u, v)$ the proof is complete by [1, Remark 19.2]. □

We next note that by the proof of Lemma 3.4(3) the set $\{\varphi|_{S_p} \mid \varphi \in \mathrm{BA}(S)\}$ is a bigenerating subset of $\mathrm{P}(S_p)$. However this set is just $\mathrm{BA}(S_p)$, and so is a T-lattice, while a bi-ideal in $\mathrm{BA}(S_p)$ is difixed if and only if it is fixed by the same point of S_p in the sense of [2]. Hence, by Proposition 2.14 we see that if $(S, \mathcal{S}, \tau, \kappa)$ is real dicompact every real bi-ideal in $\mathrm{BA}(S_p)$ is difixed, whence $(S_p, \tau_p, \kappa_p^c)$ is $\mathrm{BA}(S_p)$-bireal compact, and hence bireal compact in the sense of [2] (equivalently, in view of the characterization theorem [2, Theorem 3.3.2], real compact in the sense of Brümmer and Salbany [10]).

A similar argument for the functor $\mathfrak{K}$ shows that if (X, u, v) is bireal compact, $\mathfrak{K}(X, u, v)$ is real dicompact. Hence, using an obvious notation:

Corollary 3.6. *$\mathfrak{U} : \mathbf{ifRdiComp_2} \to \mathbf{biRComp}$ is an adjoint functor and $\mathfrak{K} : \mathbf{biRComp} \to \mathbf{ifRdiComp_2}$ the co-adjoint of $\mathfrak{U}$.*

Making a restriction to plain textures we have the following stronger results.

Theorem 3.7. *$\mathfrak{U} : \mathbf{fPCbiR_2} \to \mathbf{pCReg_{w2}}$ is an isomorphism with inverse $\mathfrak{K}$.*

Proof. We wish first to show that if $(S, \mathcal{S}, \tau, \kappa)$ is plain and $\varphi : S \to \mathbb{R}$ is pairwise continuous then $\varphi : (S, \mathcal{S}, \tau, \kappa) \to (\mathbb{R}, \mathcal{R}, \tau_{\mathbb{R}}, \kappa_{\mathbb{R}})$ is ω-preserving.

Take $u, u' \in S$ with $u \omega_S u'$ and suppose that $P_{\varphi(u')} \subseteq Q_{\varphi(u)}$. Then $\varphi(u') < \varphi(u)$ and we take $k \in \mathbb{R}$ with $\varphi(u') < k < \varphi(u)$. Setting $V = \{r \in \mathbb{R} \mid r > k\} \in \kappa_{\mathbb{R}}^c$ gives $\varphi^{-1}[V] \in \kappa^c$, whence $\varphi^{-1}[V^c] \in \kappa \subseteq \mathcal{S}$. But $P_{u'} \subseteq \varphi^{-1}[V^c]$, so $\varphi^{-1}[V^c] \not\subseteq Q_u$ and we obtain the contradiction $\varphi(u) \leqslant k$. This establishes that φ is ω-preserving and hence a **fPDitop** morphism.

Now take $(S, \mathcal{S}, \tau, \kappa) \in \mathrm{Ob}\,\mathbf{fPCbiR_2}$. Then $\mathfrak{U}(S, \mathcal{S}, \tau, \kappa) = (S, \tau, \kappa^c)$, and so $(\mathfrak{K} \circ \mathfrak{U})(S, \mathcal{S}, \tau, \kappa) = (S, \mathcal{K}_{\tau\kappa^c}, \tau, \kappa)$. We must prove that $\mathcal{S} = \mathcal{K}_{\tau\kappa^c}$. By the proof of Lemma 3.4 and the above result we have $\mathrm{BA}(S, \mathcal{S}) = \mathrm{P}(S) = \mathrm{BA}(S, \mathcal{K}_{\tau\kappa^c})$, where these sets are taken for $(S, \mathcal{S}, \tau, \kappa)$, (S, τ, κ^c) and $(S, \mathcal{K}_{\tau\kappa^c}, \tau, \kappa)$, respectively, and we denote this T-lattice by B for short.

By Corollary 2.16, ξ_B is an **fTex** isomorphism between both $(S, \mathcal{S})$ and $(S, \mathcal{K}_{\tau\kappa^c})$ and the texture induced on $\xi_B[S]$ by $(H_B, \mathcal{H}_B)$. It is clear that $\mathcal{K}_{\tau\kappa^c} \subseteq \mathcal{S}$, so take $A_0 \in \mathcal{S}$. By [6, Proposition 3.15], the mapping $A \mapsto \xi_B[A]$ from $\mathcal{K}_{\tau\kappa^c}$ to $\mathcal{H}_B|_{\xi_B[S]}$ is onto, and $\xi_B[A_0] \in \mathcal{H}_B|_{\xi_B[S]}$, so there exists $A_1 \in \mathcal{K}_{\tau\kappa^c}$ with $\xi_B[A_1] = \xi_B[A_0]$. However, $A_0, A_1 \in \mathcal{S}$ and be the same proposition the mapping $A \mapsto \xi_B[A]$ from $\mathcal{S}$ to $\mathcal{H}_B|_{\xi_B[S]}$ is one to one, so $A_0 = A_1 \in \mathcal{K}_{\tau\kappa^c}$, as required.

This gives $\mathfrak{K} \circ \mathfrak{U} = \mathrm{id}_{\mathbf{fPCbiR_2}}$, and $\mathfrak{U} \circ \mathfrak{K} = \mathrm{id}_{\mathbf{pCReg_{w2}}}$ is trivial so the proof is complete. □

Corollary 3.8. *For $(S, \mathcal{S}, \tau, \kappa) \in \mathrm{Ob}\,\mathbf{fPCbiR_2}$ we have $\mathcal{S} = \mathcal{K}_{\tau\kappa^c}$.*

Corollary 3.9. *$\mathfrak{U} : \mathbf{fPRdiComp_2} \to \mathbf{biRComp}$ is an isomorphism with inverse $\mathfrak{K}$.*

We now turn to the question of characterizing those plain real dicompact spaces whose image under $\mathfrak{U}$ is a trivial bitopology, that is a bitopology of the form (X, u, u).

Lemma 3.10. *For $(S, \mathcal{S}, \tau, \kappa) \in \mathrm{Ob}\,\mathbf{fPRdiComp_2}$, $\mathfrak{U}(S, \mathcal{S}, \tau, \kappa)$ is trivial if and only if $(S, \mathcal{S}, \tau, \kappa) = (S, \mathcal{P}(S), \tau, \tau^c)$.*

Proof. If $\mathfrak{U}(S, \mathcal{S}, \tau, \kappa) = (S, \tau, \kappa^c)$ is a trivial bitopology we have $\tau = \kappa^c$ by definition, so $\kappa = \tau^c$. By Corollary 3.8 we have $\mathcal{S} = \mathcal{K}_{\tau\kappa^c} = \mathcal{K}_{\tau\tau}$, and by (3.1) we deduce that $\mathcal{S}$ has the property $A \in \mathcal{S} \Rightarrow A^c \in \mathcal{S}$. However it is now clear that for $s \in S$ we must have $P_s = \{s\}$, so since $(S, \mathcal{S})$ is plain we obtain $\mathcal{S} = \mathcal{P}(S)$. □

Corollary 3.11. *The functor $\mathfrak{U}$ induces an isomorphism from the full subcategory of* $\mathbf{fPRdiComp_2}$ *whose objects are complemented ditopologies on a discrete texture to the full subcategory of* **biRComp** *whose objects have trivial bitopologies. Likewise, $\mathfrak{K}$ induces the inverse of this isomorphism.*

It will be noted that the full subcategory of $\mathbf{fPRdiComp_2}$ mentioned in the above Corollary is the same as the coreflective subcategory of $\mathbf{ifRdiComp_2}$ found in Theorem 3.1 by considering the joint topology.

We mention finally that if instead of restricting to plain textures we work with categories in which the morphisms are difunctions we can raise Theorem 3.5 and Corollary 3.6 to an equivalence. More specifically, defining $\mathfrak{K}_{df} : \mathbf{pCReg_{w2}} \to \mathbf{dfNpCbiR_2^*}$ by

$$\mathfrak{K}_{df}\big((X, u_X, v_X) \xrightarrow{\varphi} (Y, u_Y, v_Y)\big) = \big(X, \mathcal{K}_{u_X v_X}, u_X, v_X^c\big) \xrightarrow{(f_\varphi, F_\varphi)} \big(Y, \mathcal{K}_{u_Y v_Y}, u_Y, v_Y^c\big)$$

we have, using an obvious notation:

Theorem 3.12. $\mathfrak{K}_{df} : \mathbf{pCReg_{w2}} \to \mathbf{dfNpCbiR_2^*}$ *is an equivalence.*

Proof. Clearly $\mathfrak{K}_{df}$ may be written as the composition of the functors

$$\mathbf{pCReg_{w2}} \xrightarrow{\mathfrak{K}} \mathbf{fPCbiR_2} \xrightarrow{\mathfrak{D}} \mathbf{dfPCbiR_2} \overset{\mathfrak{E}}{\hookrightarrow} \mathbf{dfNpCbiR_2}.$$

Here $\mathfrak{K}$ is an isomorphism by Theorem 3.5, $\mathfrak{D}$ is a restriction of the isomorphism between **fPDitop** and **dfPDitop** given in [7, Theorem 2.7] and $\mathfrak{E}$ is a restriction of the embedding of **dfPDitop** in **dfNpDitop** which was shown to be an equivalence in [23, Theorem 4.3]. Hence $\mathfrak{K}_{df}$ is an equivalence. □

Corollary 3.13. $\mathfrak{K}_{df} : \mathbf{biRComp} \to \mathbf{dfRdiComp_2}$ *is an equivalence.*

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