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Research Article

Quantitative Homogenization of Attractors of Non-Newtonian Filtration Equations

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For a rapidly spatially oscillating nonlinearity g we compare solutions u^ϵ of non-Newtonian filtration equation $\partial_t\beta(u^\epsilon) - D(|Du^\epsilon|^{p-2}Du^\epsilon + \varphi(u^\epsilon)Du^\epsilon) + g(x, x/\epsilon, u^\epsilon) = f(x, x/\epsilon)$ with solutions u^0 of the homogenized, spatially averaged equation $\partial_t\beta(u^0) - D(|Du^0|^{p-2}Du^0 + \varphi(u^0)Du^0) + g^0(x, u^0) = f^0(x)$. Based on an ε-independent a priori estimate, we prove that $\|\beta(u^\epsilon) - \beta(u^0)\|_{L^1(\Omega)} \le C\epsilon e^{\rho t}$ uniformly for all $t \ge 0$. Besides, we give explicit estimate for the distance between the nonhomogenized A^ϵ and the homogenized A^0 attractors in terms of the parameter ϵ; precisely, we show fractional-order semicontinuity of the global attractors for $\epsilon \searrow 0$: $\mathrm{dist}_{L^1(\Omega)}(A^\epsilon, A^0) \le C\epsilon^\gamma$.

1. Introduction

This paper is devoted to the study of nonlinear parabolic equations related to the p-Laplacian operator. The problems related to such type of equation arise in many applications in the fields of mechanics, physics, and biology (non-Newtonian fluids, gas flow in porous media, spread of biological populations, etc.).

For example, in the study of the water in filtration through porous media, Darcy's linear relation

$$V = -K(\tau)\omega_x \tag{$*$}$$

satisfactorily describes flow conditions provided that the velocities are small. Here V represents the velocity of the water, τ is the volumetric water (moisture) content, $K(\tau)$ is the hydraulic conductivity, and ω is the total potential, which can be expressed as the sum of a hydrostatic potential $\psi(\tau)$ and a gravitational potential z: $\omega(\tau) = \psi(\tau) + z$. However, from the physical point of view, $(*)$ fails to describe the flow for large velocities. To get a more accurate description of the flow in this case, several nonlinear versions of (∗) have been proposed. One of these versions is

$$V^{\alpha} = -K(\tau)\omega_x \tag{1.1}$$

which leads us to the equation

$$u_t - \operatorname{div}\left\{|u|^m|\nabla u|^{p-2}\nabla u\right\} = f(x,t,u). \tag{$**$}$$

For the first time, nonlinear relationship instead of Darcy's relation are suggested by Dupuit and Forchheimer. This modification, known as Darcy-Forchheimer's relation, has led to much research from experimental, theoretical, and numerical points of view. For example, Darcy-Forchheimer equations are widely used in reservoir engineering and other subsurface applications.

One-dimensional (by spatial variable) variant of (∗∗) with $p \in [3/2, 2)$ and $m > p-1$ arises in the study of a turbulent flow of a gas in a one-dimensional porous medium. This phenomenon was first described by Leibenson in [1]. One of the first papers devoted to the existence problem for such type of equation was an initiating paper by Raviart [2]. In [3], the author investigated the regularity problem for more general case (which includes Leibenson's equation after changing $|u|^{\sigma}u = v$) of non-Newtonian equation as

$$\partial_t\beta(v) + \operatorname{div}\overrightarrow{B}(x,t,v,\nabla v) + B_0(x,t,v,\nabla v) = 0 \tag{1.2}$$

with some restriction on the functions $\overrightarrow{B}$ and B_0. Earlier, in [4] authors have investigated existence and the stabilization of solutions (as $t \nearrow \infty$) of nonlinear parabolic equation of mentioned type describing certain models related to turbulent flows. Also note that there is an extensive literature devoted to the existence, regularity, and the large-time behavior of solutions of (1.2) under various conditions on functions β, $\overrightarrow{B}$, and B_0 (see [2–10] and references therein).

In this paper we show that, under natural assumptions on the terms f and g, the longtime behavior of solutions of the equation

$$\partial_t\beta\left(u^0\right) - D\left(\left|Du^0\right|^{p-2}Du^0 + \varphi\left(u^0\right)Du^0\right) + g^0\left(x,u^0\right) = f^0(x) \tag{1.3}$$

can be described in terms of the global attractor A^{ϵ} of the associated dynamical system related to the equation

$$\partial_t\beta(u^{\epsilon}) - D\left(|Du^{\epsilon}|^{p-2}Du^{\epsilon} + \varphi(u^{\epsilon})Du^{\epsilon}\right) + g\left(x,\frac{x}{\epsilon},u^{\epsilon}\right) = f\left(x,\frac{x}{\epsilon}\right), \tag{1.4}$$

where functions $g(x,x/\epsilon,u^{\epsilon})$ and $f(x,x/\epsilon)$ are constructed according to some ideas previously presented in [11], where authors carry out a quantitative comparison with the averaged or homogenized equations, in particular for quasiperiodic inhomogeneities with

Diophantine frequencies (see [11, pages 176–180]). Note that a quantitative homogenization aims at determining a specific rate of convergence. Method of the construction of the homogenized equation allows us to assert that $\|\beta(u^\epsilon) - \beta(u^0)\|_{L^1(\Omega)} \leq C\epsilon e^{\rho t}$ (Theorem 3.1 below) uniformly for all $t \geq 0$. As we mentioned above, this result requires g and f to depend quasiperiodically on the rapid space variable $z = x/\epsilon$. At the same time, being interested in quantitative strong convergence not only of individual trajectories but also of global attractors, we also show that global attractors A^ϵ tend to attractor A^0 in a suitable sense providing fractional-order semicontinuity of the global attractors for $\epsilon \searrow 0$. On a related note, the author would like to mention several results on homogenization of the attractor for nonlinear parabolic equations.

In [12] the Cauchy problem for parabolic equations on Riemannian manifolds with complicated microstructure has been considered and a connection between global attractors of the initial problem of the homogenized one has been established. The asymptotical behavior of the global attractor of the boundary value problem for semilinear equation

$$u_t^\epsilon - \Delta u^\epsilon + f(u^\epsilon) = h^\epsilon(x) \tag{1.5}$$

was investigated in [13] and it was shown that this tended in a suitable sense to the finite-dimensional weak global attractor of some system of a parabolic p.d.e. coupled with an o.d.e.

Also it is necessary to note the papers in [14–16], where the media properties are assumed to be oscillatory, focusing on the homogenization of the attractor for semilinear parabolic equations (see [14]) and systems (see [15]), and of the quasilinear parabolic equations (see [16]).

Note that previous approaches on homogenization of attractors are limited to certain types of nonlinear equations. In particular, they assume the main part of the equation to be linear or monotone. The consideration of equation (1.4) is a first step in the investigation of the more general equation

$$\partial_t \beta(u^\epsilon) - D\left(a\left(\frac{x}{\epsilon}\right)\left(|Du^\epsilon|^{p-2} + \varphi(u^\epsilon)\right)Du^\epsilon\right) + g\left(x, \frac{x}{\epsilon}, u^\epsilon\right) = f\left(x, \frac{x}{\epsilon}\right), \tag{1.6}$$

where the media properties are assumed to be oscillatory. This often arises in porous media flows [17]. We also note the paper in [18] where authors deal with the homogenization problem for a one-dimensional parabolic PDE with random stationary mixing coefficients in the presence of a large zero-order term and show that the family of solutions of the studied problem converges in law.

Our work is inspired, on one hand, by results of the studies in [7, 8] which are devoted to the existence and regularity of the attractor and, on the other hand, by the paper in [11] that is related to quantitative homogenization of the global attractor for reaction-diffusion systems. For proof of the existence we use sketch of the proof of corresponding result; however, compared to the studies in [7, 8] we consider the equation with an additional nonlinear term $\varphi(s)$. Note that this term prevents us from applying the result from the paper in [4] which is often used to prove the uniqueness of the solution. Here we use a technique (see [19]) which is based on the fact that the interval $[a, b]$ may be divided into subintervals where the sign of the "difference" $u(x,t) - v(x,t)$ ($u(x,t)$ and $v(x,t)$ are solutions with the same initial condition) does not change for fixed t. Also note that the results of the paper in

[11] cannot be directly applied to prove homogenization of the attractor for (1.4) because of nonlinear terms β and φ.

Our paper is organized as follows. In Section 2 we consider the Dirichlet problem

$$\partial_t \beta(u) - D\left(|Du|^{p-2}Du + \varphi(u)Du\right) + g(x,u) = f(x), \tag{1.7}$$

$$u_{|\Gamma} = 0, \quad \Gamma = \partial\Omega \times (0,T), \tag{1.8}$$

$$u(x,0) = u_0(x), \tag{1.9}$$

where $D = \partial_x, \Omega = (a,b)$ under the following hypotheses

$(H1)$ $u_0(x)$ and $\beta(u_0)$ are in $L^2(\Omega)$,

$(H2)$ $\beta(\zeta)$ is an increasing locally Lipschitzian function from R to R, with $\beta(0) = 0$.

$(H3)$ $g(\cdot,\zeta)\zeta \leqslant c_1|\zeta|^k + c_2, \partial_\zeta g(\cdot,\zeta) > -\lambda$, (without loss of generality we suppose that $g(\cdot,0) = 0$) and there exist positive constants c_3 and c_4 such that $\operatorname{sign}(\zeta)g(x,\zeta) \geq c_3|\beta(\zeta)|^{q-1} - c_4$ for a. e. $(x,t) \in \Omega \times R$ and $q > 2$.

$(H4)$ $\varphi(\zeta)$ is a function from the space C^1 such that $\varphi(\zeta) \geq 0, \varphi(0) = 0$, and, for each M and $|\zeta| \leq M$, $|\varphi'(\zeta)| \leq (|\zeta|^l + c)(\sqrt{\beta'(\zeta)})^\alpha$ almost everywhere, for some $l, c \geq 0$ and $\alpha \geq 1$.

$(H5)$ $f(x) \in L^\infty(\Omega), p \geq 4$.

Under the mentioned condition we show the existence and uniqueness of the solution and the existence of the L^2-global attractor.
Further, in order to show more regularity of the solution we suppose the following condition.

$(H6)$ For each M and $|\zeta| \leq M$, $|g(x,\zeta) - f(x)| \leq (|\zeta|^{l_1} + c)(\sqrt{\beta'(\zeta)})^{\alpha_1}$ almost everywhere, for some $l_1, c \geq 0$ and $\alpha_1 \geq 1$.

Finally in Section 3, we derive, under conditions $(H1)$–$(H5)$, the following estimate:

$$\left\|\beta(u^\epsilon) - \beta\left(u^0\right)\right\|_{L^1(\Omega)} \leq C\epsilon e^{\rho t}. \tag{1.10}$$

Also, under additional condition on g we prove fractional-order semicontinuity of the global attractors for $\epsilon \searrow 0$, namely, the validity of the estimation $\operatorname{dist}_{L^1(\Omega)}(A^\epsilon, A^0) \leq C\epsilon^\gamma$, where A^ϵ and A^0 are global attractors of the semigroups generated by the problems (1.4), (1.8), (1.9) and (1.3), (1.8), (1.9), respectively.

2. Existence and Uniqueness

First, we suppose that ϵ in equation (1.4) is a constant. This leads us to the problem (1.7), (1.8), (1.9) where $g(x,u)$ and $f(x)$ are denoted as $g(x,x/\epsilon,u^\epsilon)$ and $f(x,x/\epsilon)$ correspondingly.

We use the standard regularization of the problem (1.7), (1.8), (1.9):

$$\partial_t \beta_\eta(u) - D\Big(\big(|Du|^{p-2} + \varphi(u) + \eta\big) Du\Big) + g(x,u) = f(x), \tag{2.1}$$

$$u_{|\Gamma} = 0, \quad \Gamma = \partial\Omega \times (0,T), \tag{1.8}$$

$$u(x,0) = u_0(x), \tag{1.9}$$

where the sequence $\beta_\eta \in C^1(R)$ is such that $\beta_\eta(0) = 0, \beta_\eta \to \beta$ in $C_{\text{loc}}(R)$, $M_\eta > \beta'_\eta > \eta$, and $|\beta_\eta| \leq |\beta|$.

Let $(u_{0\eta})_{\eta>0}$ be a sequence in $C_0^\infty(\Omega)$ such that $u_{0\eta} \to u_0$ almost everywhere in Ω and $\|u_{0\eta}\|_{L^2(\Omega)}, \|\beta_\eta(u_{0\eta})\|_{L^2(\Omega)} \leq c$, with constant $c > 0$. To show the solvability of the problem (1.8), (1.9), (2.1) we use the result for the classical solvability in the large given by Ladyžhenskaya et al. (see [20, Theorem 4.1(5.2), ch. VI]).

Our proof is based on a priori estimates and is similar to that in [7]. First, using the properties of φ and β_η, and then multiplying the equation by $|\beta_\eta(u_\eta)|^k \beta_\eta(u_\eta)$, we deduce, analogously to [7, 8], the following lemma.

Lemma 2.1. *Under the hypotheses* $(H1)$*–*$(H5)$ *for any* $\eta \in (0,1)$ *the following estimates hold:* $\|u_\eta\|_{L^\infty(\tau,T;L^\infty(\Omega))} \leqslant c(\tau,T)$,

$$\|\beta_\eta(u_\eta)\|_{L^\infty(0,T;L^2(\Omega))\cap L^q(Q_T)} \leqslant c(T), \qquad \|u_\eta\|_{L^p(0,T;W_0^{1,p}(\Omega))} \leqslant c(T). \tag{2.2}$$

Proof. Multiplying equation (2.1) by $|\beta_\eta(u_\eta)|^k \beta_\eta(u_\eta)$ and integrating by parts, we get

$$\begin{aligned} &\frac{1}{k+2}\frac{d}{dt}\int_\Omega |\beta_\eta(u_\eta)|^{k+2}dx + (k+1)\int_\Omega \big(|Du_\eta|^{p-2} + \varphi(u_\eta) + \eta\big)\, |\beta_\eta(u_\eta)|^k \beta'_\eta(u_\eta) \\ &\quad\times |Du_\eta|^2 dx + \int_\Omega g(x,u_\eta)|\beta_\eta(u_\eta)|^k \beta_\eta(u_\eta)dx = \int_\Omega f(x)|\beta_\eta(u_\eta)|^k \beta_\eta(u_\eta)dx. \end{aligned} \tag{2.3}$$

By virtue of the positivity of the $\beta'_\eta(u_\eta)$ and $\varphi(u_\eta)$ we have

$$\frac{1}{k+2}\frac{d}{dt}\int_\Omega |\beta_\eta(u_\eta)|^{k+2}dx + \int_\Omega g(x,u_\eta)|\beta_\eta(u_\eta)|^k \beta_\eta(u_\eta)dx \leq \int_\Omega f(x)|\beta_\eta(u_\eta)|^k \beta_\eta(u_\eta)dx. \tag{2.4}$$

By conditions $(H2)$ and $(H3)$ we derive

$$\begin{aligned} &\frac{1}{k+2}\frac{d}{dt}\int_\Omega |\beta_\eta(u_\eta)|^{k+2}dx + c_3\int_\Omega |\beta_\eta(u_\eta)|^{k+q}dx \\ &\quad\leq \int_\Omega f(x)|\beta_\eta(u_\eta)|^k \beta_\eta(u_\eta)dx + c_4\int_\Omega |\beta_\eta(u_\eta)|^k \beta_\eta(u_\eta)dx. \end{aligned} \tag{2.5}$$

Consequently, from $(H5)$ we obtain

$$\frac{1}{k+2}\frac{d}{dt}\int_{\Omega}\left|\beta_\eta(u_\eta)\right|^{k+2}dx + c_3\int_{\Omega}\left|\beta_\eta(u_\eta)\right|^{k+q}dx \leq (c(f)+c_4)\int_{\Omega}\left|\beta_\eta(u_\eta)\right|^{k+1}dx. \tag{2.6}$$

Further, using Holder's inequality on both sides of the latter inequality, we deduce that there exist two constants $\alpha_0 > 0$ and $\lambda_0 > 0$ such that

$$\frac{d}{dt}\left|\left|\beta_\eta(u_\eta)\right|\right|_{L^{k+2}(\Omega)} + \lambda_0\left|\left|\beta_\eta(u_\eta)\right|\right|^{q-1}_{L^{k+2}(\Omega)} \leq \alpha_0, \tag{2.7}$$

which implies from Ghidaglia's lemma [9] that

$$\left\|\beta_n(u_\eta)\right\|_{L^{k+2}(\Omega)} \leq \left(\frac{\alpha_0}{\lambda_0}\right)^{1/(q-1)} + \frac{1}{\left(\lambda_0(q-2)t\right)^{1/(q-2)}} = c(t). \tag{2.8}$$

As $k \rightarrow \infty$ in (2.8), and for $t \geq \tau > 0$, we have

$$\left|\left|\beta_\eta(u_\eta)(t)\right|\right|_{L^\infty(\Omega)} \leq c(\tau), \tag{2.9}$$

which implies that

$$\left|\left|u_\eta(t)\right|\right|_{L^\infty(\Omega)} \leq \max\left(\beta_\eta^{-1}(c(\tau)), \left|\beta_\eta^{-1}(-c(\tau))\right|\right) = \delta_\eta. \tag{2.10}$$

Since β_η converges to β in $C_{\mathrm{loc}}(\mathbb{R})$, then the sequence δ_η is bounded in $\mathbb{R}$ as $\eta \rightarrow +\infty$. Thus δ_η is bounded by $\max(\beta_\eta^{-1}(c(\tau)), |\beta_\eta^{-1}(-c(\tau))|)$, which is finite. Whence $||u_\eta||_{L^\infty(\tau,T;L^\infty(\Omega))} \leqslant c(\tau)$.

On the other hand, taking $k = 0$ in (2.3), using Holder's inequality, and integrating over $[0,T]$, we obtain the second estimate of the statement of Lemma 2.1 as

$$\left|\left|\beta_\eta(u_\eta)\right|\right|_{L^\infty(0,T;L^2(\Omega))\cap L^q(Q_T)} \leqslant c(T). \tag{2.11}$$

Further, in order to prove the latter estimate of Lemma 2.1, we multiply (2.1) by u_η and integrate over Q_t:

$$\begin{aligned} &\frac{1}{k+2}\int_0^t\int_{\Omega}\partial_t\beta_\eta(u_\eta)u_\eta dx d\theta + (k+1)\int_0^t\int_{\Omega}\left(\left|Du_\eta\right|^{p-2} + \varphi(u_\eta) + \eta\right)\left|Du_\eta\right|^2 dx d\theta \\ &\quad + \int_0^t\int_{\Omega} g(x,u_\eta)u_\eta dx d\theta = \int_0^t\int_{\Omega} f(x)u_\eta dx d\theta. \end{aligned} \tag{2.12}$$

Therefore,

$$\frac{1}{k+2}\int_0^t\int_\Omega \partial_t(\beta_\eta(u_\eta)u_\eta)dxd\theta+(k+1)\int_0^t\int_\Omega\left(|Du_\eta|^{p-2}+\varphi(u_\eta)+\eta\right)|Du_\eta|^2dxd\theta$$
$$+\int_0^t\int_\Omega g(x,u_\eta)u_\eta dxd\theta=\int_0^t\int_\Omega f(x)u_\eta dxd\theta+\frac{1}{k+2}\int_0^t\int_\Omega\beta_\eta(u_\eta)\partial_t u_\eta dxd\theta. \tag{2.13}$$

Hence,

$$\frac{1}{k+2}\int_\Omega\beta_\eta(u_\eta(t))u_\eta(t)dx-\frac{1}{k+2}\int_\Omega\beta_\eta(u_{0\eta})u_{0\eta}dx$$
$$+(k+1)\int_0^t\int_\Omega\left(|Du_\eta|^{p-2}+\varphi(u_\eta)+\eta\right)|Du_\eta|^2dxd\theta+\int_0^t\int_\Omega g(x,u_\eta)u_\eta dxd\theta$$
$$=\int_0^t\int_\Omega f(x)u_\eta dxd\theta+\frac{1}{k+2}\int_0^t\int_\Omega\frac{dB_\eta(u_\eta)}{d\tau}dxd\theta, \tag{2.14}$$

where $B_\eta(s)=\int_0^s\beta_\eta(\theta)d\theta$.

Now, taking into account that $\beta_\eta(0)=0$ and $\beta'_\eta>\eta>0$, we conclude that $B_\eta(s)\le\beta_\eta(s)s$. Thus,

$$\frac{1}{k+2}\int_\Omega\beta_\eta(u_\eta)(t)u_\eta(t)dx-\frac{1}{k+2}\int_\Omega\beta_\eta(u_{0\eta})u_{0\eta}dx$$
$$+(k+1)\int_0^t\int_\Omega\left(|Du_\eta|^{p-2}+\varphi(u_\eta)+\eta\right)|Du_\eta|^2dxd\theta+\int_0^t\int_\Omega g(x,u_\eta)u_\eta dxd\theta$$
$$\le\int_0^t\int_\Omega f(x)u_\eta dxd\theta+\frac{1}{k+2}\int_\Omega B_\eta(u_\eta(t))dx-\frac{1}{k+2}\int_\Omega B_\eta(u_{0\eta})dx\le\int_0^t\int_\Omega f(x)u_\eta dxd\theta$$
$$+\frac{1}{k+2}\int_\Omega\beta_\eta(u_\eta)(t)u_\eta(t)dx, \tag{2.15}$$

or

$$(k+1)\int_0^t\int_\Omega\left(|Du_\eta|^{p-2}+\varphi(u_\eta)+\eta\right)|Du_\eta|^2dxd\theta+\int_0^t\int_\Omega g(x,u_\eta)u_\eta dxd\theta$$
$$\le\int_0^t\int_\Omega f(x)u_\eta dxd\theta+\frac{1}{k+2}\int_\Omega\beta_\eta(u_{0\eta})u_{0\eta}dx. \tag{2.16}$$

From condition $\partial_\zeta g > -\lambda$ we obtain that $g(\cdot,\zeta) > -\lambda\zeta$ and, consequently,

$$\begin{aligned}(k+1)&\int_0^t\int_\Omega\left(|Du_\eta|^{p-2}+\varphi(u_\eta)+\eta\right)|Du_\eta|^2dxd\theta\\&\leq\int_0^t\int_\Omega|u_\eta|^2(\lambda+1)dxd\theta+\int_0^t\int_\Omega|f|^2\,dxd\theta+\frac{1}{k+2}\int_\Omega|\beta_\eta(u_{0\eta})|^2\,dx+\frac{1}{k+2}\int_\Omega|u_{0\eta}|^2\,dx.\end{aligned}\tag{2.17}$$

Therefore,

$$\int_0^T\int_\Omega|Du_\eta|^p\,dxd\theta\leq c(T).\tag{2.18}$$

Thereby assertion follows. □

Corollary 2.2. *Under condition of Lemma 2.1 there exists c such that*

$$\frac{1}{\tau}\int_t^{t+\tau}\int_\Omega|Du_\eta|^pdxd\tau\leq c\quad\text{for arbitrary }\tau.\tag{2.19}$$

(It is sufficient to carry out the proof of Lemma 2.1 using integration on the interval $(t,t+\tau)$ instead of $(0,t)$ and taking into account (2.8).)

Lemma 2.3. *Under the hypotheses* $(H1)$–$(H5)$ *there exist constants* c_i *such that for any* $\eta\in(0,1)$ *the following estimates hold:* $\|u_\eta\|_{L^\infty(\tau,T;W_0^{1,p}(\Omega))}\leqslant c(\tau,T)$,

$$\int_\tau^T\int_\Omega\left(\beta'_\eta(u_\eta)\right)^2|\partial_tu_\eta|^2dxd\theta\leq c(\tau,T),\qquad\int_t^{t+\tau}\int_\Omega\beta'_\eta(u_\eta)|\partial_tu_\eta|^2dxd\theta\leq c(\tau)\quad(t\geqslant\tau).\tag{2.20}$$

Proof. Multiplying (2.1) by ∂_tu_η and integrating over $[s,\tau+t]$, where $\tau\leq t\leq s$, we get

$$\begin{aligned}&\int_s^{t+\tau}\int_\Omega\beta'_\eta(u_\eta)|\partial_tu_\eta|^2dxd\theta-\int_s^{t+\tau}\int_\Omega D\left(\left(|Du_\eta|^{p-2}+\varphi(u_\eta)+\eta\right)Du_\eta\right)\partial_tu_\eta dxd\theta\\&\quad+\int_s^{t+\tau}\int_\Omega g(x,u_\eta)\partial_tu_\eta dxd\theta=\int_s^{t+\tau}\int_\Omega f(x)\partial_tu_\eta dxd\theta.\end{aligned}\tag{2.21}$$

Then, using integration by parts, we derive

$$\begin{aligned}&\int_s^{t+\tau}\int_\Omega\beta'_\eta(u_\eta)|\partial_tu_\eta|^2dxd\theta+\int_s^{t+\tau}\int_\Omega\left(\left(|Du_\eta|^{p-2}+\varphi(u_\eta)+\eta\right)Du_\eta\right)D\partial_tu_\eta dxd\theta\\&\quad+\int_s^{t+\tau}\int_\Omega\partial_tG(x,u)dxd\theta=\int_s^{t+\tau}\int_\Omega\partial_t(f(x)u_\eta)dxd\theta,\end{aligned}\tag{2.22}$$

where $G(\cdot,s)=\int_0^sg(\cdot,\zeta)d\zeta$.

Hence, using condition $(H4)$, we get

$$\begin{aligned}
&\int_s^{t+\tau}\int_\Omega \beta_\eta'(u_\eta)\left|\partial_t u_\eta\right|^2 dx\,d\theta \\
&\quad+\int_s^{t+\tau}\frac{d}{d\theta}\left(\frac{1}{p+2}\int_\Omega \left|Du_\eta(\theta)\right|^p dx+\frac{1}{2}\int_\Omega \varphi(u_\eta(\theta))\left|Du_\eta(\theta)\right|^2 dx+\frac{1}{2}\int_\Omega \eta\left|Du_\eta(\theta)\right|^2 dx\right. \\
&\qquad\qquad\left.+\int_\Omega G(x,u_\eta(\theta))\,dx-\int_\Omega f(x)u_\eta(\theta)\right)d\theta \\
&=\frac{1}{2}\int_s^{t+r}\int_\Omega \varphi'(u_\eta(\theta))\times\left|Du_\eta(\theta)\right|^2\partial_t u_\eta\,dx\,d\theta \\
&\le\frac{1}{2}\int_s^{t+r}\int_\Omega\left(\left|u_\eta\right|^l+c\right)\left(\sqrt{\beta'(s)}\right)^\alpha\left|Du_\eta\right|^2\left|\partial_t u_\eta\right|dx\,d\theta.
\end{aligned}\tag{2.23}$$

Further, taking into account that $\beta_\eta(u_\eta)$ is uniformly bounded by η ($u_\eta \in [-\delta,\delta]$, where δ is the bound in the proof of Lemma 2.1), it is possible to choose β_η so that $\beta_\eta' \le L$, where L is the Lipschitz constant of $\beta(\zeta)$ on $[-\delta,\delta]$. Therefore,

$$\begin{aligned}
&\frac{1}{2}\int_s^{t+\tau}\int_\Omega\left(\left|u_\eta\right|^l+c\right)\left(\sqrt{\beta'(s)}\right)^\alpha\left|Du_\eta\right|^2\left|\partial_t u_\eta\right|dx\,d\theta \\
&\quad\le\frac{\epsilon}{2}\int_s^{t+\tau}\int_\Omega\left(\beta_\eta'(u_\eta)\right)^\alpha\left|\partial_t u_\eta\right|^2 dx\,d\theta+\frac{1}{2\epsilon}\int_s^{t+\tau}\int_\Omega\left(\left|u_\eta\right|^l+c\right)^2\left|Du_\eta\right|^4 dx\,d\theta \\
&\quad\le\frac{\epsilon}{2}L^{\alpha-1}\int_s^{t+\tau}\int_\Omega\beta_\eta'(u_\eta)\left|\partial_t u_\eta\right|^2 dx\,d\theta+\frac{1}{2\epsilon}\int_s^{t+\tau}\int_\Omega\left(\left|u_\eta\right|^l+c\right)^2\left|Du_\eta\right|^4 dx\,d\theta.
\end{aligned}\tag{2.24}$$

Consequently,

$$\begin{aligned}
&\int_s^{t+\tau}\int_\Omega \beta_\eta'(u_\eta)\left|\partial_t u_\eta\right|^2 dx\,d\theta \\
&\quad+\int_s^{t+\tau}\frac{d}{d\theta}\left(\frac{1}{p+2}\int_\Omega \left|Du_\eta(\theta)\right|^p dx+\frac{1}{2}\int_\Omega \varphi(u_\eta(\theta))\left|Du_\eta(\theta)\right|^2 dx+\frac{1}{2}\int_\Omega \eta\left|Du_\eta(\theta)\right|^2 dx\right. \\
&\qquad\qquad\left.+\int_\Omega G(x,u_\eta(\theta))\,dx-\int_\Omega f(x)u_\eta(\theta)\,dx\right)d\theta \\
&\le\frac{\epsilon}{2}L^{\alpha-1}\int_s^{t+\tau}\int_\Omega\beta_\eta'(u_\eta)\left|\partial_t u_\eta\right|^2 dx\,d\theta+\frac{1}{2\epsilon}\int_s^{t+\tau}\int_\Omega\left(\left|u_\eta\right|^l+c\right)^2\left(\left|Du_\eta\right|^p+1\right)dx\,d\theta.
\end{aligned}\tag{2.25}$$

Now, using Lemma 2.1 and Corollary 2.2, we easily deduce

$$\begin{aligned}
&\frac{1}{2}\int_s^{t+\tau}\int_\Omega \beta_\eta'(u_\eta)\left|\partial_t u_\eta\right|^2 dx d\theta + \frac{1}{p+2}\int_\Omega \left|Du_\eta\right|^p (t+\tau)dx + \int_\Omega \varphi(u_\eta)\left|Du_\eta\right|^2(t+\tau)dx \\
&\quad + \frac{1}{2}\int_\Omega \eta\left|Du_\eta\right|^2(t+\tau)dx + \int_\Omega G(x,u_\eta(t+\tau))dx - \int_\Omega f(x)u_\eta(t+\tau)dx \\
&\quad \le \frac{1}{p+2}\int_\Omega \left|Du_\eta\right|^p(s)dx + \int_\Omega \varphi(u_\eta)\left|Du_\eta\right|^2(s)dx + \frac{1}{2}\int_\Omega \eta\left|Du_\eta\right|^2(s)dx \\
&\quad + \int_\Omega G(x,u_\eta(s))dx - \int_\Omega f(x)u_\eta(s)dx + \frac{1}{2\epsilon}c\tau.
\end{aligned} \tag{2.26}$$

After integrating over $[t,t+\tau]$, we derive

$$\begin{aligned}
&\tau\frac{1}{p+2}\int_\Omega \left|Du_\eta\right|^p(t+\tau)dx + \tau\int_\Omega \varphi(u_\eta)\left|Du_\eta\right|^2(t+\tau)dx + \frac{1}{2}\tau\int_\Omega \eta\left|Du_\eta\right|^2(t+\tau)dx \\
&\quad \le \frac{1}{p+2}\int_t^{t+\tau}\int_\Omega \left|Du_\eta\right|^p(s)dxds + \int_t^{t+\tau}\int_\Omega \varphi(u_\eta)\left|Du_\eta\right|^2(s)dxds \\
&\quad + \frac{1}{2}\int_t^{t+\tau}\int_\Omega \eta\left|Du_\eta\right|^2(s)dxds + \int_t^{t+\tau}\int_\Omega G(x,u_\eta(s))dxds + \int_t^{t+\tau}\int_\Omega f(x)u_\eta(s)dxds \\
&\quad - \tau\int_\Omega f(x)u_\eta(t+\tau)dx - \tau\int_\Omega G(x,u_\eta(t+\tau))dx + c(\tau)\tau.
\end{aligned} \tag{2.27}$$

Since, by virtue of conditions imposed on the function $g, -(\lambda/2)|\zeta|^2 < G(\cdot,\zeta) \le c|\zeta|^k + (\lambda/2)|\zeta|^2 + c$, then

$$\begin{aligned}
&\frac{1}{p+2}\int_\Omega \left|Du_\eta\right|^p(t+\tau)dx + \int_\Omega \varphi(u_\eta)\left|Du_\eta\right|^2(t+\tau)dx \\
&\le \frac{1}{\tau}\Bigg(\frac{1}{p+2}\int_t^{t+\tau}\int_\Omega \left|Du_\eta\right|^p(s)dxds + \int_t^{t+\tau}\int_\Omega \varphi(u_\eta)\left|Du_\eta\right|^2(s)dxds \\
&\quad + \frac{1}{2}\int_t^{t+\tau}\int_\Omega \eta\left|Du_\eta\right|^2(s)dxds + \int_t^{t+\tau}\int_\Omega \left(c\left|u_\eta\right|^k + \left(\frac{\lambda}{2}\right)\left|u_\eta\right|^2 + c\right)dxds \\
&\quad + \int_t^{t+\tau}\int_\Omega f(x)u_\eta(s)dxds\Bigg) + \int_\Omega\left(c\left|u_\eta\right|^k + \left(\frac{\lambda}{2}\right)\left|u_\eta\right|^2 + c\right)(t)dx \\
&+ \int_\Omega f(x)u_\eta(t)dx + c(\tau)
\end{aligned}$$

$$\le \frac{1}{\tau}\left(\frac{1}{p+2}\int_t^{t+\tau}\int_\Omega |Du_\eta|^p(s)dxds + \frac{1}{2}\varepsilon\int_t^{t+\tau}\int_\Omega |Du_\eta|^4(s)dxds\right.$$
$$+\frac{1}{2}\varepsilon^{-1}\int_t^{t+\tau}\int_\Omega \varphi^2(u_\eta(s))dxds + \frac{1}{2}\int_t^{t+\tau}\int_\Omega \eta|Du_\eta|^2(s)dxds$$
$$\left.+\int_t^{t+\tau}\int_\Omega\left(c|u_\eta(s)|^k + \left(\frac{\lambda}{2}\right)|u_\eta(s)|^2 + c\right)dxds + \int_t^{t+\tau}\int_\Omega |f(x)u_\eta(s)|dxds\right) + c(\tau). \tag{2.28}$$

As we noted above, $\beta'_\eta \le L$ on $[-\delta,\delta]$. Hence,

$$\int_t^{t+\tau}\int_\Omega \varphi^2(u_\eta)dxds \le \int_t^{t+\tau}\int_\Omega\left(\int_0^{u_\eta}\varphi'(s)ds\right)^2 dxds$$
$$\le \int_t^{t+\tau}\int_\Omega ||u_\eta||_{L^\infty(\tau,T;L^\infty(\Omega))}\left(||u_\eta||^l_{L^\infty(\tau,T;L^\infty(\Omega))} + c\right)L^\alpha dxds. \tag{2.29}$$

Thus, we obtain that

$$\frac{1}{p}\int_\Omega |Du_\eta|^p(t+\tau)dx + \int_\Omega \varphi(u_\eta)|Du_\eta|^2(t+\tau)dx \le c(\tau) \tag{2.30}$$

for $t \ge \tau$.

Now returning to (2.23), we have

$$\int_t^{t+\tau}\int_\Omega \beta'_\eta(u_\eta)|\partial_t u_\eta|^2 dxd\theta \le c(\tau). \tag{2.31}$$

Also, choosing β_η so that $\beta'_\eta \le L$, we derive

$$\int_\tau^T\int_\Omega \left(\beta'_\eta(u_\eta)\right)^2|\partial_t u_\eta|^2 dxd\theta \le c(\tau,T). \tag{2.32}$$

Thus, the lemma is proved. □

Passage to the Limit in (2.1).

Analogously to [7, 8], by estimates from Lemmas 2.1 and 2.3, we deduce that there exist that a subsequence u_η such that

$u_\eta \to u$ weakly in $L^p(0,T;W_0^{1,p}(\Omega))$ and *a.e.*,
$\beta_\eta(u_\eta) \to \beta(u)$ strongly in $C([0,T];L^2(\Omega))$,
$\partial_t\beta_\eta(u_\eta) \to \partial_t\beta(u)$ weakly in $L^2(Q)$,
$D((|Du_\eta|^{p-2}Du_\eta) \to \chi$ weakly in $L^{p'}(0,T;W^{-1,p'}(\Omega))$.

Further, it is easy to see that

$$g(x, u_\eta) \longrightarrow g(x, u) \quad \text{weakly in } L^2(Q). \tag{2.33}$$

Indeed, we know that $u_\eta \to u$ a.e. Besides, by virtue of the embedding $\mathfrak{R}(Q) \subset C(Q)$, where $\mathfrak{R}(Q)$ is a domain of the solvability, the operator $g : \mathfrak{R}(Q) \to L^2(Q)$ generated by the expression $g(x, u)$ is bounded. Hence, by the continuity of g we obtain that $g(x, u_\eta) \to g(x, u)$ *a.e*. By applying a known lemma from [10, (1.1, Lemma 1.3)] we can conclude that

$$g(x, u_\eta) \longrightarrow g(x, u) \quad \text{weakly in } L^2(Q). \tag{2.34}$$

Arguing similarly, by help of the lemma from [10, (1.1, Lemma 1.3)] and conditions imposed on $\varphi(\zeta)$, we obtain

$$\sqrt{\varphi(u_\eta)}Du_\eta \longrightarrow \sqrt{\varphi(u)}Du \quad \text{weakly in } L^2(Q), \tag{2.35}$$

$$\begin{gathered} \varphi(u_\eta)Du_\eta \longrightarrow \varphi(u)Du \quad \text{weakly in } L^2(Q), \\ D(\varphi(u_\eta)Du_\eta) \longrightarrow D(\varphi(u)Du) \quad \text{weakly in } L^2\left(0, T; W^{-1,2}(\Omega)\right). \end{gathered} \tag{2.36}$$

Observe now that from (2.35), in view of the known theorem,

$$\begin{aligned} \int_Q \varphi(u_\eta)\left|Du_\eta\right|^2 dxdt &= \left\|\sqrt{\varphi(u)}Du\right\|_{L^2(Q)} \\ &\le \lim_{n\nearrow\infty} \inf \left\|\sqrt{\varphi(u_\eta)}Du_\eta\right\|_{L^2(Q)} = \int_Q \varphi(u_\eta)\left|Du_\eta\right|^2 dxdt. \end{aligned} \tag{2.37}$$

Consequently, using standard monotonicity argument [10], we derive that $\chi = D(|Du|^{p-2}Du)$.

Therefore, the following theorems hold.

Theorem 2.4. *Under hypotheses* $(H1)$–$(H5)$ *the problem* (1.7), (1.8), (1.9) *has a weak solution* $u(x,t)$ *such that* $u \in L^p(0, T; W_0^{1,p}(\Omega)) \cap L^\infty(0, T; L^2(\Omega)) \cap L^\infty(\tau, T; W_0^{1,p}(\Omega))$, *for all* $\tau > 0$ *and* $\beta(u) \in C([0, T]; L^2(\Omega)) \cap L^q(Q)$.

Now, we prove that the solution of the problem is unique.

Lemma 2.5. *Suppose that the assumptions of Theorem 2.4 are fulfilled and there exists a constant* c *such that the function* $\zeta \to g(x, \zeta) + c\beta(\zeta)$ *is increasing for* $a.\ e.\ x \in \Omega$. *Then the solution of the problem* (1.7), (1.8), (1.9) *is unique.*

Proof. Let $u(x, t)$ and $v(x, t)$ be two solutions of the problem (1.7), (1.8), (1.9) with the same initial condition: $u(x, 0) = v(x, 0) = u_0(x)$. Consider the "difference" $u(x, t_0) - v(x, t_0)$, where t_0 is an arbitrary point from $(0, T)$. The above "difference" is a continuously differentiable function for almost all $t > 0$, because by virtue of estimation $\int_\tau^T \int_\Omega (\beta_\eta'(u_\eta))^2 |\partial_t u_\eta|^2 dxdt \leq c_2(\tau, T)$, using the equation, we can conclude that $|Du|^{p-2}Du \in L^2(\tau, T; W_0^{1,2}(\Omega))$ and consequently $Du(x,t) \in C(\overline{\Omega})$ for almost all t. We choose t_0 such that $u(x,t_0) \in C^1(\overline{\Omega})$. Hence, the interval $[a,b]$ may be divided into the subintervals where sign of "difference" $u(x,t_0) - v(x,t_0)$ does not change. Let (x_1,x_2) be an interval such that $u(x,t_0) - v(x,t_0) > 0$ and $u(x_i,t) = v(x_i,t)(i = \overline{1,2})$. Then from (1.7) we obtain that

$$\begin{aligned}&\int_{x_1}^{x_2} \partial_t(\beta(u) - \beta(v))dx - \int_{x_1}^{x_2} D\Big(|Du|^{p-2}Du - |Dv|^{p-2}Dv\Big)dx - \int_{x_1}^{x_2} D(\varphi(u)Du - \varphi(v)Dv)dx \\ &\qquad + \int_{x_1}^{x_2} (g(x,u) - g(x,v))dx = 0.\end{aligned} \tag{2.38}$$

By applying Newton-Leibniz formula we have

$$\begin{aligned}&\Bigg[\frac{d}{dt}\int_{x_1}^{x_2} (\beta(u) - \beta(v))dx - |Du|^{p-2}Du(x_2,t) + |Dv|^{p-2}Dv(x_2,t) + |Du|^{p-2}Du(x_1,t) \\ &\quad - |Dv|^{p-2}Dv(x_1,t) - \varphi(u)Du(x_2,t) + \varphi(v)Dv(x_2,t) + \varphi(u)Du(x_1,t) - \varphi(v)Dv(x_1,t) \\ &\quad + \int_{x_1}^{x_2} g(x,u) + c\beta(u) - g(x,v) - c\beta(v)dx - \int_{x_1}^{x_2} c\beta(u) - c\beta(v)dx\Bigg]_{t=t_0} = 0.\end{aligned} \tag{2.39}$$

Since $Du(x_1,t_0) \geq Dv(x_1,t_0), Du(x_2,t_0) \leq Dv(x_2,t_0), u(x_i,t) = v(x_i,t)(i = \overline{1,2})$, it follows that

$$\frac{d}{dt}\int_{x_1}^{x_2} (\beta(u) - \beta(v))dx\Bigg|_{t=t_0} < c\int_{x_1}^{x_2} (\beta(u) - \beta(v))dx\Bigg|_{t=t_0}. \tag{2.40}$$

Note that the functions $\varphi(t) = \int_{x_1}^{x_2} ((\beta(u) - \beta(v))dx$ and $\tilde{\varphi}(t) = \int_{x_1}^{x_2} |\beta(u) - \beta(v)|dx$ are absolutely continuous by virtue of the inclusion $\partial_t \beta(u) \in L^2(\tau, T, L^2(\Omega))$. Hence, these functions possess a derivate for almost all t. Besides, it is obvious that $\tilde{\varphi}(t) \geq \varphi(t)$, and if $\beta(u(x,t')) - \beta(v(x,t')) \geqslant 0$ for $x \in (x_1,x_2)$, then, $\tilde{\varphi}(t') = \varphi(t')$. Consequently,

$$\frac{d}{dt}\int_{x_1}^{x_2} |\beta(u) - \beta(v)|dx\Bigg|_{t=t'} \leq \frac{d}{dt}\int_{x_1}^{x_2} (\beta(u) - \beta(v))dx\Bigg|_{t=t'} \tag{2.41}$$

for almost all t'. From here, without loss of generality, we can assume that

$$\frac{d}{dt}\int_{x_1}^{x_2} |\beta(u) - \beta(v)|dx\Bigg|_{t=t_0} \leq \frac{d}{dt}\int_{x_1}^{x_2} (\beta(u) - \beta(v))dx\Bigg|_{t=t_0}. \tag{2.42}$$

It follows that

$$\frac{d}{dt}\int_{x_1}^{x_2}\left|\beta(u)-\beta(v)\right|dx\bigg|_{t=t_0} < c\int_{x_1}^{x_2}\left|\beta(u)-\beta(v)\right|dx\bigg|_{t=t_0}. \tag{2.43}$$

The same estimation holds for an arbitrary interval on which $u(x,t_0)-v(x,t_0)$ does not change its sign. Summing up similar inequalities over subintervals, we get

$$\frac{d}{dt}\int_{a}^{b}|\beta(u)-\beta(v)|dx\bigg|_{t=t_0} < c\int_{a}^{b}\left|\beta(u)-\beta(v)\right|dx\bigg|_{t=t_0} \tag{2.44}$$

almost everywhere. In view of integrability of both sides of the latter inequality ($\beta(u) \in W^{1,2}(\tau,T;L^2(\Omega))$) we have

$$\int_{t'}^{t''}\frac{d}{dt}\int_{a}^{b}\left|\beta(u)-\beta(v)\right|dx\,dt < c\int_{t'}^{t''}\int_{a}^{b}\left|\beta(u)-\beta(v)\right|dx\,dt, \tag{2.45}$$

where $t'' > t' > \tau$, or

$$\int_{a}^{b}|\beta(u)-\beta(v)|dx\bigg|_{t=t''} < \int_{a}^{b}|\beta(u)-\beta(v)|dx\bigg|_{t=t'} + c\int_{t'}^{t''}\int_{a}^{b}\left|\beta(u)-\beta(v)\right|dx\,dt. \tag{2.46}$$

Thus, by Gronwall's lemma,

$$\int_{a}^{b}\left|\beta(u)-\beta(v)\right|dx\bigg|_{t=t''} \le e^{c(t''-t')}\int_{a}^{b}\left|\beta(u)-\beta(v)\right|dx\bigg|_{t=t'}. \tag{2.47}$$

Now, taking into account that $\beta(u) \in C(0,T;L^2(\Omega))$ and $u(x,0) = v(x,0) = u_0(x)$, we obtain $u(x,t) = v(x,t)$, which finishes the proof of uniqueness. □

Remark 2.6. Note that the condition on the function $\zeta \to g(x,\zeta) + c\beta(\zeta)$ from the lemma can be changed to the following.

$(H2)'$ The function $\beta^{-1}(s)$ is a locally Lipschitzian function from R to R (it follows directly from the proof).

In this case, we can exclude the condition $(H4)$ of Theorem 2.4 and conditions $(H4)$ and $(H6)$ of Theorem 2.8 below.

So, analogously to the corresponding result from [8], we obtain that problem (1.7), (1.8), (1.9) generates a continuous semigroup $S_t : L^2(\Omega) \to L^2(\Omega) S_t u_0 = u(t,\cdot)$ and the following theorem holds.

Theorem 2.7. *Assume that assumptions of Lemma 2.5 are satisfied. Then the semigroup S_t associated with the boundary value problem* (1.7), (1.8), (1.9) *possesses a maximal attractor A which is bounded in $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$, compact, and connected in $L^2(\Omega)$.*

(For the concepts of absorbing sets and global attractors used here, we refer the reader to [9]).

Under an additional condition we can obtain more regularity for $u(x,t)$.

Theorem 2.8. *Assume that* $u_0(x) \in W^{1,\infty}(\Omega)$, $\beta(u_0) \in L^\infty(\Omega)$, *and the conditions* (H2)–(H6) *are fulfilled. Then* $Du \in L^\infty(Q)$.

Proof. We prove this fact by multiplying (2.1) on expression $|Du_\eta|^\sigma \partial_t u_\eta$, taking into account of the arbitrarity of σ

$$\begin{aligned}
&\int_0^T\int_\Omega \beta'_\eta(u_\eta)|\partial_t u_\eta|^2|Du_\eta|^\sigma dxdt - \int_0^T\int_\Omega D\Big(|Du_\eta|^{p-2}Du_\eta\Big)\partial_t u_\eta|Du_\eta|^\sigma dxdt \\
&\quad - \eta\int_0^T\int_\Omega D^2u_\eta\partial_t u_\eta|Du_\eta|^\sigma dxdt - \int_0^T\int_\Omega D(\varphi(u_\eta)Du_\eta)\partial_t u_\eta|Du_\eta|^\sigma dxdt \\
&\quad + \int_0^T\int_\Omega (g(x,u_\eta) - f(x))\partial_t u_\eta|Du_\eta|^\sigma dxdt = 0.
\end{aligned} \tag{2.48}$$

Let us estimate the terms of (2.48) separately.

Integrating by parts the second integral of (2.48), we get

$$\begin{aligned}
&-\int_0^T\int_\Omega D\Big(|Du_\eta|^{p-2}Du_\eta\Big)\partial_t u_\eta|Du_\eta|^\sigma dxdt \\
&\quad = -(p-1)\int_0^T\int_\Omega |Du_\eta|^{p-2}D^2u_\eta\partial_t u_\eta|Du_\eta|^\sigma dxdt \\
&\quad = -(p-1)\int_0^T\int_\Omega |Du_\eta|^{p+\sigma-2}D^2u_\eta\partial_t u_\eta dxdt \\
&\quad = -\frac{p-1}{p+\sigma-1}\int_0^T\int_\Omega D\Big(|Du_\eta|^{p+\sigma-2}Du_\eta\Big)\partial_t u_\eta dxdt \\
&\quad = \frac{p-1}{p+\sigma-1}\int_0^T\int_\Omega |Du_\eta|^{p+\sigma-2}Du_\eta D\partial_t u_\eta dxdt \\
&= \frac{p-1}{(p+\sigma-1)(p+\sigma)}\int_0^T\frac{d}{dt}\int_\Omega |Du_\eta|^{p+\sigma}dxdt.
\end{aligned} \tag{2.49}$$

Arguing similarly, we obtain

$$\begin{aligned}
-\eta\int_0^T\int_\Omega D^2u_\eta\partial_t u_\eta|Du_\eta|^\sigma dxdt &= -\frac{\eta}{\sigma+1}\int_0^T\int_\Omega D(|Du_\eta|^\sigma Du_\eta)\partial_t u_\eta dxdt \\
&= \frac{\eta}{\sigma+1}\int_0^T\int_\Omega |Du_\eta|^\sigma Du_\eta D\partial_t u_\eta dxdt \\
&= \frac{\eta}{(\sigma+1)(\sigma+2)}\int_0^T\frac{d}{dt}\int_\Omega |Du_\eta|^{\sigma+2}dxdt.
\end{aligned} \tag{2.50}$$

A series simple calculations give us the estimate of the third term of (2.48):

$$
\begin{aligned}
&-\int_0^T\int_\Omega D(\varphi(u_\eta)Du_\eta)\partial_t u_\eta|Du_\eta|^\sigma dxdt\\
&=\frac{1}{\sigma+2}\int_0^T\int_\Omega \varphi(u_\eta)\partial_t|Du_\eta|^{\sigma+2}dxdt+\frac{\sigma}{\sigma+1}\int_0^T\int_\Omega \varphi(u_\eta)\partial_t u_\eta D\left(|Du_\eta|^\sigma Du_\eta\right)dxdt\\
&=\frac{1}{\sigma+2}\int_0^T\int_\Omega \varphi(u_\eta)\partial_t|Du_\eta|^{\sigma+2}dxdt-\frac{\sigma}{\sigma+1}\int_0^T\int_\Omega \varphi(u_\eta)D\partial_t u_\eta|Du_\eta|^\sigma Du_\eta dxdt\\
&\quad-\frac{\sigma}{\sigma+1}\int_0^T\int_\Omega \varphi'(u_\eta)\partial_t u_\eta|Du_\eta|^{\sigma+2}dxdt=-\frac{1}{(\sigma+1)(\sigma+2)}\int_0^T\int_\Omega \varphi'(u_\eta)\partial_t u_\eta|Du_\eta|^{\sigma+2}dxdt\\
&\quad+\frac{1}{(\sigma+1)(\sigma+2)}\int_0^T\int_\Omega \partial_t\left(\varphi(u_\eta)|Du_\eta|^{\sigma+2}\right)dxdt-\frac{\sigma}{\sigma+1}\int_0^T\int_\Omega \varphi'(u_\eta)\partial_t u_\eta|Du_\eta|^{\sigma+2}dxdt\\
&=-\frac{\sigma+1}{\sigma+2}\int_0^T\int_\Omega \varphi'(u_\eta)\partial_t u_\eta|Du_\eta|^{\sigma+2}dxdt+\frac{1}{(\sigma+1)(\sigma+2)}\int_0^T\int_\Omega \partial_t\left(\varphi(u_\eta)|Du_\eta|^{\sigma+2}\right)dxdt\\
&\geq-\frac{\sigma+1}{\sigma+2}\int_0^T\int_\Omega\left(|u_\eta|^{l_1}+c\right)\left(\sqrt{\beta'_\eta(u_\eta)}\right)^{\alpha_1}|\partial_t u_\eta||Du_\eta|^{\sigma+2}dxdt+\frac{1}{(\sigma+1)(\sigma+2)}\\
&\quad\times\int_0^T\int_\Omega \partial_t\left(\varphi(u_\eta)|Du_\eta|^{\sigma+2}\right)dxdt\geq-\frac{(\sigma+1)\epsilon}{(\sigma+2)2}\int_0^T\int_\Omega\left(\beta'_\eta(u_\eta)\right)^{\alpha_1}|\partial_t u_\eta|^2|Du_\eta|^\sigma dxdt\\
&\quad-\frac{(\sigma+1)}{2\epsilon(\sigma+2)}\int_0^T\int_\Omega\left(|u_\eta|^{l_1}+c\right)^2|Du_\eta|^{\sigma+4}dxdt+\frac{1}{(\sigma+1)(\sigma+2)}\\
&\quad\times\int_0^T\int_\Omega \partial_t\left(\varphi(u_\eta)|Du_\eta|^{\sigma+2}\right)dxdt\geq-\frac{(\sigma+1)\epsilon}{(\sigma+2)2}L^{\alpha_1-1}\int_0^T\int_\Omega \beta'_\eta(u_\eta)|\partial_t u_\eta|^2|Du_\eta|^\sigma dxdt\\
&\quad-\frac{(\sigma+1)}{2\epsilon(\sigma+2)}\int_0^T\int_\Omega\left(|u_\eta|^{l_1}+c\right)^2|Du_\eta|^{\sigma+p+2}dxdt-\frac{(\sigma+1)}{2\epsilon(\sigma+2)}\int_0^T\int_\Omega\left(|u_\eta|^{l_1}+c\right)^2dxdt\\
&\quad\int_\Omega \varphi(u_\eta)|Du_\eta|^{\sigma+2}dx-\int_\Omega \varphi(u_0)|Du_0|^{\sigma+2}dx.
\end{aligned}
\tag{2.51}
$$

Further, taking into account that $\beta_\eta(u_\eta)$ is uniformly bounded by η ($u_\eta\in[-\delta,\delta]$, where δ is the bound in the proof of Lemma 2.1), it is possible to choose β_η so that $\beta'_\eta\leq L$, where L

is the Lipschitz constant of $\beta(\zeta)$ on $[-\delta, \delta]$. Therefore,

$$\begin{aligned}
&\left| \int_0^T \int_\Omega (g(x,u_\eta) - f(\mathrm{x}))\partial_t u_\eta |Du_\eta|^\sigma dxdt \right| \\
&\quad \le \left| \int_0^T \int_\Omega \left(|u_\eta|^{l_1} + c\right)\left(\sqrt{\beta'_\eta(u_\eta)}\right)^{\alpha_1} |\partial_t u_\eta||Du_\eta|^\sigma dxdt \right| \\
&\quad \le \frac{\epsilon}{2}\int_0^T \int_\Omega \left(\beta'_\eta(u_\eta)\right)^{\alpha_1} |\partial_t u_\eta|^2 |Du_\eta|^\sigma dxdt + \frac{1}{2\epsilon}\int_0^T \int_\Omega \left(|u_\eta|^{l_1} + c\right)^2 |Du_\eta|^\sigma dxdt \\
&\quad \le \frac{\epsilon}{2}L^{\alpha_1 - 1}\int_0^T \int_\Omega \beta'_\eta(u_\eta)|\partial_t u_\eta|^2 |Du_\eta|^\sigma dxdt + \frac{1}{2\epsilon}\int_0^T \int_\Omega \left(|u_\eta|^{l_1} + c\right)^2 |Du_\eta|^\sigma dxdt.
\end{aligned} \tag{2.52}$$

Also note that if $\beta(u_0) \in L^\infty(\Omega)$ then using the same arguments as in proof of (2.8) we conclude that $\|u_\eta\|_{L^\infty(Q_T)} \leqslant c$. Thus, combining (2.49)–(2.52) and choosing ϵ sufficiently small, we rewrite (2.48) in the form

$$\int_\Omega |Du_\eta|^{p+\sigma} dx \le c_1 \int_0^T \int_\Omega |Du_\eta|^{p+\sigma} dxdt + c_2 T + c_3 \int_\Omega |Du_0|^{p+\sigma} dx. \tag{2.53}$$

Applying Gronwall 's lemma, we get

$$\int_\Omega |Du_\eta|^{p+\sigma} dx \leqslant \left(c_2 T + c_3 \int_\Omega |Du_0|^{p+\sigma} dx\right) e^{c_1 T}, \tag{2.54}$$

where the constant c_i does not depend on σ. Consequently,

$$\begin{aligned}
\left(\frac{1}{|\Omega|}\int_\Omega |Du_\eta|^{p+\sigma} dx\right)^{1/(p+\sigma)} &\leqslant \left(\frac{c_3}{|\Omega|}\int_\Omega |Du_0|^{p+\sigma} dx + \frac{c_2}{|\Omega|}T\right)^{1/(p+\sigma)} e^{c_1 T/(p+\sigma)} \\
&\leqslant e^{c_1 T/(p+\sigma)} \left(\frac{c_3}{|\Omega|}\int_\Omega |Du_0|^{p+\sigma} dx\right)^{1/(p+\sigma)} + e^{c_1 T/(p+\sigma)} \left(\frac{c_2}{|\Omega|}T\right)^{1/(p+\sigma)}.
\end{aligned} \tag{2.55}$$

By letting σ tend to $+\infty$, we get necessary estimation. Theorem is proved. □

3. Quantitative Homogenization

As we mentioned earlier, our goal is to compare the global behavior of solutions $u^\epsilon(x,t)$ of (1.4) for $\epsilon \to 0$ with solutions $u = u^0(x,t)$ of the homogenized equation

$$\partial_t \beta(u) - D\left(|Du|^{p-2} Du + \varphi(u) Du\right) + g^0(x,u) = f^0(x), \tag{3.1}$$

where (3.1) and (1.4) are supplied with the same initial data $u^\epsilon(x,0) = u^0(x,0) = u_0(x)$, and homogenized nonlinearity g^0 and inhomogeneity f^0 are defined according to assumption from [11, pages 172-174].

We suppose that the function $g(x,z,\zeta)$ has the following structure:

$$g(x,z,\zeta)=\sum_{j=1}^{M} b^{j}(x,z)g_{j}(\zeta), \tag{3.2}$$

where $g_j \in C^1$ is supposed to satisfy a condition $(H3)$.

For all $j=1$, we suppose that $b^j(x,z)$ are bounded:

$$\left|b^{j}(x,z)\right| \leq C, \tag{3.3}$$

and the average $b^{0j}(x)$ of $b^j(x,x/\epsilon)$ exist in $L^{\infty}_{w^*}(\Omega)$, for $\epsilon \longrightarrow 0$:

$$\left\langle b^{j}\left(x,\frac{x}{\epsilon}\right), v(x)\right\rangle \underset{\epsilon\longrightarrow 0}{\longrightarrow} \left\langle b^{0j}(x), v(x)\right\rangle, \tag{3.4}$$

for $v(x)\in L^1(\Omega)$, where $\langle\cdot,\cdot\rangle$ indicates duality.

We also suppose that

$$\left\langle f\left(x,\frac{x}{\epsilon}\right), v(x)\right\rangle \underset{\epsilon\longrightarrow 0}{\longrightarrow} \left\langle f^{0}(x), v(x)\right\rangle, \tag{3.5}$$

for any $v(x)\in L^2(\Omega)$.

The equation $\partial_t\beta(u^0)-D((|Du^0|^{p-2}Du^0+\varphi(u^0)Du^0)+g^0(x,u^0)=f^0(x)$ is called the homogenization of equation if

$$g^{0}\left(x,u^{0}\right)=\sum_{j=1}^{M} b^{0j}(x)g_{j}\left(u^{0}\right). \tag{3.6}$$

Denoting $\widetilde{b}^j(x,z)=b^j(x,z)-b^{0j}(x)$, for $x\in\Omega, z\in R$, we assume that there exist functions $B^j(x,z)$ which are uniformly bounded for all $x\in\Omega, z\in R$ given as

$$\left|B^{j}(x,z)\right| \leq C \tag{3.7}$$

and which represent $\widetilde{b}^j$ such that $\widetilde{b}^j(x,z)=\partial_z B^j(x,z)$. With respect to the x-derivatives we assume ϵ-independent L^1-bound

$$\left\|\partial_x B^{j}\left(\cdot,\frac{\cdot}{\epsilon}\right)\right\|_{L^1(\Omega)} \leq C, \tag{3.8}$$

uniformly for all $j=1,\ldots,M$. Here ∂_x is a partial's derivatives with respect to the first argument x of the function $B^j(x,z)$.

Analogously, we denote $\widetilde{f}(x,z) = f(x,z) - f^0(x)$ and require the existence of a function $J(x,z)$ such that $\widetilde{f}(x,z)$ admits a divergence representation $\widetilde{f}(x,z) = \partial_z J(x,z)$.

Besides, we assume bounds

$$|J(x,z)| \le C, \qquad \left\| \partial_x J\left(\cdot, \frac{\cdot}{\epsilon}\right) \right\|_{L^1(\Omega)} \le C. \tag{3.9}$$

Note that sufficient conditions which guarantee the existence of divergence representations for $\widetilde{b}^j(x,z)$ and $\widetilde{f}(x,z)$ by help of $B^j(x,z)$ and $J(x,z)$, respectively, are established in [11, Theorem 3.2, pages 176-180].

The following theorem holds.

Theorem 3.1. *Let $g(x,z,w)$ and $f(x,z)$ satisfy conditions* (3.2)–(3.9) *and let assumptions of Lemma 2.5 be fulfilled. Then there exists a positive constant ρ such that the solutions $u^\epsilon(x,t)$ and $u^0(u,x)$ of the respective problems* (1.4), (1.8), (1.9) *and* (1.3), (1.8), (1.9) *with equal initial data $u_0(x) \in L^2(\Omega)$ satisfy the quantitative homogenization estimation*

$$\left\| \beta(u^\epsilon) - \beta(u^0) \right\|_{L^1(\Omega)} \le C\epsilon e^{\rho t} \quad \text{uniformly for } 0 \le t < \infty. \tag{3.10}$$

Proof. Consider the "difference" $u^\epsilon(x,t_0) - u^0(x,t_0)$, where t_0 is an arbitrary point from $(0,T)$. The above "difference" is a continuously differentiable function, in virtue of Lemma 2.3. Hence, the interval $[a,b]$ may be divided into the subintervals where sign of the "difference" $u^\epsilon(x,t_0) - u^0(x,t_0)$ does not change. Let (x_1,x_2) be an interval such that $u^\epsilon(x,t_0) - u^0(x,t_0) > 0$ and $u^\epsilon(x_i,t_0) = u^0(x_i,t_0) (i = \overline{1,2})$. Then from equations (1.3) and (1.4) we obtain that

$$\begin{aligned} &\frac{d}{dt}\int_{x_1}^{x_2} \beta(u^\epsilon) - \beta\left(u^0\right) dx - \int_{x_1}^{x_2} D\left(|Du^\epsilon|^{p-2} Du^\epsilon - \left|Du^0\right|^{p-2} Du^0 \right) dx \\ &\quad - \int_{x_1}^{x_2} D\left(\varphi(u^\epsilon) Du^\epsilon - \varphi\left(u^0\right) Du^0 \right) dx + \int_{x_1}^{x_2} \left(g^0(x,u^\epsilon) - g^0\left(x,u^0\right) \right) dx \\ &= -\int_{x_1}^{x_2} \left(g\left(x, \frac{x}{\epsilon}, u^\epsilon\right) - g^0(x,u^\epsilon) \right) dx + \int_{x_1}^{x_2} \left(f\left(x, \frac{x}{\epsilon}\right) - f^0(x) \right) dx. \end{aligned} \tag{3.11}$$

By applying Newton-Leibniz formula we have

$$\begin{aligned} &\Bigg[\frac{d}{dt}\int_{x_1}^{x_2} \left(\beta(u^\epsilon) - \beta\left(u^0\right) \right) dx - |Du^\epsilon|^{p-2} Du^\epsilon(x_2,t) + \left|Du^0\right|^{p-2} Du^0(x_2,t) \\ &\quad + |Du^\epsilon|^{p-2} Du^\epsilon(x_1,t) - \left|Du^0\right|^{p-2} Du^0(x_1,t) + \varphi(u^\epsilon) Du^\epsilon(x_2,t) - \varphi\left(u^0\right) Du^0(x_2,t) \\ &\quad - \varphi(u^\epsilon) Du^\epsilon(x_1,t) + \varphi\left(u^0\right) Du^0(x_1,t) + \int_{x_1}^{x_2} \left(g^0(x,u^\epsilon) - g^0\left(x,u^0\right) \right) dx \\ &\quad + \int_{x_1}^{x_2} \left(g\left(x, \frac{x}{\epsilon}, u^\epsilon\right) - g^0(x,u^\epsilon) \right) dx - \int_{x_1}^{x_2} \left(f\left(x, \frac{x}{\epsilon}\right) - f^0(x) \right) dx \Bigg]_{t=t_0} = 0. \end{aligned} \tag{3.12}$$

Since $Du^{\epsilon}(x_1,t_0) \geq Du^0(x_1,t_0), Du^{\epsilon}(x_2,t_0) \leq Du^0(x_2,t_0)$, and $u^{\epsilon}(x_i,t) = u^0(x_i,t)$ $(i = \overline{1,2})$, it follows that

$$\begin{aligned} \frac{d}{dt}\int_{x_1}^{x_2}\Big(\beta(u^{\epsilon}) - \beta\Big(u^0\Big)\Big)dx\bigg|_{t=t_0} &< -\int_{x_1}^{x_2}\Big(g^0(x,u^{\epsilon}) - g^0\Big(x,u^0\Big)\Big)dx \\ &\quad - \int_{x_1}^{x_2}\left(g\left(x,\frac{x}{\epsilon},u^{\epsilon}\right) - g^0(x,u^{\epsilon})\right)dx + \int_{x_1}^{x_2}\left(f\left(x,\frac{x}{\epsilon}\right) - f^0(x)\right)dx\bigg|_{t=t_0}. \end{aligned} \tag{3.13}$$

Note that

$$\begin{aligned} &-\int_{x_1}^{x_2}\left(g\left(x,\frac{x}{\epsilon},u^{\epsilon}\right) - g^0(x,u^{\epsilon})\right)dx\bigg|_{t=t_0} \\ &\quad = -\sum_{j=1}^{M}\int_{x_1}^{x_2}\left(b^j\left(x,\frac{x}{\epsilon}\right)g_j(u^{\epsilon}) - b^{0j}(x)g_j(u^{\epsilon})\right)dx\bigg|_{t=t_0} \\ &\quad = -\sum_{j=1}^{M}\int_{x_1}^{x_2}\tilde{b}^j\left(x,\frac{x}{\epsilon}\right)g_j(u^{\epsilon})dx\bigg|_{t=t_0}. \end{aligned} \tag{3.14}$$

Obviously,

$$\partial_z B^j\left(x,\frac{x}{\epsilon}\right) = \epsilon\frac{d}{dx}B^j\left(x,\frac{x}{\epsilon}\right) - \epsilon\partial_x B^j\left(x,\frac{x}{\epsilon}\right), \tag{3.15}$$

where, as we mentioned above, ∂_x indicate partial derivatives with respect to the first argument x of the function $B^j(x,z)$.

Consequently,

$$\begin{aligned} &-\int_{x_1}^{x_2}\left(g\left(x,\frac{x}{\epsilon},u^{\epsilon}\right) - g^0(x,u^{\epsilon})\right)dx\bigg|_{t=t_0} \\ &\quad = -\sum_{j=1}^{M}\int_{x_1}^{x_2}\left(\partial_z B^j\left(x,\frac{x}{\epsilon}\right)g_j(u^{\epsilon})\right)dx\bigg|_{t=t_0} \\ &\quad = -\epsilon\sum_{j=1}^{M}\int_{x_1}^{x_2}\frac{d}{dx}B^j\left(x,\frac{x}{\epsilon}\right)g_j(u^{\epsilon})dx + \epsilon\sum_{j=1}^{M}\int_{x_1}^{x_2}\partial_x B^j\left(x,\frac{x}{\epsilon}\right)g_j(u^{\epsilon})dx\bigg|_{t=t_0}. \end{aligned} \tag{3.16}$$

Therefore, in view of condition $(H3)$ and (3.7)

$$
\begin{aligned}
&-\int_{x_1}^{x_2}\left(g\left(x,\frac{x}{\epsilon},u^{\epsilon}\right)-g^0(x,u^{\epsilon})\right)dx\bigg|_{t=t_0} \\
&\quad\le \epsilon\sum_{j=1}^{M}\int_{x_1}^{x_2}\left|B^j\left(x,\frac{x}{\epsilon}\right)\right|\left|\partial_x g_j(u^{\epsilon})\right|dx+\ \epsilon\sum_{j=1}^{M}\left\|\partial_x B^j\left(\cdot,\frac{\cdot}{\epsilon}\right)\right\|_{L^1(x_1,x_2)}\left\|g_j(u^{\epsilon})\right\|_{L^{\infty}(x_1,x_2)}\bigg|_{t=t_0} \\
&\quad\le C\epsilon\sum_{j=1}^{M}\int_{x_1}^{x_2}\left|\partial_x g_j(u^{\epsilon})\right|dx+\epsilon\sum_{j=1}^{M}\left\|\partial_x B^j\left(\cdot,\frac{\cdot}{\epsilon}\right)\right\|_{L^1(x_1,x_2)}\left\|g_j(u^{\epsilon})\right\|_{L^{\infty}(x_1,x_2)}\bigg|_{t=t_0}.
\end{aligned}
\tag{3.17}
$$

Observe now that the approximate solution u_{η}^{ϵ} is bounded in the space $L^{\infty}(\tau,T;W_0^{1,p}(\Omega))$ uniformly with respect to ϵ, because in the proof of this Lemma 2.3 we use the same constant c_i and λ (condition $(H3)$) to estimate every $g_j(\omega)$ from (3.2), and from condition (3.5) and $(H5)$ (using known theorem on boundedness of the weakly convergence sequence in normed space) we can conclude that $|f(x,x/\epsilon)|\le C$. Thus, $\|u^{\epsilon}\|_{L^{\infty}(\tau,T;W_0^{1,p}(\Omega))}\le C$. Consequently, $\sum_{j=1}^{M}\int_{x_1}^{x_2}|\partial_x g_j(u^{\epsilon})|dx\le C$ and (3.17) becomes

$$
-\int_{x_1}^{x_2}\left(g\left(x,\frac{x}{\epsilon},u^{\epsilon}\right)-g^0(x,u^{\epsilon})\right)dx\bigg|_{t=t_0}\le\ C\epsilon+C\epsilon\sum_{j=1}^{M}\left\|\partial_x B^j\left(\cdot,\frac{\cdot}{\epsilon}\right)\right\|_{L^1(x_1,x_2)}\bigg|_{t=t_0}. \tag{3.18}
$$

Analogously, by (3.9), using the same arguments as in proof of (3.18), we deduce that

$$
\int_{x_1}^{x_2}\left(f\left(x,\frac{x}{\epsilon}\right)-f^0(x)\right)dx\le C\epsilon. \tag{3.19}
$$

Now, bearing in mind that the function $\zeta\ \rightarrow\ g^0(x,\zeta)+c\beta(\zeta)$ is increasing and combining (3.18) and (3.19), we rewrite (3.13) in the form

$$
\begin{aligned}
&\frac{d}{dt}\int_{x_1}^{x_2}\left(\beta(u^{\epsilon})-\beta\left(u^0\right)\right)dx\bigg|_{t=t_0} \\
&\quad< C\int_{x_1}^{x_2}\left|\beta(u^{\epsilon})-\beta\left(u^0\right)\right|dx\bigg|_{t=t_0}+C\epsilon+\ C\epsilon\sum_{j=1}^{M}\left\|\partial_x B^j\left(\cdot,\frac{\cdot}{\epsilon}\right)\right\|_{L^1(x_1,x_2)}\bigg|_{t=t_0}.
\end{aligned}
\tag{3.20}
$$

Further, note that the functions $\psi(t) = \int_{x_1}^{x_2}((\beta(u^\epsilon) - \beta(u^0))dx$ and $\widetilde{\psi}(t) = \int_{x_1}^{x_2} |\beta(u^\epsilon) - \beta(u^0)|dx$ are absolutely continuous $(\beta(u^\epsilon)_t \in L^2(\tau, T, L^2(\Omega))$. Besides, it is obvious that $\widetilde{\psi}(t) \geq \psi(t)$ and $\widetilde{\psi}(t_0) = \psi(t_0)$. Consequently,

$$\begin{aligned} \frac{d}{dt}\int_{x_1}^{x_2}\left|\beta(u^\epsilon) - \beta\left(u^0\right)\right|dx\bigg|_{t=t_0} &\leq \frac{d}{dt}\int_{x_1}^{x_2}\left(\beta(u^\epsilon) - \beta\left(u^0\right)\right)dx\bigg|_{t=t_0} \\ &\leq C\int_{x_1}^{x_2}\left|\beta(u^\epsilon) - \beta\left(u^0\right)\right|dx\bigg|_{t=t_0} + C\epsilon + C\epsilon\sum_{j=1}^{M}\left\|\partial_x B^j\left(\cdot, \frac{\cdot}{\epsilon}\right)\right\|_{L^1(x_1,x_2)}\Bigg|_{t=t_0}. \end{aligned} \tag{3.21}$$

Hence, by condition (3.8)

$$\frac{d}{dt}\int_{x_1}^{x_2}\left|\beta(u^\epsilon) - \beta\left(u^0\right)\right|dx\bigg|_{t=t_0} \leq C\int_{x_1}^{x_2}\left|\beta(u^\epsilon) - \beta\left(u^0\right)\right|dx\bigg|_{t=t_0} + C\epsilon. \tag{3.22}$$

The same estimation holds for an arbitrary interval on which $u^\epsilon(x,t) - u^0(u,x)$ does not change its sign. Summing up similar inequalities over subintervals, we get

$$\frac{d}{dt}\int_{a}^{b}\left|\beta(u^\epsilon) - \beta\left(u^0\right)\right|dx\bigg|_{t=t_0} < C\int_{a}^{b}\left|\beta(u^\epsilon) - \beta\left(u^0\right)\right|dx\bigg|_{t=t_0} + C\epsilon. \tag{3.23}$$

Consequently,

$$\int_{a}^{b}\left|\beta(u^\epsilon) - \beta\left(u^0\right)\right|dx\bigg|_{t=t''} \leq C\epsilon e^{\rho(t''-t')}\left(t'' > t' > \tau\right). \tag{3.24}$$

Thus, taking into account that $\beta(u^\epsilon) \in C(0,T;L^2(\Omega))$,

$$\int_{a}^{b}|\beta(u^\epsilon) - \beta(u^0)|dx\bigg|_{t} \leq C\epsilon e^{\rho t}. \tag{3.25}$$

Thereby, assertion follows. □

Remark 3.2. It is easy to see that the condition on the function $\zeta \rightarrow g(x,\zeta) + c\beta(\zeta)$ in Lemma 2.5, which we use in Theorem 3.1, can be changed to the $(H2)'$. In this case we can exclude condition $(H4)$.

Now, note that if $\sum_{j=1}^{M} b^{0j}(x) < \widetilde{c} < 0$ and condition $(H2)'$ is fulfilled then we derive, using simple reasoning, the following exponential attraction with exponential rate $\nu > 0$: $\mathrm{dist}_{L^1(\Omega)}(S_t^0 u_0; A_0) \leqslant ce^{-\nu t}$.

Indeed, we know that solution of the corresponding stationary problem belongs to the attractor, that is, it belongs to A_0. Denoting this solution by v, we will use the same arguments as in proof of Lemma 2.5.

Defining $u = S_t^0 u_0$, we have

$$\begin{aligned}
&\frac{d}{dt}\int_{x_1}^{x_2}\beta(u)-\beta(v)dx-\int_{x_1}^{x_2}D\left(|Du|^{p-2}Du-|Dv|^{p-2}Dv\right)dx-\int_{x_1}^{x_2}D(\varphi(u)Du-\varphi(v)Dv)dx\\
&\quad+\int_{x_1}^{x_2}\left(g^0(x,u)-g^0(x,v)\right)dx=0,
\end{aligned}\tag{3.26}$$

where x_1 and x_2 are such that $\beta(u)-\beta(v)\geq 0$ on the interval (x_1,x_2). Hence,

$$\begin{aligned}
&\frac{d}{dt}\int_{x_1}^{x_2}\beta(u)-\beta(v)dx-\int_{x_1}^{x_2}D\left(|Du|^{p-2}Du-|Dv|^{p-2}Dv\right)dx\\
&\quad-\int_{x_1}^{x_2}D(\varphi(u)Du-\varphi(v)Dv)dx+\sum_{j=1}^{M}b^{0j}(x)\int_{x_1}^{x_2}(g_j(u)-g_j(v))dx=0.
\end{aligned}\tag{3.27}$$

Bearing into mind that $g_j(w)$ is supposed to satisfy condition $(H3)$ $(\partial_\zeta g_j > -\lambda)$, we have

$$\frac{d}{dt}\int_{x_1}^{x_2}\beta(u)-\beta(v)dx\leq-\lambda\tilde{c}\int_{x_1}^{x_2}(u-v)dx.\tag{3.28}$$

Using $(H2)'$, we derive

$$\frac{d}{dt}\int_{x_1}^{x_2}\beta(u)-\beta(v)dx\leq-\lambda\tilde{c}\int_{x_1}^{x_2}(\beta(u)-\beta(v))dx.\tag{3.29}$$

Hence, from(2.42)

$$\frac{d}{dt}\int_{x_1}^{x_2}\left|\beta(u)-\beta(v)\right|dx\leq-\lambda\tilde{c}\int_{x_1}^{x_2}\left|\beta(u)-\beta(v)\right|dx.\tag{3.30}$$

Further, arguing similarly to the proof of Lemma 2.5, we arrive to

$$\frac{d}{dt}\int_{a}^{b}\left|\beta(u)-\beta(v)\right|dx<-\lambda\tilde{c}\int_{a}^{b}\left|\beta(u)-\beta(v)\right|dx,\tag{3.31}$$

or

$$\int_{a}^{b}\left|\beta(u)-\beta(v)\right|dx<c(u_0,v)e^{-\lambda\tilde{c}t}.\tag{3.32}$$

Consequently, by condition $(H2)'$

$$\int_{x_1}^{x_2}|u-v|dx\leq L_1\int_{x_1}^{x_2}\left|\beta(u)-\beta(v)\right|dx\leq L_1c(u_0,v)e^{-\lambda\tilde{c}t}=ce^{-\nu t}.\tag{3.33}$$

Thus, we obtain that $\mathrm{dist}_{L^1(\Omega)}(S_t^0 u_0; A_0) \leqslant ce^{-\nu t}$, where $\nu = \lambda\tilde{c}, L_1$ is the Lipschitz constant.

Hence, using Remarks 2.6 and 3.2, with the help of Lemma 4.1 of the study in [11], we obtain the estimate for the distance between the nonhomogenized A^{ϵ} and the homogenized A^0 attractors in terms of the parameter $\epsilon\, \mathrm{dist}_{L^1(\Omega)}(A^{\epsilon}, A^0) \le C\epsilon^{\gamma}$. So, the following theorem holds.

Theorem 3.3. *Let $g(x,z,w)$ and $f(x,z)$ satisfy conditions* (3.2)–(3.9), *and assumptions* $(H1)$–$(H3)$, $(H2)'$, *and* $(H5)$ *are fulfilled. Also suppose that $\sum_{j=1}^{M} b^{0j}(x) < \tilde{c} < 0$. Then the global attractors A^{ϵ} of the problem* (1.4), (1.8), (1.9) *satisfy an upper semicontinuity distance estimate of the form* $\mathrm{dist}_{L^1(\Omega)}(A^{\epsilon}, A^0) \le C\epsilon^{\gamma}$.

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