

Research Article

Solution of Effective-Mass Dirac Equation with Scalar-Vector and Pseudoscalar Terms for Generalized Hulthén Potential

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We find the exact bound state solutions and normalization constant for the Dirac equation with scalar-vector-pseudoscalar interaction terms for the generalized Hulthén potential in the case where we have a particular mass function $m(x)$. We also search the solutions for the constant mass where the obtained results correspond to the ones when the Dirac equation has spin and pseudospin symmetry, respectively. After giving the obtained results for the nonrelativistic case, we search then the energy spectra and corresponding upper and lower components of Dirac spinor for the case of PT-symmetric forms of the present potential.

1. Introduction

The Hulthén potential [1] is one of the best known potentials in physics, as a short-range potential [2]. In the present work, we deal with the following form [3, 4]:

$$V(x) = V_0 \frac{e^{-2\beta x}}{qe^{-2\beta x} - 1}, \quad (1)$$

where the parameter V_0 can be written as $Z(2\beta)$ with the constant Z and β is the screening parameter (in atomic units) [2] with deformation parameter q . The constant Z is related to the atomic number if one uses this potential in atomic physics. Basically, the Hulthén potential is a special form of the Eckart potential [2, 5].

As a short-range potential, the Hulthén potential has a great advantage because the Schrödinger equation can be solved exactly for this potential with $\ell = 0$. With this advantage, the Hulthén potential has been used in different areas of physics, such as in solid-state physics [6], nuclear and particle physics [7], atomic physics [8], and chemical physics [9], and investigated with various techniques [2, 10–17].

In this work, we search the bound state solutions of the generalized Hulthén potential which can be written also in a complex form identifying the PT-symmetric case in a closed form for the case where the mass depends on spatial coordinate and extend the Dirac equation including the scalar,

vector, and pseudoscalar interaction terms to this case. After the works by Von Roos [18], and Lévy-Leblond [19], the solutions of relativistic and nonrelativistic wave equations with a position-dependent mass have received great attention in literature [3, references therein]. In [20], the bound state solutions of the Klein-Gordon (KG) and Dirac equations with the Hulthén potential by using the approach proposed by Biedenharn have been worked where the scattering state solutions have been also presented. In [21], the analytical results for the bound states of the Dirac equation with the generalized Hulthén potential as a tensor term have been studied within the concept of the SUSYQM. In the present work, we extend the search including the solutions of the Dirac equation having a pseudoscalar interaction term, as in [22–25], for the q -parameter Hulthén potential within the position-dependent mass (PDM) formalism. This formalism gives an opportunity such as writing the analytical results for the case where the mass is constant. This means that our results are also available for the cases where the Dirac equation has pseudospin and spin symmetry. Our generic results which will be given below makes it possible to give the “wave functions” with their normalization constants for both the cases of PDM formalism and constant mass. We search here also the analytical results for the PT-symmetric/non-Hermitian and PT-symmetric/pseudo-Hermitian form of the Hulthén potential for both of upper and lower component which are presented

again within the PDM formalism. These give us also the results for the case where the mass is constant if necessary. Among the above results, because of the q -parameter in potential, we apply our results for three different forms of the potential as special cases.

The organization of this work is as follows. In Section 2, we write the Dirac equation with scalar ($V_S(x)$), vector ($V_V(x)$), and pseudoscalar ($V_P(x)$) potentials in 1 + 1 dimension for the case where the mass is a spatially coordinate function. In Section 3, we search the bound state solutions for upper and lower component of the Dirac spinor separately and give the normalization constant. We construct a relation between the mass function and the potentials to reduce the Dirac equation to an analytically solvable form of second-order differential equation. We give also the results for the case where the mass is constant and observe that the results obtained for this case correspond to the solutions when the spin and pseudospin symmetry occur in Dirac equation. The spin symmetry appears when the difference of the scalar and vector potentials is constant; that is, $\Delta(x) = \text{const.}$, and the pseudospin symmetry appears when the sum of the scalar and vector potentials is constant; that is, $\Sigma(x) = \text{const.}$ [20, 21, 26]. Finally, we obtain the nonrelativistic result for the bound state solution for the generalized Hulthén potential. The present work can also be seen as an application of the parametric generalization of the Nikiforov-Uvarov method which will be given in Appendix briefly [27, 28]. In Section 4, we write the generalized Hulthén potential in a complex form which corresponds to the PT-symmetric form of the potential and find the energy levels for upper and lower component of the Dirac spinor with normalized wave functions. The PT-symmetric formulation with non-Hermitian Hamiltonians having real or complex spectra of quantum mechanics has received a great attention in literature after the work by Bender and Boettcher [29–31]. In Section 5, we collect briefly our analytical results for special values of the parameter q corresponding to the standard Hulthén potential ($q = 1$), to the Woods-Saxon potential ($q = -1$) and to the exponential potential ($q = 0$) while the mass depends on spatially coordinate function. We give our conclusions in Section 6.

2. Dirac Equation in 1+1 Dimension

The time-independent Dirac equation for a spin-1/2 particle subjected to scalar, vector, and pseudoscalar potentials in terms of the $\Sigma(x) = V_V(x) + V_S(x)$, and $\Delta(x) = V_V(x) - V_S(x)$ is given by ($\hbar = c = 1$) [22, 23, 32–36]

$$\left(\sigma_1 p + \sigma_3 m(x) + \frac{1 + \sigma_3}{2} \Sigma(x) + \frac{1 - \sigma_3}{2} \Delta(x) + \sigma_2 V_P(x) \right) \Psi(x) = E \Psi(x), \quad (2)$$

where σ_1, σ_2 , and σ_3 are the Pauli spin matrices, and we write the mass as $m(x)$. By taking the Dirac spinor as $\Psi = (\phi_1, \phi_2)^t$,

where t indicates the transpose, we obtain the following first-order coupled equations for upper and lower components:

$$\begin{aligned} -i \frac{d\phi_1(x)}{dx} - [m(x) - \Delta(x) + E] \phi_2(x) + iV_P \phi_1(x) \\ = 0, \end{aligned} \quad (3)$$

$$\begin{aligned} -i \frac{d\phi_2(x)}{dx} + [m(x) + \Sigma(x) - E] \phi_1(x) - iV_P \phi_2(x) \\ = 0. \end{aligned} \quad (4)$$

Writing $\phi_2(x)$ in terms of $\phi_1(x)$ with the help of (3) and inserting it into (4) give us

$$\begin{aligned} \left(\frac{d^2 \phi_1(x)}{dx^2} - \frac{dV_P(x)}{dx} \phi_1(x) - V_P(x) \frac{d\phi_1(x)}{dx} \right) \\ \cdot [m(x) - \Delta(x) + E] - \left(\frac{dm(x)}{dx} - \frac{d\Delta(x)}{dx} \right) \\ \cdot \left(\frac{d\phi_1(x)}{dx} - V_P(x) \phi_1(x) \right) + V_P(x) \\ \cdot \left(\frac{d\phi_1(x)}{dx} - V_P(x) \phi_1(x) \right) [m(x) - \Delta(x) + E] \\ - [m(x) - \Delta(x) + E]^2 [m(x) + \Sigma(x) - E] \phi_1(x) \\ = 0. \end{aligned} \quad (5)$$

We write here the equality $dm(x)/dx = d\Delta(x)/dx$ between the mass function and the potentials to reduce the above complicated equation to a simpler one which can be solved analytically. This relation gives us also the opportunity about finding the mass function explicitly and the second-order equation for upper component $\phi_1(x)$ as

$$\left\{ \frac{d^2}{dx^2} - \frac{dV_P(x)}{dx} - V_P^2(x) - [m(x) + E - \Delta(x)] [m(x) + \Sigma(x) - E] \right\} \phi_1(x) = 0. \quad (6)$$

By following similar steps and using the equality for the mass function as $dm(x)/dx = -d\Sigma(x)/dx$, we obtain the second-order equation for lower component $\phi_2(x)$ as

$$\left\{ \frac{d^2}{dx^2} + \frac{dV_P(x)}{dx} - V_P^2(x) - [m(x) + E - \Delta(x)] [m(x) + \Sigma(x) - E] \right\} \phi_2(x) = 0. \quad (7)$$

Equations (6) and (7) can be solved by using the parametric generalization of the Nikiforov-Uvarov method which

is given in Appendix briefly. In the next section, we solve the above equations for the scalar, vector, and pseudoscalar potentials by identifying them in terms of the Hulthén potential given in (1). First, we find the appropriate mass function by using the equalities, and then we write the bound state solutions with the corresponding normalized wave functions.

3. Bound States for Generalized Hulthén Potential

We are now in a position to identify the potentials in terms of the generalized Hulthén potential. We tend to write them as follows [3, 4]:

$$\begin{aligned} V_V(x) &= V_0 \frac{e^{-2\beta x}}{qe^{-2\beta x} - 1}, \\ V_S(x) &= -S_0 \frac{e^{-2\beta x}}{qe^{-2\beta x} - 1}, \\ V_P(x) &= V_i \frac{e^{-2\beta x}}{qe^{-2\beta x} - 1}, \end{aligned} \quad (8)$$

where $i = 1, 2$, V_1 is for the upper component $\phi_1(x)$, and V_2 is for the lower component $\phi_2(x)$. We obtain (7) for $\phi_2(x)$ by replacement $V_2 \leftrightarrow -V_1$ in (6). In addition, we can handle the explicit form of the wave function for $\phi_2(x)$ by doing $\beta \leftrightarrow -\beta$ in $\phi_1(x)$.

From (8), we have

$$\begin{aligned} \Sigma(x) &= (V_0 - S_0) \frac{e^{-2\beta x}}{qe^{-2\beta x} - 1}, \\ \Delta(x) &= (V_0 + S_0) \frac{e^{-2\beta x}}{qe^{-2\beta x} - 1}. \end{aligned} \quad (9)$$

By using (8) and (9), we write the mass function from the equality obtained for $\phi_1(x)$ as $m(x) = m_0 + m_1(e^{-2\beta x}/(qe^{-2\beta x} - 1))$, where the parameter m_0 is basically the integral constant, and we denote it as “constant mass”; the other parameter m_1 is obtained as $m_1 = V_0 + S_0$. This means that the mass parameter m_1 contains the contributions coming from vector and scalar potentials. The equality obtained for $\phi_2(x)$ gives us the mass function as $m(x) = m_0 + m_2(e^{-2\beta x}/(qe^{-2\beta x} - 1))$ with $m_2 = V_0 - S_0$ including the contributions coming from vector and scalar potentials. So, we can combine these two mass functions in a single form as $m(x) = m_0 + m_i(e^{-2\beta x}/(qe^{-2\beta x} - 1))$ which will be used in computation below.

By using a new variable as $s = 1/(1 - qe^{-2\beta x})$ ($-\infty < x < +\infty \rightarrow 0 \leq s \leq 1$), using (8)-(9), and inserting the mass function $m(x)$, we have the following representative equation for both components:

$$\left\{ \frac{d^2}{ds^2} + \frac{1-2s}{s(1-s)} \frac{d}{ds} - \frac{1}{s^2(1-s)^2} \left[\frac{A}{4\beta^2} + \frac{2B}{4\beta^2}s + \frac{C}{4\beta^2}s^2 \right] \right\} \phi_i(s) = 0, \quad (10)$$

where

$$\begin{aligned} A &= m_0^2 - E^2 + 2QV_0(E + m_0) + a_i, \\ B &= -QV_0(E + m_0) - b_i, \\ C &= 2\beta QV_i + Q^2V_i^2 + Q^2[(m_i - S_0)^2 - V_0^2], \end{aligned} \quad (11)$$

with

$$\begin{aligned} a_i &= 2Qm_0m_i + Q^2V_i^2 + Q^2[(m_i - S_0)^2 - V_0^2] \\ &\quad - 2Qm_0(V_0 + S_0), \\ b_i &= QV_i(\beta + QV_i) + Q^2[(m_i - S_0)^2 - V_0^2] \\ &\quad - Qm_0(S_0 + V_0) + Qm_0m_i, \end{aligned} \quad (12)$$

and $Q = 1/q$. Equation (10) can be solved by using the parametric Nikiforov-Uvarov method. For this aim, we compare (10) with (A.1) in Appendix, and with the help of (A.3) we obtain the parameter set

$$\begin{aligned} \alpha_1 &= 1, \\ \alpha_2 &= 2, \\ \alpha_3 &= 1, \\ \xi_1 &= \frac{C}{4\beta^2}, \\ \xi_2 &= -\frac{2B}{4\beta^2}, \\ \xi_3 &= \frac{A}{4\beta^2}, \\ \alpha_4 &= \alpha_5 = 0, \\ \alpha_6 &= \xi_1, \\ \alpha_7 &= -\xi_2, \\ \alpha_8 &= \xi_3, \\ \alpha_9 &= \xi_1 - \xi_2 + \xi_3, \\ \alpha_{10} &= 1 + 2\sqrt{\xi_3}, \\ \alpha_{11} &= 2 + 2\left(\sqrt{\xi_1 - \xi_2 + \xi_3} + \sqrt{\xi_3}\right), \\ \alpha_{12} &= \sqrt{\xi_3}, \\ \alpha_{13} &= -\left(\sqrt{\xi_1 - \xi_2 + \xi_3} + \sqrt{\xi_3}\right). \end{aligned} \quad (13)$$

With the help of (A.2) in Appendix, we write the energy spectrum of the Dirac equation with scalar-vector-pseudoscalar generalized Hulthén potential within the position-dependent mass formalism as

$$\begin{aligned} & \left[\sqrt{m_0^2 - E^2 + 2QV_0(E + m_0) + a_i} + \sqrt{m_0^2 - E^2} \right. \\ & \left. + \beta(2n + 1) \right]^2 - [(\beta + QV_i)^2 \\ & + Q^2[(m_i - S_0)^2 - V_0^2]] = 0, \end{aligned} \quad (14)$$

which can be solved numerically to get the energy eigenvalues.

In order to handle the generic wave function for the upper and lower component of the Dirac spinor, we use (A.4) in Appendix which gives

$$\phi_i(s) = N_i s^{\alpha'/2} (1-s)^{\beta'/2} P_n^{(\alpha', \beta')}(1-2s), \quad (15)$$

with $\alpha' = 2\sqrt{\xi_3}$, $\beta' = 2\sqrt{\xi_1 - \xi_2 + \xi_3}$, and the normalization constant N_i . Let us now find the normalization constant. Using a new variable $z = 1 - 2s$ ($0 \leq s \leq 1 \rightarrow +1 \leq z \leq -1$), and writing the normalization condition $\int_{-\infty}^{+\infty} |\phi_i(x)|^2 dx = 1$ as

$$\int_{-1}^{+1} |\phi_i(z)|^2 \frac{dz}{\beta(1-z)(1+z)} = 1. \quad (16)$$

We get

$$\begin{aligned} & \frac{|N_i|^2}{\beta} \frac{1}{2^{\alpha'+\beta'}} \\ & \cdot \int_{-1}^{+1} (1-z)^{\alpha'-1} (1+z)^{\beta'-1} \left[P_n^{(\alpha', \beta')}(z) \right]^2 dz = 1. \end{aligned} \quad (17)$$

The condition to be satisfied in (20) gives an upper limit for ℓ as $\ell < A/\beta$ which can be used to determine the greatest integer value for the quantum number n as $n < [A/\beta]$.

We obtain the formal analytical solutions for the problem under consideration giving the results in terms of representative equations (14) and (15). Now we move on to consider the upper and lower components of the Dirac spinor separately and summarize the results for the case where the mass is constant and the case of nonrelativistic limit for the present problem.

By using the following representation of the Jacobi polynomials [22]

$$\begin{aligned} P_n^{(\alpha', \beta')}(z) &= \frac{1}{n!} \sum_{\ell=0}^n \frac{1}{\ell!} (-n)_\ell (n + \alpha' + \beta' + 1)_\ell \\ &\cdot (n + \alpha' + 1)_\ell \left(\frac{1-z}{2} \right)^\ell, \end{aligned} \quad (18)$$

and (17) is written as

$$\begin{aligned} & \frac{|N_i|^2}{\beta} \frac{1}{2^{\alpha'+\beta'+\ell}} \frac{1}{n!} \\ & \cdot \sum_{\ell=0}^n \frac{1}{\ell!} (-n)_\ell (n + \alpha' + \beta' + 1)_\ell (n + \alpha' + 1)_\ell \\ & \cdot \int_{-1}^{+1} (1-z)^{\alpha'-1+\ell} (1+z)^{\beta'-1} P_n^{(\alpha', \beta')}(z) dz = 1, \end{aligned} \quad (19)$$

where $(n)_r$ is the Pochhammer symbol [24]. With the help of the following integral equation including a Jacobi polynomial written in terms of the hypergeometric function ${}_3F_2(-n, a, b; c, d; y)$ [24]

$$\begin{aligned} & \int_{-1}^{+1} (1-y)^\rho (1+y)^\sigma P_n^{(\alpha', \beta')}(y) dy \\ &= \frac{2^{\rho+\sigma+1} \Gamma(\rho+1) \Gamma(\sigma+1) \Gamma(n+1+\alpha')}{n! \Gamma(\rho+\sigma+2) \Gamma(\alpha'+1)} {}_3F_2(-n, \\ & n + \alpha' + \beta' + 1, \rho + 1; \alpha' + 1, \rho + \sigma + 2; 1), \end{aligned} \quad (20)$$

with the conditions $\text{Re } \rho > -1$ and $\text{Re } \sigma > -1$; the normalization constant is computed

$$N_i = \sqrt{\Gamma' \Gamma''}, \quad (21)$$

where

$$\begin{aligned} \Gamma' &= \frac{2\beta(n!)^2 \Gamma(\alpha' + \beta') \Gamma(\alpha' + 1)}{\Gamma(\beta) \Gamma(n + \alpha' + 1)}, \\ \Gamma'' &= \left[\sum_{\ell=0}^n \frac{1}{\ell!} (-n)_\ell (n + \alpha' + \beta' + 1)_\ell (n + \alpha' + 1)_\ell {}_3F_2(-n, n + \alpha' + \beta' + 1, \alpha' + \ell; \alpha' + 1, \alpha' + \beta' + \ell; 1) \right]^{-1}. \end{aligned} \quad (22)$$

3.1. Results. For the upper component ($i = 1$), we write the potential parameter as $m_1 = V_0 + S_0$ which gives the following energy eigenvalue equation from (14) as

$$\begin{aligned} & \left[\sqrt{m_0^2 - E^2 + 2QV_0(E + m_0) + a_1} + \sqrt{m_0^2 - E^2} \right. \\ & \left. + \beta(2n + 1) \right]^2 - (\beta + QV_1)^2 = 0, \end{aligned} \quad (23)$$

TABLE 1: The variation of energy eigenvalues with different n for $\beta = 1$, $m_0 = 50$, $V_1 = V_2 = 1.5$, $Q = 100$, $V_0 = 1$, $S_0 = 2$.

n	Energy values for $i = 1$	Energy values for $i = 2$
0	-15.97700	-28.37410
1	-17.93500	-30.26090
2	-19.78040	-32.01920
3	-21.52680	-33.66200
4	-23.18510	-35.19950

with $a_1 = Q^2 V_1^2$ and $b_1 = Q V_1 (\beta + Q V_1)$. The corresponding wave functions are given by

$$\phi_1(s) = N_1 s^{\alpha'/2} (1-s)^{\beta'/2} P_n^{(\alpha', \beta')} (1-2s), \quad (24)$$

with $\alpha' = (1/\beta) \sqrt{m_0^2 - E^2 + 2QV_0(E + m_0) + Q^2 V_1^2}$, and $\beta' = (1/\beta) \sqrt{m_0^2 - E^2}$.

$$\alpha' = \frac{1}{\beta} \sqrt{m_0^2 - E^2 + 2QV_0(E + m_0) - 4Qm_0S_0 + Q^2 V_1^2 - 4Q^2 S_0 (V_0 - S_0)},$$

$$\beta' = \frac{1}{\beta} \sqrt{m_0^2 - E^2}. \quad (27)$$

Before going further, we tend to give some numerical results obtained from (23) and (25) in Table 1 where one observes that the energy values for upper component are larger than the ones for lower component, and numerical values for both components decrease while quantum number n increases.

Now we can modify our results to the case where the mass is constant. Let us first write $m_1 = 0$ which means $V_0 = -S_0$. This situation corresponds to the spin symmetric case for the Dirac equation in 3 + 1 dimension [20, 21, 26]. We write the energy eigenvalue equation for the Dirac equation with the generalized Hulthén potential as

$$\left[\sqrt{m_0^2 - E^2 + 2QV_0(E + m_0) + a_1} + \sqrt{m_0^2 - E^2} + \beta(2n+1) \right]^2 - (\beta + QV_1)^2 = 0, \quad (28)$$

with $a_1 = Q^2 V_1^2$, and $b_1 = Q V_1 (\beta + Q V_1)$. Here, one has to choose the positive eigenvalues because in the case of the spin symmetry only the bound states with positive energy occur [26]. For the constant mass, the wave functions with normalization constant given in (21) are

$$\phi_1(s) = N_1 s^{\alpha'/2} (1-s)^{\beta'/2} P_n^{(\alpha', \beta')} (1-2s), \quad (29)$$

with $\alpha' = (1/\beta) \sqrt{m_0^2 - E^2 + 2QV_0(E + m_0) + Q^2 V_1^2}$, and $\beta' = (1/\beta) \sqrt{m_0^2 - E^2}$. The case where $m_2 = 0$ giving $V_0 = +S_0$ corresponds to the pseudospin symmetric situation for

For the lower component ($i = 2$), we have the following energy eigenvalue equation from (14) with the potential parameter $m_2 = V_0 - S_0$

$$\left[\sqrt{m_0^2 - E^2 + 2QV_0(E + m_0) + a_2} + \sqrt{m_0^2 - E^2} + \beta(2n+1) \right]^2 - [(-\beta + QV_2)^2 - 4Q^2 S_0 (V_0 - S_0)] = 0, \quad (25)$$

with $a_2 = -4Qm_0S_0 + Q^2 V_1^2 - 4Q^2 S_0 (V_0 - S_0)$ and $b_2 = -2Qm_0 - 4Q^2 S_0 (V_0 - S_0) + QV_1 (\beta + QV_1)$. The corresponding wave functions are written as

$$\phi_2(s) = N_2 s^{\alpha'/2} (1-s)^{\beta'/2} P_n^{(\alpha', \beta')} (1-2s), \quad (26)$$

with

the Dirac equation [20, 21, 26], and the energy eigenvalue equation becomes

$$\left[\sqrt{m_0^2 - E^2 + 2QV_0(E + m_0) + a_2} + \sqrt{m_0^2 - E^2} - \beta(2n+1) \right]^2 - (\beta + QV_1)^2 = 0, \quad (30)$$

with $a_2 = -4Qm_0S_0 + Q^2 V_1^2$, and $b_2 = -4Qm_0S_0 + QV_1 (\beta + QV_1)$. The last equation can give negative or positive eigenvalues, but one uses only negative energy eigenvalues because negative energy states can exist in the case of pseudospin symmetry [26]. The corresponding wave functions are given as

$$\phi_2(s) = N_2 s^{\alpha'/2} (1-s)^{\beta'/2} P_n^{(\alpha', \beta')} (1-2s), \quad (31)$$

with

$$\alpha' = \frac{1}{\beta} \sqrt{m_0^2 - E^2 + 2QV_0(E + m_0) - 4Qm_0S_0 + Q^2 V_1^2},$$

$$\beta' = \frac{1}{\beta} \sqrt{m_0^2 - E^2}. \quad (32)$$

The pseudospin symmetry, as a hidden symmetry in atomic nuclei, has been suggested firstly by Arima and coworkers [29, 30]. After the pseudospin symmetry has found a place as a relativistic symmetry in literature, some special features, spin symmetry, for example, have been studied [31].

There have been many efforts about the recent progress on pseudospin and spin symmetry in different systems such as stable, exotic, deformed, and spherical nuclei. These efforts extend the subject of “hidden symmetries” in atomic nuclei to include different perspectives such as perturbative study of the pseudospin symmetry and SUSY approach to hidden symmetries combining with similarity renormalization group and studying the source of some particular states which intrude from the major shell above to the shell below forming the nuclear magic numbers 28, 50, 82, and so forth [31].

Finally, we tend to give only the eigenvalue equation for the nonrelativistic limit which can be obtained by using $E - m_0 \sim E$ and $E + m_0 \sim 2m_0$ in (14) ($\hbar = c = 1$)

$$\left[\sqrt{-2m_0E + 4Qm_0V_0 + a_i} + \sqrt{-2m_0E + \beta(2n+1)} \right]^2 - [(\beta + QV_i)^2 + Q^2[(m_i - S_0)^2 - V_0^2]] = 0. \quad (33)$$

The last equation gives two different results for energy eigenvalues, and one should choose the appropriate one.

4. Bound States for PT-Symmetric Forms

Let us now study the case where the potential parameter β is pure imaginary which means that the potential has a complex form as follows:

$$V(x) = QV_0 \frac{\cos(2\beta x) + i \sin(2\beta x)}{\cos(2\beta x) + i \sin(2\beta x) - Q}, \quad (34)$$

with $i = \sqrt{-1}$. This form of the potential in (1) is PT-symmetric because it satisfies

$$[V(-x)]^* = V(x), \quad (35)$$

which is non-Hermitian [4]. The bound state spectra of the generalized, PT-symmetric Hulthén potential can be found from (14), and we write it explicitly as

$$\sqrt{m_0^2 - E^2 + 2QV_0(E + m_0) + a_i} + \sqrt{m_0^2 - E^2 + i\beta n'} + \lambda \sqrt{\Gamma_i} = 0, \quad (36)$$

with

$$\begin{aligned} n' &= 2n + 1, \\ \Gamma_i &= (i\beta + QV_i)^2 + Q^2[(m_i - S_0)^2 - V_0^2], \end{aligned} \quad (37)$$

where $\lambda = \pm 1$. The obtained result says that four different solutions can be possible, and we expect that one of them, at least, gives a real spectra for the PT-symmetric Hulthén potential [4]. The corresponding upper and lower components of the Dirac spinor are written with the help of (15) as

$$\phi_i(s) \sim s^{\alpha''/2} (1-s)^{\beta''/2} P_n^{(\alpha'', \beta'')} (1-2s), \quad (38)$$

where $\alpha'' = -\alpha'$, and $\beta'' = (1/\beta)\sqrt{E^2 - m_0^2} = -\beta'$. We write the upper and lower spinor component without the normalization constant, but it can be computed in a similar way

given in the above section by using a modified normalization condition written for the non-Hermitian quantum systems [25, 37, 38].

An interesting form of the potential can be obtained if all potential parameters are taken as pure imaginary; namely, $V_0 \rightarrow iV_0(S_0 \rightarrow iS_0)$, $\beta \rightarrow i\beta$, $q \rightarrow iq$, giving

$$V(x) = V_0 \frac{q - \sin(\beta x) - i \cos(\beta x)}{q^2 - 2q \sin(\beta x) + 1} = V^* \left(\frac{\pi}{2} - x \right), \quad (39)$$

which is PT-symmetric but non-Hermitian (and also pseudo-Hermitian) [4, 25]. The energy spectra for this form of the potential are written as

$$\begin{aligned} \sqrt{m_0^2 - E^2 + 2QV_0(E + m_0) + a_i} + \sqrt{m_0^2 - E^2 + i\beta n'} \\ + \lambda \sqrt{\Gamma_i} = 0, \end{aligned} \quad (40)$$

with

$$\begin{aligned} n' &= 2n + 1, \\ \Gamma_i &= -(\beta - QV_i)^2 + Q^2[(m_i - iS_0)^2 + V_0^2], \\ a_i &= -2m_0Q(V_0 + S_0) - 2iQm_0m_i \\ &\quad - Q^2[(m_i - iS_0)^2 + V_0^2] - Q^2V_i^2. \end{aligned} \quad (41)$$

It is worthwhile to say that one has to choose the result giving a real spectrum obtained from (40) for the above form of the generalized Hulthén potential.

5. Solutions for Specific q -Values

The value of $q = +1$ corresponds to the standard Hulthén potential for which the energy equation is obtained from (14), and the upper and lower components of Dirac spinor from (15). For $q = -1$, the generalized Hulthén potential gives

$$V(x) = -V_0 \frac{e^{-2\beta x}}{e^{-2\beta x} + 1}, \quad (42)$$

which is the Woods-Saxon potential. The energy levels and upper and lower components of Dirac spinor for this form are obtained from (14) and (15), respectively.

For $q = 0$, we have

$$V(x) = -V_0 e^{-2\beta x}, \quad (43)$$

which is the exponential potential, and it is known that there is no explicit expression for the bound states for nonrelativistic, and relativistic wave equations [39–41]. Hence we have to reconsider the problem by using the new variable $s = e^{-2\beta x}$ giving

$$\begin{aligned} \left\{ \frac{d^2}{ds^2} + \frac{1}{s} \frac{d}{ds} - \frac{1}{s^2} \left[\frac{A'}{4\beta^2} - \frac{2B'}{4\beta^2} s + \frac{C'}{4\beta^2} s^2 \right] \right\} \phi_i(s) \\ = 0, \end{aligned} \quad (44)$$

with

$$\begin{aligned} A' &= m_0^2 - E^2, \\ B' &= m_0 m_i - m_0 S_0 + EV_0 + \beta V_i, \\ C' &= V_i^2 + (m_i - S_0)^2 - V_0^2. \end{aligned} \quad (45)$$

We compare (44) with (A.1) in Appendix, and with the help of (A.3) we obtain the parameter set

$$\begin{aligned} \alpha_1 &= 1, \\ \alpha_2 &= \alpha_3 = 0, \\ \xi_1 &= \frac{C'}{4\beta^2}, \\ \xi_2 &= \frac{2B'}{4\beta^2}, \\ \xi_3 &= \frac{A'}{4\beta^2}, \\ \alpha_4 &= \alpha_5 = 0, \\ \alpha_6 &= \xi_1, \\ \alpha_7 &= -\xi_2, \\ \alpha_8 &= \xi_3, \\ \alpha_9 &= \xi_1, \\ \alpha_{10} &= 1 + 2\sqrt{\xi_3}, \\ \alpha_{11} &= 2\sqrt{\xi_1}, \\ \alpha_{12} &= \sqrt{\xi_3}, \\ \alpha_{13} &= \sqrt{\xi_1}. \end{aligned} \quad (46)$$

Equation (A.10) gives the upper and lower component of Dirac spinor for exponential potential in (43)

$$\begin{aligned} \phi_i(s) & \\ & \sim s^{(1/2\beta)\sqrt{m_0^2-E^2}} e^{(1/2\beta)\sqrt{(m_i-S_0)^2-V_0^2+V_i^2}} L_n^{(1/\beta)\sqrt{m_0^2-E^2}} \left(2\sqrt{\xi_1} s \right). \end{aligned} \quad (47)$$

6. Conclusions

We have analyzed the analytical solutions of the Dirac equation with scalar-vector-pseudoscalar generalized Hulthén potential in 1 + 1 dimension within the position-dependent mass formalism. We have reduced the two extended effective-mass versions of coupled equations written for the upper and lower component to a form of analytical solvable equations by relating the mass function with the potentials. We have given both energy eigenvalue equations and normalized wave functions in closed forms. We have also computed the results for

the case where the mass is constant which correspond to spin and pseudospin symmetric cases in Dirac equation. We have written the results for the bound states in the nonrelativistic case. We have studied the bound state spectrum and the corresponding normalized upper and lower component of Dirac spinor for the complex, generalized Hulthén potential which are PT-symmetric, non-Hermitian forms of the potential.

Appendix

The general form of a second-order differential equation which is solved by using the parametric generalization of the Nikiforov-Uvarov method [27] is as follows:

$$\begin{aligned} \frac{d^2 F(s)}{ds^2} + \frac{\alpha_1 - \alpha_2 s}{s(1 - \alpha_3 s)} \frac{dF(s)}{ds} - \frac{\xi_1 s^2 - \xi_2 s + \xi_3}{[s(1 - \alpha_3 s)]^2} F(s) \\ = 0, \end{aligned} \quad (A.1)$$

with the quantization rule

$$\begin{aligned} \alpha_2 n - (2n + 1) \alpha_5 + (2n + 1) (\sqrt{\alpha_9} + \alpha_3 \sqrt{\alpha_8}) \\ + n(n - 1) \alpha_3 + \alpha_7 + 2\alpha_3 \alpha_8 + 2\sqrt{\alpha_8 \alpha_9} = 0, \end{aligned} \quad (A.2)$$

where $n = 0, 1, 2, \dots$

The parameters α'_i 's within this approach are defined as

$$\begin{aligned} \alpha_4 &= \frac{1}{2} (1 - \alpha_1), \\ \alpha_5 &= \frac{1}{2} (\alpha_2 - 2\alpha_3), \\ \alpha_6 &= \alpha_5^2 + \xi_1, \\ \alpha_7 &= 2\alpha_4 \alpha_5 - \xi_2, \\ \alpha_8 &= \alpha_4^2 + \xi_3, \\ \alpha_9 &= \alpha_3 (\alpha_7 + \alpha_3 \alpha_8) + \alpha_6. \end{aligned} \quad (A.3)$$

The corresponding wave functions are given in terms of the parameters α_i [27]

$$\begin{aligned} F(s) &= N s^{\alpha_{12}} (1 - \alpha_3 s)^{-\alpha_{12} - \alpha_{13}/\alpha_3} \\ & \cdot P_n^{(\alpha_{10}-1, \alpha_{11}/\alpha_3 - \alpha_{10}-1)} (1 - 2\alpha_3 s), \end{aligned} \quad (A.4)$$

where

$$\begin{aligned} \alpha_{10} &= \alpha_1 + 2\alpha_4 + 2\sqrt{\alpha_8}, \\ \alpha_{11} &= \alpha_2 - 2\alpha_5 + 2(\sqrt{\alpha_9} + \alpha_3 \sqrt{\alpha_8}), \\ \alpha_{12} &= \alpha_4 + \sqrt{\alpha_8}, \\ \alpha_{13} &= \alpha_5 - (\sqrt{\alpha_9} + \alpha_3 \sqrt{\alpha_8}), \end{aligned} \quad (A.5)$$

with the Jacobi polynomials $P_n^{(\sigma_1, \sigma_2)}(s)$ and a normalization constant N .

For the second independent solution the quantization condition is given by

$$\alpha_2 n + (1 - 2n) \alpha_5 + (2n + 1) (\sqrt{\alpha_9} - \alpha_3 \sqrt{\alpha_8}) + n(n - 1) \alpha_3 + \alpha_7 + 2\alpha_3 \alpha_8 - 2\sqrt{\alpha_8 \alpha_9} = 0, \quad (\text{A.6})$$

with the corresponding wave functions

$$F(s) = N s^{\alpha_{12}^*} (1 - \alpha_3 s)^{-\alpha_{12}^* - \alpha_{13}^* / \alpha_3} \cdot P_n^{(\alpha_{10}^* - 1, \alpha_{11}^* / \alpha_3 - \alpha_{10}^* - 1)} (1 - 2\alpha_3 s), \quad (\text{A.7})$$

where

$$\begin{aligned} \alpha_{10}^* &= \alpha_1 + 2\alpha_4 - 2\sqrt{\alpha_8}, \\ \alpha_{11}^* &= \alpha_2 - 2\alpha_5 - 2(\sqrt{\alpha_9} - \alpha_3 \sqrt{\alpha_8}), \\ \alpha_{12}^* &= \alpha_4 - \sqrt{\alpha_8}, \\ \alpha_{13}^* &= \alpha_5 - (\sqrt{\alpha_9} - \alpha_3 \sqrt{\alpha_8}). \end{aligned} \quad (\text{A.8})$$

If a situation is appearing in the problem such as $\alpha_3 = 0$, then the quantization rule in (A.2) becomes

$$\begin{aligned} (\alpha_2 - 2\alpha_5) n + (2n + 1) (\sqrt{\alpha_9} - \alpha_3 \sqrt{\alpha_8}) + n(n - 1) \alpha_3 + \alpha_7 + 2\alpha_3 \alpha_8 - 2\sqrt{\alpha_8 \alpha_9} + \alpha_5 \\ = 0, \end{aligned} \quad (\text{A.9})$$

with the corresponding wave functions

$$F(s) = N s^{\alpha_{12}} e^{\alpha_{13} s} L_n^{\alpha_{10} - 1}(\alpha_{11} s), \quad (\text{A.10})$$

when the limits become $\lim_{\alpha_3 \rightarrow 0} P_n^{(\alpha_{10} - 1, \alpha_{11} / \alpha_3 - \alpha_{10} - 1)}(1 - 2\alpha_3 s) = L_n^{\alpha_{10} - 1}(\alpha_{11} s)$ and $\lim_{\alpha_3 \rightarrow 0} (1 - \alpha_3 s)^{-\alpha_{12} - \alpha_{13} / \alpha_3} = e^{\alpha_{13} s}$ with generalized Laguerre polynomials $L_n^{\alpha}(s)$.

Disclosure

Present address for the author is Department of Mathematical Science, City University London, UK.

Competing Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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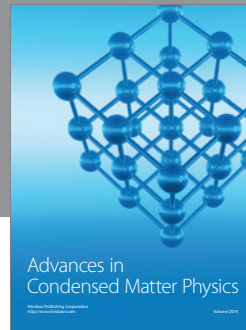
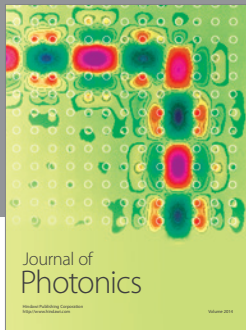
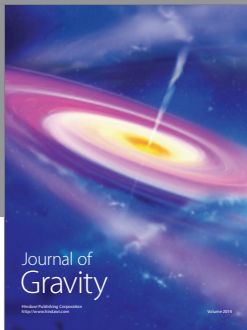
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