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A novel traveling wave solution for Ostrovsky equation using Exp-function method

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ABSTRACT

In this paper, we predict a new traveling wave solution of Ostrovsky equation by using He's Exp-function method. The method is straightforward and concise and its application shows an imminent prospective for the nonlinear problems in mathematical physics.

1. Introduction

Various physical phenomena in engineering and physics may be described by some nonlinear evolution equations (NLEEs). Nonlinear wave phenomena of dissipation, diffusion, reaction and convection are very important and can be represented with a variety of nonlinear wave equations. In order to get some information about the physical system the approximate-exact solution of these equations must be given, but the solution methods are quite few and have some limitations.

In recent years, quite a few methods for obtaining traveling and solitary wave solutions of nonlinear evolution equations (NLEEs) have been proposed. Among them, inverse scattering method [1–3], bilinear transformation [4], tanh–sech method [5,6], extended tanh method [7], hyperbolic tangent method [8] sine–cosine method [9,10], homogeneous balance method [11,12], pseudo-spectral method [13] and exponential rational function method [8] have been used to search nonlinear dispersive and dissipative problems.

This paper deals with the nonlinear evolution equation

$$(u_t + c_0 u_x + puu_x + qu_{xxx})_x = \gamma u \tag{1}$$

which was first presented by Ostrvsky [8,14] to describe nonlinear surface and internal waves in rotating ocean where c_0 is the velocity of dispersionless linear waves, p is the nonlinear coefficient, q is the Boussinesq dispersion and γ is the Coriolis dispersion coefficients. This equation is presently known as Ostrovsky equation [15] and combines the effect of small nonlinearity of hydrodynamics type, with weak dispersion which marks not only at small scales, but also at large scales.

When Coriolis dispersion is negligible, ($\gamma = 0$), Eq. (1) reduces to the KdV equation. But, as a limiting case of very long waves for which Boussinesq dispersion vanishes, ($q = 0$), Eq. (1) also makes sense and reads as

$$(u_t + c_0 u_x + puu_x)_x = \gamma u. \tag{2}$$

This equation was first considered by Ostrovsky [14]. Later, Vakhnenko and Parker [16,17] demonstrated that the reduced Ostrovsky equation (2) can be transformed to the new integrable equation as follows

$$uu_{xxt} - u_x u_{xt} + u^2 u_t = 0. \tag{3}$$

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Nonlinear wave phenomena can be better understood with the help of approximate-exact solutions when they exist for some particular values of the parameters involved in the equation. On the other hand, finding approximate-exact solutions of a nonlinear evolution equations (NLEEs) is a complicated task. Moreover, it is not always possible. There are many alternative approaches obtaining closed form solutions of nonlinear partial differential equations but almost all of them are so cumbersome to calculate or have some limitations. It is necessary to look for new techniques as simple as possible in order to find approximate-exact solutions of partial differential equations of physical significance.

Recently, He and Wu [18–20] proposed a novel method, so called Exp-function method, which is easy, concise and an effective method to implement to nonlinear evolution equations (NLEEs) arising in mathematical physics. The Exp-function method has been successfully applied to many kinds of NLEEs [21–35] and it has some merits as follows:

- The solution process of the method, by the help of symbolic computation is entirely simple [18,21,23].
- The Exp-function method leads to not only generalized solitary solutions but also periodic solutions [18,21,23].
- The Exp-function method may be employed in both the straightforward way and sub-equation way [24].

The method also has unifying frame in which other methods could benefit as complimentary view points. Because, some known solutions obtained by the most existing methods such as Adomian decomposition method, tanh–(coth) method, algebraic method, Jacobi elliptic function expansion method, F-expansion method, auxiliary equation method and others can be recovered as special cases [18,21]. Moreover, the Exp-function method for discrete NLEEs is more powerful than hyperbolic function method [21].

On the other hand, Classical Jacobi elliptic function expansion method, tanh–(coth) method and F-expansion method cannot be applied to NLEEs in which the odd- and even-order derivative terms coexist [25,24]. Ebaid [25] showed that Burgers equation in which the odd- and even-order derivative terms coexist can be solved by the Exp-function method.

The aim of this paper is to extend the Exp-function method introduced by He [18–20] to find new approximate-exact solution for Ostrovsky equation given in Eq. (3).

2. Exp-function method

Recently, Yusufoğlu and Bekir [8], by applying hyperbolic tangent method and exponential rational function method, demonstrated that Ostrovsky equation admits traveling wave solution. We seek traveling wave solution of Ostrovsky equation (3) by assuming the solution in the following frame:

$$u = U(\xi), \quad \xi = \alpha(x - \beta t) \tag{4}$$

where α, β are constants.

We substitute Eq. (4) into Eq. (3) to obtain nonlinear ordinary differential equation

$$\alpha^2 UU''' - \alpha^2 U'U'' + U^2U' = 0. \tag{5}$$

In view of the Exp-function method, we assume that the solution of Eq. (5) can be expressed in the form

$$U(\xi) = \frac{\sum_{n=-c}^{d} a_n \exp(n\xi)}{\sum_{m=-p}^{q} b_m \exp(m\xi)}, \tag{6}$$

where c, d, p and q are positive integers which are unknown to be determined later, a_n and b_m are unknown constants.

Eq. (6) can be re-written in an alternative form as follows

$$U(\xi) = \frac{a_c \exp(c\xi) + \cdots + a_{-d} \exp(-d\xi)}{b_p \exp(p\xi) + \cdots + b_{-q} \exp(-q\xi)}. \tag{7}$$

In order to determine values c and p, we balance the terms UU''' and $U'U''$ with U^2U' in Eq. (5),we have

$$UU''' = \frac{c_1 \exp[(2c + 4p)\xi] + \cdots}{c_2 \exp[6p\xi] + \cdots}, \tag{8}$$

$$U^2U' = \frac{c_3 \exp[(3c + 3p)\xi] + \cdots}{c_4 \exp[6p\xi] + \cdots}, \tag{9}$$

where c_i are determined coefficients. By balancing highest order of Exp-function in Eqs. (8) and (9), we have

$$3c + 3p = 2c + 4p, \tag{10}$$

which leads to the limit

$$p = c. \tag{11}$$

Proceeding the same manner as illustrated above, we can determine values of d and q. Balancing the terms

$$U'U'' = \frac{\cdots + d_1 \exp[-(2d+4q)\xi]}{\cdots + d_2 \exp[-6q\xi]}, \tag{12}$$

$$U^2U' = \frac{\cdots + d_3 \exp[-(3d+3q)\xi]}{\cdots + d_4 \exp[-6q\xi]}, \tag{13}$$

where d_i are determined coefficients for simplicity, we have

$$-[2d+4q] = -[3d+3q],$$

which leads to the limit

$$q = d. \tag{14}$$

We set, for convenience $p = c = 1$ and $d = q = 1$, then Eq. (7) leads to

$$U(\xi) = \frac{a_1 \exp(\xi) + a_0 + a_{-1}\exp(-\xi)}{b_1 \exp(\xi) + b_0 + b_{-1}\exp(-\xi)}. \tag{15}$$

Substituting Eq. (15) into Eq. (5), and using the *Maple*, we obtain

$$\begin{aligned} &\frac{1}{A}[k_4\exp(4\xi) + k_3\exp(3\xi) + k_2\exp(2\xi) + k_1\exp(\xi) + k_0 + k_{-1}\exp(-\xi) \\ &\quad + k_{-2}\exp(-2\xi) + k_{-3}\exp(-3\xi) + k_{-4}\exp(-4\xi)] = 0 \end{aligned} \tag{16}$$

where

$$\begin{aligned}
A &= (b_1\exp(\xi) + b_0 + b_{-1}\exp(-\xi))^5,\\
k_0 &= -35\alpha^2 a_1 a_0 b_1 b_{-1}^2 + 35\alpha^2 a_{-1} a_0 b_1^2 b_{-1} - 5a_0^2 a_{-1} b_1 b_0 - 5a_{-1}^2 a_1 b_1 b_0 - 5\alpha^2 a_1^2 b_{-1}^2 b_0 + 5\alpha^2 a_{-1}^2 b_1^2 b_0\\
&\quad + 5a_1^2 a_{-1} b_{-1} b_0 + 5a_0^2 a_1 b_{-1} b_0 + 5a_1^2 a_0 b_{-1}^2 - 5a_{-1}^2 a_0 b_1^2 - 5\alpha^2 a_0 a_{-1} b_1 b_0^2 + 5\alpha^2 a_0 a_1 b_{-1} b_0^2 k^2 a_1 b_0^2\\
&\quad - \alpha a_1 b_0^2 + 3\gamma a_1 a_0^2 + 4k^2 a_{-1} b_1^2 - \alpha a_{-1} b_1^2 + 3\gamma a_1^2 a_{-1} - k^2 a_0 b_1 b_0 - 2\alpha a_0 b_1 b_0 - 4k^2 a_1 b_1 b_{-1} - 2\alpha a_1 b_1 b_{-1}\\
k_1 &= 2a_1^3 b_{-1}^2 - 4a_0^2 a_{-1} b_1^2 - a_0^3 b_1 b_0 - 4a_{-1}^2 a_1 b_1^2 + a_1^2 a_{-1} b_0^2 + a_0^2 a_1 b_0^2 - 24\alpha^2 a_1^2 b_1 b_{-1}^2 + 24\alpha^2 a_1 a_{-1} b_1^2 b_{-1}\\
&\quad + 16\alpha^2 a_0^2 b_1^2 b_{-1} + 2a_1^2 a_{-1} b_1 b_{-1} + 2a_0^2 a_1 b_1 b_{-1} + 7a_1^2 a_0 b_{-1} b_0 + \alpha^2 a_0 a_{-1} b_1^2 b_0 - 6\alpha^2 a_0 a_1 b_1 b_{-1} b_0\\
&\quad + \alpha^2 a_0 a_1 b_0^3 - \alpha^2 a_1^2 b_{-1} b_0^2 - \alpha^2 a_0^2 b_1 b_0^2 - 10\alpha^2 a_1 a_{-1} b_1 b_0^2 - 6a_1 a_0 a_{-1} b_1 b_0\\
k_2 &= 3a_1^3 b_{-1} b_0 + 2a_1^2 a_0 b_0^2 - a_0^3 b_1^2 - a_1^2 a_{-1} b_1 b_0 - a_0^2 a_1 b_1 b_0 + 4a_1^2 a_0 b_1 b_{-1} + 25\alpha^2 a_1 a_0 b_1^2 b_{-1} - 3\alpha^2 a_{-1} a_0 b_1^3\\
&\quad - 3\alpha^2 a_1 a_0 b_1 b_0^2 - 12\alpha^2 a_1^2 b_{-1} b_1 b_0 + 3\alpha^2 a_0^2 b_1^2 b_0 - 10\alpha^2 a_1 a_{-1} b_1^2 b_0 - 6a_1 a_0 a_{-1} b_1^2\\
k_3 &= 2a_1^3 b_1 b_{-1} - 2a_1^2 a_{-1} b_1^2 - 2a_0^2 a_1 b_1^2 + a_1^2 a_0 b_1 b_0 + 8\alpha^2 a_1^2 b_1^2 b_{-1} + a_1^3 b_0^2 - 8\alpha^2 a_1 a_{-1} b_1^3\\
&\quad + 3\alpha^2 a_1 a_0 b_1^2 b_0 - 3\alpha^2 a_1^2 b_1 b_0^2\\
k_4 &= a_1^3 b_1 b_0 - a_1^2 a_0 b_1^2 - \alpha^2 a_1 a_0 b_1^3 + \alpha^2 a_1^2 b_1^2 b_0\\
k_{-1} &= -a_0^2 a_{-1} b_0^2 + a_0^3 b_{-1} b_0 - a_{-1}^2 a_1 b_0^2 + 4a_1^2 a_{-1} b_{-1}^2 + 4a_0^2 a_1 b_{-1}^2 - 2a_{-1}^3 b_1^2 - 2a_0^2 a_{-1} b_1 b_{-1} - 2a_{-1}^2 a_1 b_1 b_{-1}\\
&\quad + 24\alpha^2 a_{-1}^2 b_1^2 b_{-1} - 7a_{-1}^2 a_0 b_1 b_0 - 24\alpha^2 a_1 a_{-1} b_1 b_{-1}^2 - 16\alpha^2 a_0^2 b_1 b_{-1}^2 + 6\alpha^2 a_0 a_{-1} b_1 b_{-1} b_0 - \alpha^2 a_0 a_{-1} b_0^3\\
&\quad - \alpha^2 a_0 a_1 b_{-1}^2 b_0 + \alpha^2 a_{-1}^2 b_1 b_0^2 + \alpha^2 a_0^2 b_{-1} b_0^2 + 10\alpha^2 a_1 a_{-1} b_{-1} b_0^2 + 6a_1 a_0 a_{-1} b_{-1} b_0\\
k_{-2} &= a_0^3 b_{-1}^2 - 3a_{-1}^3 b_1 b_0 - 2a_{-1}^2 a_0 b_0^2 + a_0^2 a_{-1} b_{-1} b_0 + a_{-1}^2 a_1 b_{-1} b_0 - 4a_{-1}^2 a_0 b_1 b_{-1} + 3\alpha^2 a_1 a_0 b_{-1}^3\\
&\quad - 25\alpha^2 a_{-1} a_0 b_1 b_{-1}^2 + 3\alpha^2 a_0 a_{-1} b_{-1} b_0^2 + 12\alpha^2 a_{-1}^2 b_1 b_{-1} b_0 - 3\alpha^2 a_0^2 b_{-1}^2 b_0 + 10\alpha^2 a_1 a_{-1} b_{-1}^2 b_0 + 6a_1 a_0 a_{-1} b_{-1}^2\\
k_{-3} &= -a_{-1}^3 b_0^2 + 2a_0^2 a_{-1} b_{-1}^2 + 2a_{-1}^2 a_1 b_{-1}^2 - 2a_{-1}^3 b_1 b_{-1} - a_{-1}^2 a_0 b_{-1} b_0 + 8\alpha^2 a_1 a_{-1} b_{-1}^3 - 8\alpha^2 a_{-1}^2 b_1 b_{-1}^2\\
&\quad - 3\alpha^2 a_0 a_{-1} b_{-1}^2 b_0 + 3\alpha^2 a_{-1}^2 b_{-1} b_0^2\\
k_{-4} &= -a_{-1}^3 b_{-1} b_0 + a_{-1}^2 a_0 b_{-1}^2 + \alpha^2 a_{-1} a_0 b_{-1}^3 - \alpha^2 a_{-1}^2 b_{-1}^2 b_0
\end{aligned}$$

and equating to zero the coefficients of all powers of $\exp(n\xi)$ yield a set of algebraic equations

$$\begin{cases} k_0 = 0, \quad k_1 = 0, \quad k_2 = 0, \quad k_3 = 0, \quad k_4 = 0 \\ k_{-1} = 0, \quad k_{-2} = 0, \quad k_{-3} = 0, \quad k_{-4} = 0. \end{cases} \tag{17}$$

Solving the system of algebraic equations, Eq. (15), with the aid of *Maple* we obtain:

$$\text{(i)} \quad b_{-1} = \frac{b_0^2}{4b_1}, \qquad a_{-1} = 0, \qquad a_1 = 0, \qquad a_0 = 3\alpha^2 b_0, \tag{18}$$

and

$$\text{(ii)}\quad b_{-1}=\frac{b_0^2}{4b_1},\qquad a_1=-\alpha^2 b_1,\qquad a_0=2\alpha^2 b_0,\qquad a_{-1}=\frac{-\alpha^2 b_0^2}{4b_1}. \tag{19}$$

So, substituting Eq. (18) into (15) we obtain the following solution:

$$U(\xi)=\frac{3\alpha^2 b_0}{b_1\exp(\xi)+b_0+\frac{b_0^2}{4b_1}\exp(-\xi)} \tag{20}$$

where $\exp(\xi)=\cosh\xi+\sinh\xi$, b_0, b_1 are free parameters. If we set $b_0=2b_1$ or $b_0=1$, $b_1=1/2$, Eq. (20) becomes

$$U=\frac{3\alpha^2}{\cosh\xi+1}=\frac{3\alpha^2}{\cosh\alpha(x-\beta t)+1} \tag{21}$$

where $\xi=\alpha(x-\beta t)$.

Substituting Eq. (19) into (15) we obtain the following solution:

$$U(\xi)=\frac{-\alpha^2 b_1\exp(\xi)+2\alpha^2 b_0-\frac{\alpha^2 b_0^2}{4b_1}\exp(-\xi)}{b_1\exp(\xi)+b_0+\frac{b_0^2}{4b_1}\exp(-\xi)} \tag{22}$$

where $\exp(\xi)=\cosh\xi+\sinh\xi$, b_0, b_1 are free parameters. If we set $b_0=2b_1$ or $b_0=1$, $b_1=1/2$

$$U(\xi)=-\alpha^2+\frac{3\alpha^2 b_0}{b_1\exp(\xi)+b_0+\frac{b_0^2}{4b_1}\exp(-\xi)} \tag{23}$$

or

$$U(\xi)=-\alpha^2+\frac{3\alpha^2}{\cosh\xi+1} \tag{24}$$

The solutions given in (23) and (24) reduce to

$$U=-\alpha^2+\frac{1}{(\cosh\alpha(x-\beta t)+1)} \tag{25}$$

where $\xi=\alpha(x-\beta t)$. It can be easily seen that the solutions obtained in Eqs. (21) and (25) differ only by a constant.

3. Conclusion

In this paper, the Exp-function method has been tested by applying it successfully to the Ostrovsky equation. The Exp-function method leads to traveling wave solutions with some free parameters connecting the known solutions in the literature. The free parameters may imply some physical meaningful results for the problem considered. The free parameters, of course, might be related to initial conditions as well.

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