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Note

Anti-Ramsey number of matchings in hypergraphs

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ABSTRACT

A k-matching in a hypergraph is a set of k edges such that no two of these edges intersect. The *anti-Ramsey number* of a k-matching in a complete s-uniform hypergraph $\mathcal{H}$ on n vertices, denoted by $\mathrm{ar}(n, s, k)$, is the smallest integer c such that in any coloring of the edges of $\mathcal{H}$ with exactly c colors, there is a k-matching whose edges have distinct colors. The *Turán number*, denoted by $\mathrm{ex}(n, s, k)$, is the the maximum number of edges in an s-uniform hypergraph on n vertices with no k-matching. For $k \geq 3$, we conjecture that if $n > sk$, then $\mathrm{ar}(n, s, k) = \mathrm{ex}(n, s, k-1) + 2$. Also, if $n = sk$, then $\mathrm{ar}(n, s, k) = \begin{cases} \mathrm{ex}(n, s, k-1) + 2 & \text{if } k < c_s \\ \mathrm{ex}(n, s, k-1) + s + 1 & \text{if } k \geq c_s \end{cases}$, where c_s is a constant dependent on s. We prove this conjecture for $k = 2$, $k = 3$, and sufficiently large n, as well as provide upper and lower bounds.

1. Introduction

A *hypergraph* $\mathcal{H}$ consists of a set $V(\mathcal{H})$ of *vertices* and a family $\mathcal{E}(\mathcal{H})$ of nonempty subsets of $V(\mathcal{H})$ called *edges* of $\mathcal{H}$. If each edge of $\mathcal{H}$ has exactly s vertices then $\mathcal{H}$ is *s-uniform*. A *complete s-uniform* hypergraph is a hypergraph whose edge set is the set of all s-subsets of the vertex set. A *matching* is a set of edges in a (hyper)graph in which no two edges have a common vertex. We call a matching with k edges a *k-matching* and a matching containing all vertices a *perfect matching*. In an edge-coloring of a (hyper)graph $\mathcal{H}$, a sub(hyper)graph $\mathcal{F} \subseteq \mathcal{H}$ is *rainbow* if all edges of $\mathcal{F}$ have distinct colors. The *anti-Ramsey number* of a graph G, denoted by $\mathrm{ar}(G, n)$, is the minimum number of colors needed to color the edges of K_n so that, in any coloring, there exists a rainbow copy of G. The *Turán number* of a graph G, denoted by $\mathrm{ex}(n, G)$, is the maximum number of edges in a graph on n vertices that does not contain G as a subgraph. The *anti-Ramsey number* of a k-matching, denoted by $\mathrm{ar}(n, s, k)$, is the minimum number of colors needed to color the edges of a complete s-uniform hypergraph on n vertices so that there exists a rainbow k-matching in any coloring. The *Turán number* of a k-matching, denoted by $\mathrm{ex}(n, s, k)$, is the maximum number of edges in an s-uniform hypergraph on n vertices that contains no k-matching.

In 1973, Erdős, Simonovits, and Sós [6] showed that $\mathrm{ar}(K_p, n) = \mathrm{ex}(n, K_{p-1}) + 2$ for sufficiently large n. More recently, Montellano-Ballesteros and Neumann-Lara [10] extended this result to all values of n and p with $n > p \geq 3$. A history of results and open problems on this topic was given by Fujita, Magnant, and Ozeki [8]. The Turán number $\mathrm{ex}(n, 2, k)$ was determined by Erdős and Gallai [4] as

$$\mathrm{ex}(n, 2, k) = \max\left\{\binom{2k-1}{2}, \binom{k-1}{2} + (k-1)(n-k+1)\right\}$$

for $n \geq 2k$ and $k \geq 1$. Schiermeyer [11] proved that $\mathrm{ar}(n, 2, k) = \mathrm{ex}(n, 2, k-1) + 2$ for $k \geq 2$ and $n \geq 3k + 3$. Later, Chen, Li, and Tu [2] and independently Fujita, Kaneko, Schiermeyer, and Suzuki [7] showed that $\mathrm{ar}(n, 2, k) = \mathrm{ex}(n, 2, k-1) + 2$

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for $k \geq 2$ and $n \geq 2k + 1$. The value

$$\mathrm{ar}(n, 2, k) = \begin{cases} \mathrm{ex}(n, 2, k-1) + 2 & \text{if } k < 7 \\ \mathrm{ex}(n, 2, k-1) + 3 & \text{if } k \geq 7 \end{cases}$$

was determined for $n = 2k$ in [2] and by Haas and the second author [9], independently.

The same ideas implying a lower bound for the anti-Ramsey number of graphs given in [6] provide a lower bound for $\mathrm{ar}(n, s, k)$.

Proposition 1. *For all n,* $\mathrm{ar}(n, s, k) \geq \mathrm{ex}(n, s, k-1) + 2$.

Proof. Let $\mathcal{H}$ be a complete s-uniform hypergraph on n vertices. Let $\mathcal{G}$ be a subhypergraph of $\mathcal{H}$ with $\mathrm{ex}(n, s, k-1)$ edges such that $\mathcal{G}$ does not contain a $(k-1)$-matching. Color each edge of $\mathcal{G}$ with distinct colors and color all of the remaining edges of $\mathcal{H}$ the same, using an additional color. If there is a rainbow k-matching in this coloring, then it uses $k-1$ edges from $\mathcal{G}$ which is a contradiction. Therefore, this coloring has no rainbow k-matching. □

For k-matchings the Turán number $\mathrm{ex}(n, s, k)$ is still not known for $k \geq 3$ and $s \geq 3$. Erdős [3] conjectured in 1965 the value of $\mathrm{ex}(n, s, k)$ as follows. Let $g(n, s, k-1)$ be the number of s-sets of $\{1, \ldots, n\}$ that intersect $\{1, \ldots, k-1\}$. By definition, $g(n, s, k-1) = \binom{n}{s} - \binom{n-k+1}{s}$.

Conjecture 2 (*Erdős [3]*)**.** *For* $n \geq sk$, $s \geq 2$, *and* $k \geq 2$,

$$\mathrm{ex}(n, s, k) = \max\left\{\binom{sk-1}{s}, g(n, s, k-1)\right\}. \tag{1}$$

Erdős, Ko, and Rado [5] proved that $\mathrm{ex}(n, s, 2) = \binom{n-1}{s-1} = g(n, s, 1)$ for $n \geq 2s$. This conjecture is true for $s = 2$, as shown by Erdős and Gallai [4]. Erdős [3] proved that

$$\mathrm{ex}(n, s, k) = g(n, s, k-1) = \binom{n}{s} - \binom{n-k+1}{s} \tag{2}$$

for sufficiently large n. Later, Bollobás, Daykin, and Erdős [1] sharpened this result by showing that (2) holds for $n > 2s^3(k-1)$.

In Section 2, we provide bounds on $\mathrm{ar}(n, s, k)$ and show that anti-Ramsey number and Turán number of a k-matching differ at most by a constant. In Section 3, we determine the value of $\mathrm{ar}(n, s, k)$ for $k \in \{2, 3\}$ and show that $\mathrm{ar}(n, s, k) = \mathrm{ex}(n, s, k-1) + 2$ for $k \in \{2, 3\}$ and $n > ks$. The claim also holds for $n = ks$ when $k = 3$. We conjecture that this is true for all k.

Conjecture 3. *Let* $k \geq 3$. *If* $n > sk$, *then* $\mathrm{ar}(n, s, k) = \mathrm{ex}(n, s, k-1) + 2$. *Also, if* $n = sk$, *then*

$$\mathrm{ar}(n, s, k) = \begin{cases} \mathrm{ex}(n, s, k-1) + 2 & \text{if } k < c_s \\ \mathrm{ex}(n, s, k-1) + s + 1 & \text{if } k \geq c_s \end{cases}$$

where c_s is a constant dependent on s.

Finally, in Section 4, we give the exact value of $\mathrm{ar}(n, s, k)$ when n is sufficiently large.

We introduce some notation for hypergraphs used in the remaining sections. For a set X, $\binom{X}{s}$ denotes all s-subsets of X. We call a hypergraph an *intersecting family* if every two edges intersect. For a vertex x in a hypergraph $\mathcal{H}$, we call the number of edges of $\mathcal{H}$ containing x the *degree* of x written $deg_{\mathcal{H}}(x)$. The maximum degree of a hypergraph $\mathcal{H}$ is denoted by $\Delta(\mathcal{H})$.

2. General bounds on the anti-Ramsey number

The following constructions provide a lower bound for $\mathrm{ar}(n, s, k)$ in Corollary 6.

Construction 4. *Let $\mathcal{H}$ be the complete s-uniform hypergraph with vertex set $\{v_1, \ldots, v_n\}$, where $n = sk$. Let $A = \{v_1, \ldots, v_{s+1}\}$ and $c = \binom{n-s-1}{s} + s$. Define a c-coloring h of $\mathcal{E}(\mathcal{H})$ as follows. For any edge $E \in \mathcal{E}$, if $v_1 \in E$, then let $h(e) = \min\{i : v_i \notin E\}$. If $E \cap A \neq \emptyset$ but $v_1 \notin E$, then let $h(E) = \min\{i : v_i \in E\}$. Assign distinct other colors to the remaining edges.*

Assume there is a rainbow perfect matching $\mathcal{M}$ in this coloring. Since $n = sk$, at least two edges of M intersect A. Let E be the edge of $\mathcal{M}$ that contains v_1. Let $j = \min\{i : v_i \notin V(E)\}$ and let E' be the edge of $\mathcal{M}$ that contains v_j. By the above construction, E and E' both have color j.

Construction 5. *Let $\mathcal{H}$ be a complete s-uniform hypergraph on $n \geq sk$ vertices. Let S be a subset of $V(\mathcal{H})$ with $k-2$ vertices and color the edges containing any vertex from S with distinct colors. Color all of the remaining edges the same with an additional color. The number of colors used is $\binom{n}{s} - \binom{n-k+2}{s} + 1$.*

This construction has no rainbow k-matching, since at least two edges among any k must lie completely outside S. Constructions 4 and 5 establish lower bounds for the anti-Ramsey number:

Corollary 6. *If $n \geq sk$, then* $\mathrm{ar}(n, s, k) \geq \begin{cases} \max\left\{\binom{n}{s} - \binom{n-k+2}{s} + 2, \binom{n-s-1}{s} + s + 1\right\} & \text{if } n = sk, \\ \binom{n}{s} - \binom{n-k+2}{s} + 2 & \text{otherwise.} \end{cases}$

Theorem 7. *If $n \geq sk + (s-1)(k-1)$, then* $\mathrm{ar}(n, s, k) \leq \mathrm{ex}(n, s, k-1) + k$.

Proof. Let $\mathcal{H}$ be a complete s-uniform hypergraph on n vertices whose edges are colored with $\mathrm{ex}(n, s, k-1) + k$ colors. Since taking exactly one edge of each color gives a subhypergraph with $\mathrm{ex}(n, s, k-1) + k$ edges, there exists a rainbow $(k-1)$-matching $\mathcal{M}$. Let the colors of the edges in $\mathcal{M}$ be $\alpha_1, \ldots, \alpha_{k-1}$. Let $A = V(\mathcal{H}) \setminus V(\mathcal{M})$. Note that every edge induced by A has a color in $\{\alpha_1, \ldots, \alpha_{k-1}\}$, otherwise, there is a rainbow k-matching containing the edges of $\mathcal{M}$.

Remove all edges of $\mathcal{H}$ that have color α_i for $1 \leq i \leq k-1$ and let $\mathcal{G}$ be the remaining hypergraph (with colors preserved). In this coloring, there are at least $\mathrm{ex}(n, s, k-1) + 1$ colors and therefore a rainbow $(k-1)$-matching exists; call it $\mathcal{M}'$. Since no edge of $\mathcal{G}$ is induced by A, $|V(\mathcal{M}') \cap A| \leq (k-1)(s-1)$. Together with the assumed lower bound on n, this yields $|A \setminus V(\mathcal{M}')| = |V(\mathcal{H}) \setminus (V(\mathcal{M} \cup \mathcal{M}'))| \geq n - s(k-1) - (s-1)(k-1) \geq s$. Hence some edge induced by A intersects no edge in $\mathcal{M}'$ and completes a rainbow k-matching with $\mathcal{M}$ induced by A that does not intersect any edge in $\mathcal{M}'$. The color of e is α_i for some i, $1 \leq i \leq k-1$ and there is a rainbow k-matching using the edges in $\mathcal{M}'$ and e. □

3. Anti-Ramsey numbers for *k*-matchings, $k \in \{2, 3\}$

Theorem 8. *If $n \geq 2s$, then*

$$\mathrm{ar}(n, s, 2) = \begin{cases} \frac{1}{2}\binom{n}{s} + 1 & n = 2s \\ 2 & n > 2s. \end{cases}$$

Proof. Let $\mathcal{H}$ be a complete s-uniform hypergraph on n vertices. If $n = 2s$, then by coloring complementary edges with the same color and using distinct colors for all such pairs, we can obtain a coloring without a rainbow 2-matching. If $\mathcal{H}$ is colored by at least $\frac{1}{2}\binom{n}{s} + 1$ colors then, by the pigeonhole principle, one of the vertex-disjoint edge pairs has distinct colors.

Now, let $n \geq 2s + 1$ and consider a coloring of the edge set of $\mathcal{H}$ with 2 colors such that there is no rainbow 2-matching. This requires disjoint edges to have the same color. Hence in the Kneser graph $K(n, s)$, where the vertices are the edges of $\mathcal{H}$ and two vertices are adjacent when the corresponding edges of $\mathcal{H}$ are disjoint, all edges in the same component must have the same color. It is well known that the Kneser graph is connected when $n \geq 2s + 1$, so only one color can be used when avoiding a rainbow 2-matching. □

Theorem 9. *If $n \geq 3s$, then* $\mathrm{ar}(n, s, 3) = \binom{n-1}{s-1} + 2 = \mathrm{ex}(n, s, 2) + 2$.

Proof. Let $\mathcal{H}$ be a complete s-uniform hypergraph on n vertices with edge set $\mathcal{E}$. We consider a coloring of $\mathcal{E}$ using $\binom{n-1}{s-1} + 2$ colors, such that there is no rainbow 3-matching. Fix a vertex v and let $E(v)$ denote the set of edges that contain v. Choose Q as a subset of $\mathcal{E} \setminus E(v)$ such that the edges of Q do not have any color in common with the edges of $E(v)$ and each color not used on $E(v)$ is the color of exactly one edge in Q. This implies that $|Q| \geq 2$, since $|E(v)| = \binom{n-1}{s-1}$.

Note that any pair of edges E_1 and E_2 in Q have nonempty intersection, otherwise there is a rainbow 3-matching containing E_1, E_2, and any edge of $E(v)$ that does not intersect E_1 and E_2. Let $A, B \in Q$ and $C, D \in E(v)$ We use (A, B) to denote an unordered pair of edges A and B. We write $(A, B) \diamond (C, D)$ if

$$\begin{array}{llll} A \cap D = \emptyset, & B \cap C = \emptyset, & \text{and} & A \cup D = B \cup C \\ & & \text{or} & \\ A \cap C = \emptyset, & B \cap D = \emptyset, & \text{and} & A \cup C = B \cup D. \end{array} \tag{3}$$

An example of the configuration of A, B, C and D is shown in Fig. 1.

We define an auxiliary bipartite graph G with vertex set $V(G) = X \cup Y$, where $X = \binom{Q}{2}$, $Y = \binom{E(v)}{2}$ and the edge set of G is defined as $E(G) = \{(A, B)(C, D) : (A, B) \diamond (C, D),\ (A, B) \in X,\ (C, D) \in Y\}$. In the proof of Claim 10, we use the following result of Erdős, Ko and Rado [5] which gives an upper bound on the size of an s-uniform intersecting family on n vertices.

$$\mathrm{ex}(n, s, 2) = \binom{n-1}{s-1}, \quad \text{for } n \geq 2s. \tag{4}$$

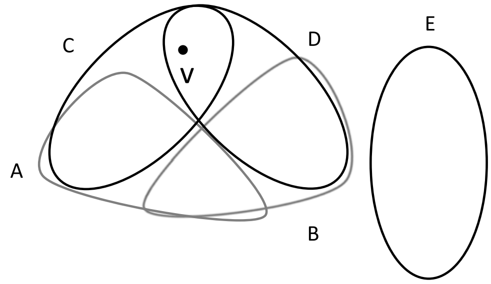

Fig. 1. The edges A, B, C, D and E.

Claim 10. *There is a matching in G whose vertex set contains all vertices in $X = \binom{Q}{2}$.*

Recall that Q is an intersecting subfamily. The degree $deg_G(A, B)$ is the number of vertices (C, D) in Y that satisfy the relation in (3). Therefore, the number of neighbors of (A, B) are given by the number of choices for the set $(C \cap D) \setminus \{v\}$. Let $\ell = |A \cap B|$, where $1 \le \ell \le s - 1$. Since $|C \cap D| = \ell$, each vertex in X has the same degree given by

$$deg_G((A, B)) = \binom{n - (2s - \ell) - 1}{\ell - 1}. \tag{5}$$

Now, by the same observations as above, the degree of a vertex (C, D) in Y can be bounded above. Let (A, B) and (A', B'), where $(A', B') \neq (A, B)$, be neighbors of (C, D). By definition of the relation $\diamond$, the edges A, A', B, and B' are all distinct. Since Q is an intersecting family, $A \cap B$ and $A' \cap B'$ cannot be vertex-disjoint. Therefore the collection of $A \cap B$'s that satisfy $(A, B) \diamond (C, D)$ for a fixed vertex (C, D) in Y with $|C \cap D| = \ell$ is an ℓ-uniform intersecting family on the vertex set $V \setminus (C \cup D)$ which has $n - (2s - \ell)$ vertices. By using (4), we obtain an upper bound on the degree of (C, D) as

$$deg_G((C, D)) \le \binom{n - (2s - \ell) - 1}{\ell - 1}. \tag{6}$$

Let G' be a connected component of G. A result of the definition of the edge set of G is that if $(U_1, U_2), (V_1, V_2) \in V(G')$ and $|U_1 \cap U_2| = \ell$, then $|V_1 \cap V_2| = \ell$. Let $T \subseteq (V(G') \cap X)$ and $N(T) \subseteq (V(G') \cap Y)$ be the neighborhood of T. Since (5) and (6) also hold for G' we have

$$\begin{aligned} |T| \binom{n - (2s - \ell) - 1}{\ell - 1} &= \sum_{(A,B) \in T} deg_{G'}((A, B)) \\ &\le \sum_{(C,D) \in N(T)} deg_{G'}((C, D)) \\ &\le |N(T)| \binom{n - (2s - \ell) - 1}{\ell - 1}. \end{aligned}$$

Therefore, $|T| \le |N(T)|$ for any $T \subseteq (V(G') \cap X)$ and by Hall's Theorem, there is a matching containing each vertex in $G' \cap X$. Applying this to each component of G completes the proof of the claim.

Claim 11. *Let $(A, B) \in \binom{Q}{2}$ and $(C, D) \in \binom{E(v)}{2}$ with $(A, B) \diamond (C, D)$. Then the edges C and D have the same color.*

Let S be the subset of $V(\mathcal{H})$ that is vertex-disjoint from these four edges, thus $|S| = n - 2s \ge s$. Let E be an edge induced by S. Let A, B, C and D be related as in (3) such that without loss of generality $\{A, D, E\}$ and $\{B, C, E\}$ are matchings. If E has the same color as A or B then $\{B, C, E\}$ or $\{A, D, E\}$, respectively, must be a rainbow matching. Therefore, E must have the same color as C and D, since there are no rainbow 3-matchings. Hence, C and D have the same color.

We define another auxiliary graph G_v with vertex set $E(v)$ and edge set $\{CD : C, D \in E(v) \text{ and } deg_G((C, D)) > 0\}$. Let $|Q| = q$ and p be the number of components of G_v. By Claim 11, each component of G_v corresponds to a subset of $E(v)$ whose members have the same color. Therefore, $p \ge \binom{n-1}{s-1} + 2 - q$.

One can find an injective mapping $f : \binom{Q}{2} \to \binom{E(v)}{2}$ defined by using the adjacencies of vertices in a matching of G given by Claim 10. Therefore there are at least $\binom{q}{2}$ edges in G_v. The maximum number of components of a graph with fixed number of vertices and edges is attained in the case when all edges are in a single component with minimum number of vertices and remaining components are isolated vertices. Thus, $p \le \binom{n-1}{s-1} - q + 1$. This is a contradiction with the lower bound of p given above. $\square$

4. Anti-Ramsey number for large n

By following the same ideas of the proof of (2) in [1] and [3], one can prove Theorem 12. For completeness, we provide its proof here.

Theorem 12. *For fixed s and k and $n \geq 2s^3k$,* $\mathrm{ar}(n, s, k) = \mathrm{ex}(n, s, k-1) + 2$.

Proof of Theorem 12. Let $\mathcal{H}$ be a complete s-uniform hypergraph on n vertices. The lower bound for $\mathrm{ar}(n, s, k)$ is provided by Construction 5. To prove the upper bound, we proceed by induction on k. Theorem 9 deals with the base case when $k = 3$ and $n \geq 3s$.

For the inductive case, color the edges of $\mathcal{H}$ with exactly $c = \binom{n}{s} - \binom{n-k+2}{s} + 2 = \sum_{i=1}^{k-2} \binom{n-i}{s-1} + 2$ colors. We show that $\mathcal{H}$ has a rainbow k-matching. Let $\mathcal{G}$ be a subgraph of $\mathcal{H}$ with c edges such that each color appears on exactly one edge of $\mathcal{G}$. Let v be a vertex such that $deg_{\mathcal{G}}(v) = \Delta(\mathcal{G})$.

Note that there are at least $c - \binom{n-1}{s-1}$ colors on the edges of the complete subhypergraph $\mathcal{H} \setminus \{v\}$ and the inductive hypothesis implies that $c - \binom{n-1}{s-1} = \mathrm{ar}(n-1, s, k-1)$ and there is a rainbow $(k-1)$-matching in $\mathcal{H} \setminus \{v\}$. Call this matching $\mathcal{M}$ and modify $\mathcal{G}$ to obtain a new hypergraph $\mathcal{G}'$ such that the edge set of $\mathcal{G}'$ consists of the edges of $\mathcal{M}$ and all edges of $\mathcal{G}$ except the ones that have a color from $\mathcal{M}$. By this definition, $\mathcal{G}$ and $\mathcal{G}'$ have the same number of colors and each color on $\mathcal{H}$ appears exactly once on $\mathcal{G}'$. The only difference is that $deg_{\mathcal{G}'}(v) \geq \Delta(\mathcal{G}') - (k-1)$ and v may not be a vertex with maximum degree in $\mathcal{G}'$, but its degree is still high enough.

We analyze the two cases depending on the maximum degree in $\mathcal{G}'$. If $\Delta(\mathcal{G}') < c/((k-1)s)$ then the number of edges containing a vertex in $\mathcal{M}$ is less than c and there is an edge of $\mathcal{G}'$ that is vertex-disjoint from $\mathcal{M}$ and we are done. Otherwise, $\Delta(\mathcal{G}') \geq c/((k-1)s)$. The number of edges of $\mathcal{G}'$ containing both v and a vertex of $\mathcal{M}$ is at most $(k-1)s\binom{n-2}{s-2}$. For $n \geq 2s^3k$, we have

$$deg_{\mathcal{G}'}(v) \geq \Delta(\mathcal{G}') - (k-1) \geq \frac{c}{(k-1)s} - (k-1) = \frac{\binom{n}{s} - \binom{n-k+2}{s} + 2}{(k-1)s} - (k-1) > (k-1)s\binom{n-2}{s-2},$$

where the last inequality will be proved as Claim 13. Therefore, there is an edge of $\mathcal{G}'$ that contains v and does not intersect any edge of $\mathcal{M}$, which implies that there is a rainbow k-matching.

Claim 13. *For $n \geq 2s^3k$,*

$$\binom{n}{s} - \binom{n-k+2}{s} + 2 > (k-1)^2 s\left(s + \binom{n-2}{s-2}^{-1}\right)\binom{n-2}{s-2}.$$

Below, we first present the observations that will be used later.
Note that for $r \leq m \leq n$,

$$\binom{m}{r} \geq \left(\frac{m-r+1}{n-r+1}\right)^r \binom{n}{r} = \left(1 - \frac{n-m}{n-r+1}\right)^r \binom{n}{r}.$$

By using the fact that $(1-x)^a \geq 1 - ax$ for $0 \leq x < 1$, the relation above gives that

$$\binom{m}{r} \geq \left(1 - \frac{r(n-m)}{n-r+1}\right)\binom{n}{r}. \tag{7}$$

Observe that

$$\binom{n}{s} - \binom{n-k+2}{s} + 2 = \sum_{i=1}^{k-2}\binom{n-i}{s-1} + 2 > (k-2)\frac{n-k+2}{s-1}\binom{n-k+1}{s-2}.$$

By (7) and the inequality above, we obtain

$$\binom{n}{s} - \binom{n-k+2}{s} + 2 > (k-2)\frac{n-k+2}{s-1}\left(1 - \frac{(s-2)(k-3)}{n-s+1}\right)\binom{n-2}{s-2}. \tag{8}$$

Assume that our claim does not hold. Then, (8) implies that

$$(k-1)^2 s\left(s + \binom{n-2}{s-2}^{-1}\right) > (k-2)\frac{n-k+2}{s-1}\left(1 - \frac{(s-2)(k-3)}{n-s+1}\right).$$

One can check that this is a contradiction for $n \geq 2s^3k$ and we are done. □

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