Journal of Algebra **212,** 743–752 (1999)
Article ID jabr.1998.7636, available online at http://www.idealibrary.com on IDEAL®

A Characterization of Prime Submodules

Yücel Tıraş and Abdullah Harmancı

Department of Mathematics, Hacettepe University, Ankara, Turkey
E-mail: ytiras@eti.cc.hun.edu.tr

and

P. F. Smith

Department of Mathematics, University of Glasgow G12 8QW, Scotland, United Kingdom

Communicated by Kent R. Fuller

Received January 5, 1998

INTRODUCTION

Let R be a commutative domain and let M be an R-module. It is proved that to every prime submodule of M there corresponds a prime ideal of R and a set of linear equations of a certain type, and conversely. In particular, in case M is a finitely generated R-module generated by n elements, for some positive integer n, then the prime submodules of M are given by prime ideals of R and certain finite systems of equations containing at most n equations.

PRELIMINARIES AND RESULTS

Throughout this article all rings are commutative with identity and all modules are unital. Let R be a ring and let M be an R-module. For any submodule N of M let $(N: M) = \{r \in R: rM \subseteq N\}$. Clearly $(N: M)$ is an ideal of R. A submodule N of M is called prime if $N \neq M$ and given, $r \in R$, $m \in M$, then $rm \in N$ implies $m \in N$ or $r \in (N: M)$. (For more information about prime submodules, see [1–4]). The following lemma is well known (see, for example, [4]).

0021-8693/99 $30.00

LEMMA 1. *Let M be an R-module. Then a submodule N of M is prime if and only if $P = (N: M)$ is a prime ideal of R and the (R/P)-module M/N is torsionfree.*

Let M be an R-module which is generated by elements $m_i (i \in I)$, where the index set I need not be finite. Then every element of M can be written in the form $\sum_{i \in I} r_i m_i$ where $r_i \in R (i \in I)$ and $r_i \neq 0$ for at most a finite number of elements $i \in I$. It will be convenient to write the elements of M in this form.

Let I be a nonempty index set. By an $I \times I$ column-finite matrix (a_{ij}) over a ring R we mean a collection of elements $a_{ij} \in R$ $(i, j \in I)$ such that for each $j \in I$ the set $\{i \in I: a_{ij} \neq 0\}$ is empty or finite.

LEMMA 2. *Let R be a domain with field of fractions K and let M be a free R-module with basis $\{m_i: i \in I\}$. Let N be a proper submodule of M such that M/N is a torsionfree R-module. Then there exists a nonzero $I \times I$ column-finite matrix (a_{ij}) over K such that*

$$N = \left\{ \sum_{i \in I} r_i m_i \in M: r_i \in R, (i \in I) \text{ and } \sum_{j \in I} a_{ij} r_j = 0, (i \in I) \right\}.$$

Proof. Without loss of generality we can consider M as an R-module of the K-vector space V with basis $\{m_i: i \in I\}$. Now KN is a subspace of V and $N = KN \cap M$ because M/N is torsionfree. Thus KN is a proper subspace of V and hence $V = KN \oplus W$ for some nonzero K-submodule W of V.

Let $\pi: V \to W$ denote the canonical projection with kernel KN. For each $j \in I$, $\pi(m_j) = \sum_{i \in I} a_{ij} m_i$ for some $a_{ij} \in K$ $(i \in I)$ such that $\{i \in I: a_{ij} \neq 0\}$ is empty or finite. Clearly (a_{ij}) is an $I \times I$ column-finite matrix over K and is nonzero because W, and hence π, is nonzero.

Let $m \in M$. Then $m = \sum_{j \in I} s_j m_j$ for some $s_j \in R$ where $s_j \neq 0$ for at most finite number of elements $j \in I$. It follows that

$$\pi(m) = \sum_{j \in I} s_j \pi(m_j) = \sum_{j \in I} s_j \left(\sum_{i \in I} a_{ij} m_i \right) = \sum_{i \in I} \left(\sum_{j \in I} a_{ij} s_j \right) m_i.$$

Now

$$\begin{aligned} N = M \cap KN &= \{m \in M: \pi(m) = 0\} \\ &= \left\{ \sum_{j \in L} s_j m_j \in M: \sum_{j \in I} a_{ij} s_j = 0 \ (i \in I) \right\}. \end{aligned}$$

∎

COROLLARY 3. *Let R be a domain and let M be a free R-module with basis $\{m_1, \dots, m_n\}$, for some positive integer n. Let N be a proper submodule of M such that M/N is a torsionfree $R-$ module. Then there exist elements $b_{ij} \in R$ for $1 \leq i, j \leq n$, not all zero, such that*

$$N = \left\{ r_1 m_1 + \cdots + r_n m_n : r_i \in R, (1 \leq i \leq n) \text{ and } \sum_{j=1}^{n} b_{ij} r_j = 0, (1 \leq i \leq n) \right\}.$$

Proof. In Lemma 2, $I = \{1, \dots, n\}$. For each $1 \leq i, j \leq n$, there exist $b_{ij} \in R$, $0 \neq c_{ij} \in R$ such that $a_{ij} = b_{ij}/c_{ij}$. Without loss of generality, there exists $0 \neq c \in R$ such that $c_{ij} = c$ $(1 \leq i, j \leq n)$. The result now follows by Lemma 2. ∎

Note that, in general, in Lemma 2 we cannot assume that $a_{ij} \in R$ for all $i, j \in I$, as the following example shows.

EXAMPLE 4. Let $\mathbb{Z}$ denote the ring of integers and let $M = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \cdots$ denote the free $\mathbb{Z}$-module of countably infinite rank. Let

$$N = \left\{ (r_1, r_2, r_3, \dots) \in M : \tfrac{1}{2} r_1 + \tfrac{1}{4} r_2 + \tfrac{1}{8} r_3 + \cdots = 0 \right\}.$$

Then N is a proper submodule of M and M/N is a torsionfree $\mathbb{Z}$-module. However there do not exist elements $a_i \in \mathbb{Z}$ $(i \geq 1)$, not all zero, such that

$$N \subseteq \left\{ (r_1, r_2, r_3, \dots) : \sum_{i \geq 1} a_i r_i = 0 \right\}.$$

Proof. It is easy to check that N is a proper submodule of M and that M/N is a torsionfree $\mathbb{Z}$ module. Suppose that there exist elements $a_i \in \mathbb{Z}$ $(i \geq 1)$, not all zero, such that $N \subseteq \{(r_1, r_2, r_3, \dots) : \sum_{i \geq 1} a_i r_i = 0\}$. There exists a positive integer k such that $a_k \neq 0$. Let t be any positive integer with $t > k$. Then $x = (0, 0, \dots, 0, -1, 0, 0, \dots, 0, 2^{t-k}, 0, 0, \dots)$ belongs to N, where -1 is the kth component and 2^{t-k} is the th component. Then $a_k(-1) + a_t 2^{t-k} = 0$, i.e., $a_k = 2^{t-k} a_t$. Thus $a_k \in \bigcap_{n=1}^{\infty} \mathbb{Z} 2^n = 0$, a contradiction. ∎

Let R be a domain with field of fractions K. Let M be an R-module with ordered generating set $G = \{m_i : i \in I\}$, i.e., $M = \sum_{i \in I} R m_i$, where I is some ordered index set. Let $A = (a_{ij})$ be an $I \times I$ column-finite matrix over K. Then we say that A is *G-compatible* if whenever $r_i \in R$ $(i \in I)$ with $r_i \neq 0$ for at most a finite number of elements $i \in I$ and $\sum_{i \in I} r_i m_i = 0$ then $\sum_{j \in I} a_{ij} r_j = 0$ $(i \in I)$. We illustrate this concept in the following proposition.

PROPOSITION 5. *Let A be a G-compatible $\mathbb{N} \times \mathbb{N}$ column-finite matrix over $\mathbb{Q}$ for the $\mathbb{Z}$-module $\mathbb{Q}$ with ordered generating set $G = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\}$ where $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{Q}$ denote the natural numbers, integers and rational numbers, respectively. Then*

$$A = \begin{bmatrix} q_1 & \dfrac{q_1}{2} & \dfrac{q_1}{3} & \cdots \\ \vdots & \vdots & \vdots & \\ q_n & \dfrac{q_n}{2} & \dfrac{q_n}{3} & \cdots \\ \vdots & \vdots & \vdots & \end{bmatrix},$$

for some positive integer n and nonzero $q_i \in \mathbb{Q}$ $(1 \le i \le n)$.

Proof. Suppose that $A = (a_{ij})$ where $i, j \in \mathbb{N}$. Let $m \in \mathbb{N} \setminus \{1\}$. Then $1 - m(1/m) = 0$ so that

$$a_{i1}1 + a_{im}(-m) = 0, \qquad (i \in I).$$

Thus $a_{im} = a_{i1}/m$ for all $i, m \in \mathbb{N}$. The result follows. ∎

LEMMA 6. *Let R be a domain with field of fraction K and let M be an R-module with ordered generating set $G = \{m_i : i \in I\}$. Then N is a proper submodule of M such that M/N is a torsionfree R module if and only if there exists a nonzero G-compatible $I \times I$ column-finite matrix (a_{ij}) over K such that*

$$N = \left\{ \sum_{i \in I} r_i m_i \in M : r_i \in R, (i \in I) \text{ and } \sum_{j \in I} a_{ij} r_j = 0, (i \in I) \right\}.$$

Proof. Suppose that (a_{ij}) is a nonzero G-compatible $I \times I$ column-finite matrix over K and $N = \{\sum_{i \in I} r_i m_i \in M : r_i \in R \ (i \in I)$ and $\sum_{j \in I} a_{ij} r_j = 0$ $(i \in I)\}$. Note that if $m \in M$ such that $m = \sum_{i \in I} r_i m_i$ and $m = \sum_{i \in I} s_i m_i$ where $r_i, \ s_i \in R$ $(i \in I)$ and neither of the set $\{i \in I : r_i \neq 0\}$ and $\{i \in I : s_i \neq 0\}$ is infinite then $\sum_{i \in I}(r_i - s_i) m_i = 0$ so that $\sum_{j \in J} a_{ij}(r_j - s_j) = 0$ $(i \in I)$, i.e., $\sum_{j \in J} a_{ij} r_j = 0$ $(i \in I)$ $\Leftrightarrow$ $\sum_{j \in J} a_{ij} s_j = 0$ $(i \in I)$. Thus N is well defined and it is easy to check that N is a submodule of M. There exist $i', j' \in I$ such that $a_{i'j'} \neq 0$. Then $m_{j'} \notin N$. Thus N is a proper submodule of M. It is clear that the module M/N is torsionfree.

Conversely, suppose that N is a proper submodule of M and M/N is a torsionfree R-module. There exist a free R-module F with basis $\{f_i : i \in I\}$ and an epimorphism $\varphi : F \to M$ such that $\varphi(f_i) = m_i$ $(i \in I)$. Let $H = \varphi^{-1}(N)$. It can easily be checked that H is a proper submodule of F and F/H is a torsionfree $R-$ module. By Lemma 2, there exists a nonzero

$I \times I$ column-finite matrix (a_{ij}) over K such that

$$H = \left\{ \sum_{i \in I} r_i f_i \in F \colon r_i \in R,\ (i \in I) \text{ and } \sum_{j \in I} a_{ij} r_j = 0,\ (i \in I) \right\}.$$

Let $s_i \in R\ (i \in I)$ such that $s_i \neq 0$ for at most a finite number of elements $i \in I$ and $\sum_{i \in I} s_i m_i = 0$. Then $\sum_{i \in I} s_i f_i \in \operatorname{Ker} \varphi \leq H$ so that $\sum_{j \in J} a_{ij} s_j = 0$, $(i \in I)$. Thus the matrix (a_{ij}) is G-compatible. Finally,

$$N = \varphi(H) = \left\{ \sum_{i \in I} r_i m_i \in M \colon r_i \in R,\ (i \in I) \text{ and } \sum_{j \in I} a_{ij} r_j = 0,\ (i \in I) \right\}.$$

∎

To illustrate Lemma 6, consider the $\mathbb{Z}$-module $\mathbb{Q}$ with ordered generating set $G = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\}$. By Proposition 5 and Lemma 6, N is a proper submodule of $\mathbb{Q}$ and $\mathbb{Q}/N$ is torsionfree if and only if

$$N = \left\{ \sum_{n \in \mathbb{N}} \frac{r_n}{n} \colon \sum_{n \in \mathbb{N}} \frac{r_n}{n} = 0 \right\}, \quad \text{i.e., } N = 0.$$

There is an analogue of Lemma 6 in case I is finite, say $I = \{1, \dots, n\}$, for some $n \in \mathbb{N}$. In this case the elements a_{ij} can be replaced by elements $b_{ij} \in R$, $(1 \leq i, j \leq n)$ (compare Corollary 3).

Let $\mathbb{R}$ be a domain with field of fractions K and let P be a prime ideal of R. Let R_P denote the localization of R at P. Then R_P is the subring of K consisting of all elements r/c where $r \in R$, $c \in R \setminus P$. Let M be an R-module with ordered generating set $G = \{m_i \colon i \in I\}$. Let $A = (a_{ij})$ be an $I \times I$ column-finite matrix over K. Then we say that A is (G, P)-compatible if whenever $r_i \in R$, $(i \in I)$ with $r_i \neq 0$ for at most a finite number of elements $i \in I$ and $\sum_{i \in I} r_i m_i \in PM$ then $\sum_{j \in I} a_{ij} r_j \in R_P P$, $(i \in I)$. Note that A is $(G, 0)$-compatible if and only if A is G-compatible.

EXAMPLE 7. Let M denote the $\mathbb{Z}$-module $(\mathbb{Z}/\mathbb{Z}2 \oplus \mathbb{Z}/\mathbb{Z}3)$ with ordered generating set $G = \{(1 + \mathbb{Z}2, 0 + \mathbb{Z}3), (0 + \mathbb{Z}2, 1 + \mathbb{Z}3)\}$.

(i) The zero 2×2 matrix is the only $(G, 0)$-compatible matrix.

(ii) For any prime $p \neq 2, 3$, a 2×2 matrix A is (G, P)-compatible if and only if each entry of A belongs to $\mathbb{Z}_p P$ when $P = \mathbb{Z}_p$.

(iii) $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ is a $(G, \mathbb{Z}2)$-compatible matrix.

(iv) $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ is a $(G, \mathbb{Z}3)$-compatible matrix.

Proof. (i) Let $m_1 = (1 + \mathbb{Z}2, 0 + \mathbb{Z}3)$, $m_2 = (0 + \mathbb{Z}2, 1 + \mathbb{Z}3)$. Let $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ be a $(G, 0)$-compatible matrix. Then $2m_1 = 0$ gives

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

so that $a = 0$, $c = 0$, and $3m_2 = 0$ gives

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

so that $b = 0$, $d = 0$.

(ii) Now let p be a prime integer, $p \neq 2, 3$ and set $P = \mathbb{Z}p$. Then $PM = M$. Let $A = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ where a, b, c, and $d \in \mathbb{Q}$. Clearly if $a, b, c, d \in \mathbb{Z}_p P$ then A is (G, P)-compatible. Conversely, suppose that A is (G, P)-compatible. Then $m_1 \in M = PM$ gives $a1 + b0 \in \mathbb{Z}_p P$, $c1 + d0 \in \mathbb{Z}_p P$, i.e., $a, c \in \mathbb{Z}_p P$. Similarly $m_2 \in M = PM$ gives $b, d \in \mathbb{Z}_p P$.

(iii) Note that $2M = (0 \oplus \mathbb{Z}/\mathbb{Z}3)$. Let $r, s \in \mathbb{Z}$ such that $rm_1 + sm_2 \in 2M$. Then $rm_1 = 0$ so that r is even. Then

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} r \\ 0 \end{bmatrix} = \begin{bmatrix} r \\ 0 \end{bmatrix},$$

where $r \in \mathbb{Z}2 \subseteq \mathbb{Z}_2 2$. Thus $\left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right]$ is a $(G, \mathbb{Z}2)$-compatible matrix.

(iv) Similar to (iii). ∎

THEOREM 8. *Let R be a domain and let M be an R-module with ordered generating set $G = \{m_i : i \in I)$. Then N is a prime submodule of M if and only if there exist a prime ideal P of R and a (G, P)-compatible $I \times I$ column-finite matrix (a_{ij}) over the local ring R_P such that $a_{ij} \notin R_P P$ for some $i, j \in I$ and*

$$N = \left\{ \sum_{i \in I} r_i m_i \in M : r_i \in R, (i \in I) \text{ and } \sum_{j \in I} a_{ij} r_j \in R_P P, (i \in I) \right\}.$$

In this case $P = (N: M)$.

Proof. Suppose first that there exists a prime ideal P of R and a (G, P)-compatible $I \times I$ column-finite matrix (a_{ij}) over R_P such that N has the stated form. By hypothesis, $PM \subseteq N$ and N is a submodule of M. There exist $i', j' \in I$ such that $a_{i'j'} \notin R_P P$ and then $m_{j'} \notin N$. Thus N is a proper submodule of M. Let $m \in M$, $c \in R \setminus P$ such that $cm \in N$. There exist elements $r_i \in R$ $(i \in I)$ such that $r_i \neq 0$ for at most a finite number of elements $i \in I$ and $m = \sum_{i \in I} r_i m_i$. Then $cm \in N$ implies that

$\sum_{j\in I} a_{ij}(cr_j) \in R_P P$ $(i \in I)$ and hence $c(\sum_{j\in I} a_{ij} r_j) \in R_P P$ $(i \in I)$ and $(\sum_{j\in I} a_{ij} r_j) \in R_P P$ $(i \in I)$. Thus $m \in N$. It follows that the (R/P)-module M/N is torsionfree. By Lemma 1, N is a prime submodule of M. Clearly $P = (N: M)$.

Conversely, suppose that N is a prime submodule of the $R -$ module M. By Lemma 1, $P = (N: M)$ is a prime ideal of R and M/N is a torsionfree (R/P)-module. Now $\overline{M} = M/PM$ has ordered generating set $\overline{G} = \{\overline{m}_i: i \in I\}$ where $\overline{m}_i = m_i + PM$. Let K denote the field of fractions of the domain R/P. By Lemma 6, there exists a nonzero $\overline{G}$-compatible $I \times I$ column-finite matrix (b_{ij}) over K such that

$$\frac{N}{PM} = \left\{ \sum_{i\in I} (r_i + P)\overline{m}_i \in \overline{M}: r_i \in R\ (i \in I) \text{ and } \sum_{j\in I} b_{ij}(r_i + P) = 0, (i \in I) \right\}.$$

Let $x \in N$. Then $x + PM = \sum_{i\in I}(r_i + P)\overline{m}_i$ where $r_i \in R$ $(i \in I)$, there are at most a finite number of elements $i \in I$ such that $r_i \notin P$ and $\sum_{j\in I} b_{ij}(r_j + P) = 0$ $(i \in I)$. Let $J = \{i \in I: r_i \notin P\}$. Then $x + PM = \sum_{i\in J}(r_i + P)\overline{m}_i = (\sum_{i\in J} r_i m_i) + PM$ so that there exist a finite subset J' of I and elements $p_i \in P$ $(i \in J')$ such that

$$x = \sum_{i\in J} r_i m_i + \sum_{i\in J'} p_i m_i.$$

Let $N' = \{\sum_{i\in I} s_i m_i \in M$: $s_i \in R$ $(i \in I)$ and $\sum_{j\in I} b_{ij}(s_j + P) = 0$ $(i \in I)\}$. We have shown that $x \in N'$ and hence $N \subseteq N'$. But it is clear that $N'/PM \subseteq N/PM$ and hence $N' \subseteq N$. Thus $N' = N$.

For each $i, j \in I$, $b_{ij} = (c_{ij} + P)^{-1}(f_{ij} + P)$ for some $f_{ij} \in R$, $c_{ij} \in R \setminus P$. For each $i, j \in I$ such that $f_{ij} \in P$ we set $a_{ij} = 0$. Note that $f_{ij} \notin P$ for some $i, j \in I$. Let i be any element of I such that $f_{ij'} \notin P$ for some $j' \in I$. Consider the equation $\sum_{j\in I} b_{ij}(r_j + P) = 0$ (in K) where $r_j \in R$ $(j \in I)$ and $r_j \neq 0$ for at most a finite number of elements $j \in I$. Let $J'' = \{j \in J: r_j \neq 0\}$. Then J'' is finite. Let $c = \prod_{j\in J''} c_{ij} \in R \setminus P$. Then $\sum_{j\in J} b_{ij}(r_j + P) = 0$ gives $\sum_{j\in J}(c_{ij} + P)^{-1}(f_{ij} + P)(r_j + P) = 0$ and hence, multiplying through by $c + P$, we have

$$\sum_{j\in J}\left(\prod_{k\in J\setminus\{j\}}(c_{ik} + P)\right)(f_{ij} + P)(r_j + P) = 0,$$

so that

$$\sum_{j\in J}\left(\prod_{k\in J\setminus\{j\}} c_{ik}\right) f_{ij} r_j \in P.$$

Now multiplying through by c^{-1} we have

$$\sum_{j \in J} c_{ij}^{-1} f_{ij} r_j \in R_P P,$$

and hence

$$\sum_{j \in I} c_{ij}^{-1} f_{ij} r_j \in R_P P.$$

Let $a_{ij} = c_{ij}^{-1} f_{ij} \in R_P$ for all $j \in I$ such that $f_{ij} \notin P$. Clearly (a_{ij}) is an $I \times I$ column-finite matrix over the ring R_P.

Now suppose that $t_i \in R$ $(i \in I)$ such that $t_i \neq 0$ for at most a finite number of elements $i \in I$ and $\sum_{i \in I} t_i m_i \in PM$. Then $\sum_{i \in I}(t_i + P)\overline{m}_i = 0$. Thus $\sum_{i \in I} b_{ij}(t_i + P) = 0$. By the preceding argument, $\sum_{i \in I} a_{ij} t_i \in R_P P$. Thus the matrix (a_{ij}) is (G, P)-compatible. It is now clear that

$$N = \left\{ \sum_{i \in I} r_i m_i \in M \colon r_i \in R \ (i \in I) \text{ and } \sum_{j \in I} a_{ij} r_j \in R_P P \ (i \in I) \right\},$$

as required. ∎

Let R be any ring. By a maximal prime submodule of an $R-$ module M we mean a prime submodule N such that N is maximal in $\{L\colon L$ is a prime submodule of M and $(L\colon M) = (N\colon M)\}$. Theorem 8 has the following corollary.

COROLLARY 9. *Let R be a domain and let M be an R-module with generating set $G = \{m_i\colon i \in I\}$. Then N is a maximal prime submodule of M if and only if there exists a prime ideal P of R and elements $a_i \in R_P$, $(i \in I)$, not all in $R_P P$, such that*

(i) *whenever $r_i \in R$ $(i \in I)$ such that $r_i \neq 0$ for at most a finite number of elements $i \in I$ and $\sum_{i \in I} r_i m_i \in PM$ then $\sum_{i \in I} a_i r_i \in R_P P$ and*

(ii) $N = \{\sum_{i \in I} r_i m_i \in M\colon r_i \in R \ (i \in I)$ *and* $\sum_{i \in I} a_i r_i \in R_P P\}$.

Proof. Suppose that N is a maximal prime submodule of M and $P = (N\colon M)$. By Theorem 8 there exists a (G, P)-compatible $I \times I$ column-finite matrix (a_{ij}) over R_P such that $a_{ij} \notin R_P P$ for some $i, j \in I$ and

$$N = \left\{ \sum_{i \in I} r_i m_i \in M \colon r_i \in R \ (i \in I) \text{ and } \sum_{j \in I} a_{ij} r_j \in R_P P \ (i \in I) \right\}.$$

Suppose that $i', j' \in I$ such that $a_{i'j'} \notin R_P P$ and let

$$L = \left\{ \sum_{i \in I} r_i m_i \in M : r_i \in R \ (i \in I) \text{ and } \sum_{j \in I} a_{i'j} r_j \in R_P P \right\}.$$

By Theorem 8, L is a prime submodule of M and $(L : M) = P$. Clearly $N \subseteq L$. Therefore $N = L$ and N satisfies (i) and (ii).

Conversely, suppose that N satisfies (i) and (ii). Define a mapping,

$$\theta : \frac{M}{PM} \to \frac{R_P}{R_P P} \text{ by } \theta\left(\sum_{i \in I} \overline{r_i m_i} \right) = \sum_{i \in I} a_i r_i + R_P P.$$

By (i), θ is well defined. Clearly, θ is an R-homomorphism and by (ii) $N = \ker \theta$. Let $\bar{\theta}$: $M/N \to R_P/R_P P$ be the induced monomorphism and let φ: $R_P \otimes (M/N) \to R_P/R_P P$ be the induced R_P-homomorphism. Because $R_P/R_P P$ is a simple R_P-module and $\varphi \neq 0$ it follows that $R_P \otimes (M/N)$ is a simple R_P-module. It follows easily that N is a maximal prime submodule of M with $(N : M) = P$. ∎

The situation for finitely generated modules is a good deal more straightforward. We have the following analogue of Theorem 8.

THEOREM 10. *Let R be a ring and let $M = \sum_{i=1}^{n} R m_i$ be a finitely generated R-module. Then N is a prime submodule of M if and only if there exist a prime ideal P of R and elements $a_{ij} \in R$ $(1 \leq i, j \leq n)$, not all in P, such that*

(i) *given elements $r_i \in R$ $(1 \leq i \leq n)$, $\sum_{i=1}^{n} r_i m_i \in PM$ implies that $\sum_{j=1}^{n} a_{ij} r_j \in P$ for all $1 \leq i \leq n$, and*

(ii) *$N = \{ \sum_{i=1}^{n} s_i m_i \in M$: $s_i \in R$ $(1 \leq i \leq n)$ and $\sum_{j=1}^{n} a_{ij} s_j \in P$, $(1 \leq i \leq n)\}$. In this case, $P = (N : M)$.*

Proof. Suppose first that N satisfies (i) and (ii). Then the proof of Theorem 8 shows that N is a prime submodule of M with $P = (N : M)$. Conversely, suppose that N is a prime submodule of M. Let $P = (N : M)$. Let $\overline{R} = R/P$, $\overline{M} = M/PM$, $\overline{N} = N/PM$, $\bar{r} = r + P$ for all r in R and $\overline{m} = m + PM$ for all m in M. Then $\overline{M}/\overline{N}$ is a torsionfree $\overline{R}$-module. By Lemma 6 there exist elements $a_{ij} \in R$ $(1 \leq i, j \leq n)$, not all in P and $c \in R \setminus P$ such that

(i)′ whenever $r_i \in R$ $(1 \leq i \leq n)$ with $\sum_{i=1}^{n} \bar{r}_i \overline{m}_i = \bar{0}$ then $\sum_{j=1}^{n} (\bar{a}_{ij} \bar{c}^{-1}) \bar{r}_j = \bar{0}$ for all $1 \leq i \leq n$ and

(ii)′ $\overline{N} = \{ \sum_{i=1}^{n} \bar{s}_i \overline{m}_i$: $s_i \in R$ $(i \in I)$ and $\sum_{j=1}^{n} (\bar{a}_{ij} \bar{c}^{-1}) \bar{s}_j = \bar{0}$, $(1 \leq i \leq n)\}$.

It is now clear that the elements $\{a_{ij}$: $1 \leq i, j \leq n\}$ satisfy (i) and (ii). ∎

There is an analogue of Corollary 9 for finitely generated modules.

COROLLARY 11. *Let R be a ring and let $M = \sum_{i=1}^{n} Rm_i$ be a finitely generated $R-$ module. Then N is a maximal prime submodule of M if and only if there exist a prime ideal P of R and elements $a_i \in R$ $(1 \leq i \leq n)$, not all in P, such that*

(i) *given elements $r_i \in R$, $(1 \leq i \leq n)$, $\sum_{i=1}^{n} r_i m_i \in PM$ implies that $\sum_{i=1}^{n} a_i r_i \in P$, and*

(ii) $N = \{\sum_{i=1}^{n} s_i m_i\colon s_i \in R \ (1 \leq i \leq n)$ *and* $\sum_{i=1}^{n} a_i s_i \in P\}$.

Proof. By the proof of Corollary 9.

REFERENCES

1. S. M. George, R. Y. McCasland, and P. F. Smith, A principal ideal theorem analogue for modules over commutative rings, *Comm. Algebra* **22** (1994), 2083–2099.
2. J. Jenkins and P. F. Smith, On the prime radical of a module over a commutative ring, *Comm. Algebra* **20** (1992), 3593–3602.
3. R. L. McCasland and M. E. Moore, Prime submodules, *Comm. Algebra* **20** (1992), 1803–1817.
4. R. L. McCasland and P. F. Smith, Prime submodules of Noetherian modules, *Rocky Mountain J. Math.* **23** (1993), 1041–1062.