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Annulus criteria for mixed nonlinear elliptic differential equations

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ABSTRACT

New oscillation criteria are obtained for forced second order elliptic partial differential equations with damping and mixed nonlinearities of the form

$$\nabla \cdot \left(A(x)|\nabla y|^{\alpha-1}\nabla y\right) + \langle b(x), |\nabla y|^{\alpha-1}\nabla y\rangle + f(x,y) = e(x), \quad x \in \Omega$$

where Ω is an exterior domain in $\mathbb{R}^N$,

$$f(x,y) = c(x)|y|^{\alpha-1}y + c_1(x)|y|^{\beta-1}y + c_2(x)|y|^{\gamma-1}y$$

and

$$\beta > \alpha > \gamma > 0.$$

It is assumed that $A = (a_{ij})_{N\times N}$ is a real symmetric positive definite matrix function, $b = (b_i)_{\mathbb{N}\times 1}$ is a real vector function, $a_{ij} \in C^{1+\mu}_{\text{loc}}(\Omega, \mathbb{R})$, and $b_i, c, c_1, c_2, e \in C^{\mu}_{\text{loc}}(\Omega, \mathbb{R})$ for all i, j, for some $\mu \in (0, 1)$.

Examples are given to illustrate the results.

1. Introduction

Let $\Omega(r_0) = \{x \in \mathbb{R}^N : |x| \geq r_0\}$, where $r_0 \geq 0$ is a fixed real number, N is a positive integer, and $|x|$ is the Euclidean norm in $\mathbb{R}^N$. Denote by $\langle , \rangle$ the usual scalar product in $\mathbb{R}^N$, and $\nabla = (\partial/\partial x_i)_{\mathbb{N}\times 1}$ as usual.

We consider the second order elliptic partial differential equation with mixed nonlinearities of the form

$$\nabla \cdot (A(x)|\nabla y|^{\alpha-1}\nabla y) + \langle b(x), |\nabla y|^{\alpha-1}\nabla y\rangle + f(x,y) = e(x), \quad x \in \Omega(r_0), \tag{1.1}$$

with

$$f(x,y) = c(x)|y|^{\alpha-1}y + c_1(x)|y|^{\beta-1}y + c_2(x)|y|^{\gamma-1}y, \tag{1.2}$$

$$\beta > \alpha > \gamma > 0, \tag{1.3}$$

where

(i) $A = (a_{ij})_{N\times N}$ is a real symmetric positive definite matrix function,
(ii) $b = (b_i)_{\mathbb{N}\times 1}$ is a real vector function,
(iii) $a_{ij} \in C^{1+\mu}_{\text{loc}}(\Omega(r_0), \mathbb{R})$ for some $\mu \in (0, 1)$,
(iv) $b_i, c, c_1, c_2, e \in C^{\mu}_{\text{loc}}(\Omega(r_0), \mathbb{R})$.

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A function $y \in C^{1+\mu}_{\mathrm{loc}}(\Omega(r_0), \mathbb{R})$ with the property $a_{ij}|\nabla y|^{\alpha-1}\partial y/\partial x_i \in C^{1+\mu}_{\mathrm{loc}}(\Omega(r_0), \mathbb{R})$ is said to be a solution of Eq. (1.1) in $\Omega(r_0)$ provided that $y(x)$ satisfies Eq. (1.1) for all $x \in \Omega(r_0)$. We restrict our attention only to solutions $y(x)$ of Eq. (1.1) which satisfy $\sup\{|y(x)| : x \in \Omega(r)\} > 0$ for any $r \geq r_0$. Such a solution is called oscillatory if the set $\{x \in \Omega(r) : y(x) = 0\}$ is unbounded; otherwise it is said to be nonoscillatory. Eq. (1.1) is oscillatory if all solutions are oscillatory. For the existence and as well as the oscillation theory of nonlinear elliptic differential equations, we refer the reader in particular to the monographs [1,2].

The oscillation of second-order elliptic equations has been investigated by many authors via employing the known techniques used in second-order ordinary differential equations. Below we provide a short review of the results available in the literature.

Noussair and Swanson [3] first extended Wintner-type oscillation criteria to the semilinear elliptic equation

$$\nabla \cdot (A(x)\nabla y) + p(x)f(y) = 0, \quad x \in \Omega(r_0) \tag{1.4}$$

based on N-dimensional vector partial Riccati type transformation

$$w(x) = -\frac{\alpha(|x|)}{f(y(x))}(A\nabla y)(x)$$

where $\alpha \in C^2(0, \infty)$ is an arbitrary positive function, see also [4] for some related results.

Zuang et al. [5] presented Kamenev-type oscillation results for damped elliptic equations of the form

$$\nabla \cdot (A(x)\nabla y) + b^T(x)\nabla y + C(x, y) = 0 \tag{1.5}$$

in the special case $C(x, y) = c(x)f(y)$. Other types of criteria such as Philos, Leighton and Hille-type for Eq. (1.5) have also been established. For some recent contributions, we refer the reader to the papers [6,7] and references cited therein.

Marik [8], by using a radialization method, derived oscillation criteria for half-linear partial differential equations of the form

$$\nabla \cdot (A(x)|\nabla y|^{\alpha-1}\nabla y) + \langle b(x), |\nabla y|^{\alpha-1}\nabla y\rangle + c(x)|y|^{\alpha-1}y = 0. \tag{1.6}$$

The method is based on a comparison with half-linear ordinary differential equations.

Yoshida [9] has studied the oscillation of super-half-linear-sub-half-linear damped elliptic equations of the form

$$\nabla \cdot (A(x)|\nabla y|^{\alpha-1}\nabla y) + (\alpha + 1)B(x)(|\nabla y|^{\alpha-1}\nabla y) + C(x)|y|^{\beta-1}y + D(x)|y|^{\gamma-1}y = f(x) \tag{1.7}$$

where $0 < \gamma < \alpha < \beta$, by utilizing a Picone-type inequality. In particular, it is shown that each nonoscillatory solution $y(x)$ satisfies $\liminf_{|x|\to\infty} |y(x)| = 0$ under some hypotheses. Yoshida's work was motivated by that of Li and Li [10] on second order nonlinear ordinary differential equations. See also Jaros et al. [11] for a related work when $B(x) \equiv 0$ and $f(x) \equiv 0$. The approach is to reduce the multi-dimensional oscillation problems to one-dimensional oscillation problems for half-linear ordinary differential equations.

All of the results mentioned above involve the integral of the coefficients appearing in the equation and hence require the information of the coefficients on the entire exterior domain Ω. However, as it is observed for ordinary differential equations the oscillation is only an interval property by the Sturm Separation Theorem. Due to this fact there have appeared several works making use of information of the coefficient functions on a union of intervals rather than on an infinite interval. This type of a criterion is called an interval oscillation criterion for ordinary differential equations, see [12–17] and the references cited therein. Therefore, our aim is to make a contribution in this direction by establishing oscillation criteria which are based on a sequence of annuluses

$$\{x \in \mathbb{R}^N : a_i \leq |x| \leq b_i,\ i \in \mathbb{N}\}.$$

To the best of our knowledge only a few works exist for partial differential equations concerning annulus oscillation criteria. For a sampling of works done, we may refer in particular to the following investigations.

Zhuang [18,19] has extended the interval oscillation criteria given by Yang [20] to forced elliptic equations of the form

$$\nabla \cdot (A(x)\nabla y) + q(x)f(y) = e(x) \tag{1.8}$$

and

$$\nabla \cdot (A(x)\nabla y) + B^T(x)\nabla y + q(x)f(y) = e(x). \tag{1.9}$$

In [21,22], Kamenev-type oscillation criteria are given for

$$\nabla \cdot (A(x)\nabla y) + B^T(x)\nabla y + C(x, y) = e(x), \tag{1.10}$$

where $C(x, y)$ has mixed nonlinearities in y.

By using a Picone-type inequality, Yoshida [23] has studied

$$\nabla \cdot (a(x)|\nabla y|^{\alpha-1}\nabla y) + (\alpha + 1)B(x)(|\nabla y|^{\alpha-1}\nabla y) + C(x)|y|^{\beta-1}y = f(x) \tag{1.11}$$

when C is nonnegative on a bounded domain G. The domain is divided into subdomains in such a way that $f(x) > 0$ in G_1 and $f(x) < 0$ in G_2.

Compared to Eqs. (1.7)–(1.11), there seems to be nothing known on the annulus oscillation of Eq. (1.1). As there is a demand for further research on oscillation of elliptic differential equations as such equations arise from a variety of physical phenomena, we aim to fill a gap by establishing annulus oscillation criteria for elliptic differential equations of the form Eq. (1.1).

2. Preliminaries

First we introduce some notation for use in the next section. Let

$$S(r) = \{x \in \mathbb{R}^N : |x| = r\},$$
$$\Omega[a,b] = \{x \in \mathbb{R}^N : a \le |x| \le b\},$$

and

$$\Omega(a,b) = \{x \in \mathbb{R}^N : a < |x| < b\}.$$

$A^{-1}(x)$ denotes the inverse of the matrix $A(x)$, $\nu(x) = x/|x|$ is the outside unit normal vector to the sphere $S(|x|)$, $\mathrm{d}\sigma$ represents the area element of the sphere $S(|x|)$, $\lambda_{\max}$ and $\lambda_{\min}$ denote the largest and smallest eigenvalue of the matrix $A(x)$, respectively. Finally, $|A(x)|$ is the matrix norm induced by the vector norm in $\mathbb{R}^N$, i.e., $|A(x)| = \sup_{v\neq 0} |A(x)v|/|v|$, $v \in \mathbb{R}^N$.

We need the following preparatory lemmas.

Lemma 2.1 (*Arithmetic–Geometric Mean Inequality*). *If $p_i \ge 0$ and $q_i > 0$ for all $i = 1, 2, \dots, m$, and $\sum_{i=0}^m q_i = 1$, then*

$$\sum_{i=0}^{m} p_i q_i \ge \prod_{i=0}^{m} p_i^{q_i}. \tag{2.1}$$

Lemma 2.2 (*Young's Inequality*). *If $p > 1$ and $q > 1$ are conjugate numbers, i.e. $\frac{1}{p} + \frac{1}{q} = 1$, then for any $u, v \in R$*

$$\frac{|u|^p}{p} + \frac{|v|^q}{q} \ge |uv|$$

and equality holds iff $u = |v|^{q-2}v$.

Let $\beta > \gamma$. Put $u = A^{\gamma/\beta}y^\gamma$, $p = \beta/\gamma$, and $v = (B\gamma)^{1-\gamma/\beta}(\beta-\gamma)^{\gamma/\beta-1}$. It follows from Lemma 2.2 that

$$Ay^\beta + B \ge \beta\gamma^{-\gamma/\beta}(\beta-\gamma)^{(\gamma/\beta)-1}A^{\gamma/\beta}B^{1-\gamma/\beta}y^\gamma \tag{2.2}$$

for all $A, B, y \ge 0$. Rewriting the above inequality, we also have

$$Cy^\gamma - D \le \beta^{-\beta/\gamma}\gamma(\beta-\gamma)^{(\beta/\gamma)-1}C^{\beta/\gamma}D^{1-\beta/\gamma}y^\beta \tag{2.3}$$

for all $C, y \ge 0$ and $D > 0$.

3. The main results

For any $[a,b] \subset [r_0, \infty)$, we define

$$D(a,b) = \{u \in C^1([a,b],\mathbb{R}) : u(t) \not\equiv 0 \quad \forall t \in (a,b), u(a) = 0 = u(b)\}.$$

Theorem 3.1. *Let $\eta_0 > 0$ be fixed so that $\beta\eta_0 < \beta - \alpha$. Let*

$$\eta_1 = \frac{\alpha - \gamma(1-\eta_0)}{\beta-\gamma}, \qquad \eta_2 = \frac{\beta(1-\eta_0)-\alpha}{\beta-\gamma}.$$

Suppose that for any given $r \ge r_0$, there exist a_1, b_1, a_2, b_2 such that $r \le a_1 < b_1$, $r \le a_2 < b_2$ and that

$$c_i(x) \ge 0 \quad \text{for } x \in \Omega[a_1,b_1] \cup \Omega[a_2,b_2], \ (i = 1,2) \tag{3.1}$$

and

$$(-1)^k e(x) \ge 0 (\not\equiv 0) \quad \text{for } x \in \Omega[a_k,b_k], \ (k = 1,2). \tag{3.2}$$

If there exists a function $u \in D(a_k,b_k)$ for $k = 1, 2$ such that

$$\int_{\Omega[a_k,b_k]}\left[C(x)u^{\alpha+1}(|x|) - \frac{1}{(\alpha+1)^{\alpha+1}\lambda_{\min}^{\alpha}(x)}|A(x)|^{\alpha+1}|H(x)|^{\alpha+1}\right]\mathrm{d}x > 0 \tag{3.3}$$

where

$$H(x) = (\alpha + 1)\nabla u(|x|) - b^T(x)A^{-1}(x)u(|x|)$$

and

$$C(x) = c(x) + (|e(x)|/\eta_0)^{\eta_0}(c_1(x)/\eta_1)^{\eta_1}(c_2(x)/\eta_2)^{\eta_2},$$

then Eq. (1.1) is oscillatory.

Proof. To arrive at a contradiction, let us assume that there is a nonoscillatory solution y of (1.1). Without loss of generality, we take $y(x)$ to be positive for all $x \in \Omega(a_0)$ for some $a_0 \geq r_0$. In view of (3.1) and (3.2), we can choose $a_1 \geq a_0$ sufficiently large so that $c_1(x) \geq 0$, $c_2(x) \geq 0$ and $e(x) \leq 0$ for all $x \in \Omega[a_1, b_1]$.

Let $x \in \Omega[a_1, b_1]$. Then, setting

$$u_0 = \frac{1}{\eta_0}\frac{|e(x)|}{y^\alpha(x)}, \qquad u_1 = \frac{1}{\eta_1}c_1(x)y^{\beta-\alpha}(x), \qquad u_2 = \frac{1}{\eta_2}c_2(x)y^{\gamma-\alpha}(x),$$

we may write

$$-e(x) + c_1(x)y^\beta + c_2(x)y^\gamma = y^\alpha(\eta_0 u_0 + \eta_1 u_1 + \eta_2 u_2),$$

and hence by Lemma 2.1, we have

$$-e(x) + c_1(x)y^\beta + c_2(x)y^\gamma \geq |e(x)|^{\eta_0} y^\alpha \left(\prod_{i=0}^{2} \eta_i{}^{-\eta_i}\right)\prod_{i=1}^{2} c_i^{\eta_i}(x).$$

Using this inequality in (1.1) results in

$$\nabla \cdot (A(x)|\nabla y|^{\alpha-1}\nabla y) + \langle b(x), |\nabla y|^{\alpha-1}\nabla y\rangle + C(x)y^\alpha \leq 0. \tag{3.4}$$

Now we make use of a Riccati-like transformation

$$w(x) = \frac{1}{y^\alpha(x)}A(x)|\nabla y|^{\alpha-1}\nabla y$$

in (3.4) to get

$$\nabla \cdot w(x) \leq -C(x) - \langle A^{-1}(x)b(x), w(x)\rangle - \frac{\alpha}{y^{\alpha+1}}\langle A(x)|\nabla y|^{\alpha-1}\nabla y, \nabla y\rangle. \tag{3.5}$$

Noting that

$$|w(x)| \leq |A(x)|\frac{|\nabla y|^\alpha}{y^\alpha}, \quad (\nabla y)^T A(x)\nabla y \geq \lambda_{\min}(x)|\nabla y|^2,$$

we obtain from (3.5),

$$\nabla \cdot w(x) \leq -C(x) - \langle A^{-1}(x)b(x), w(x)\rangle - \frac{\alpha\lambda_{\min}(x)}{|A(x)|^{\frac{\alpha+1}{\alpha}}}|w(x)|^{\frac{\alpha+1}{\alpha}}. \tag{3.6}$$

Multiplying (3.6) by $u^{\alpha+1}(|x|)$ and integrating over the annulus $\Omega[a_1, b_1]$, we get

$$\begin{aligned}\int_{\Omega[a_1,b_1]} C(x)u^{\alpha+1}(|x|)\mathrm{d}x \leq &-\int_{\Omega[a_1,b_1]} u^{\alpha+1}(|x|)\nabla \cdot w(x)\mathrm{d}x - \int_{\Omega[a_1,b_1]} b^T(x)A^{-1}(x)u^{\alpha+1}(|x|)w(x)\mathrm{d}x \\ &- \int_{\Omega[a_1,b_1]} \frac{\alpha\lambda_{\min}(x)}{|A(x)|^{\frac{\alpha+1}{\alpha}}}u^{\alpha+1}(|x|)|w(x)|^{\frac{\alpha+1}{\alpha}}\mathrm{d}x.\end{aligned}$$

On the other hand, we may write that

$$\begin{aligned}\int_{\Omega[a_1,b_1]} u^{\alpha+1}(|x|)\nabla \cdot w(x)\mathrm{d}x &= \int_{a_1}^{b_1} u^{\alpha+1}(r)\left(\frac{\mathrm{d}}{\mathrm{d}r}\int_{S(r)}\langle \nu^T, w\rangle \mathrm{d}\sigma\right)\mathrm{d}r \\ &= -\int_{\Omega[a_1,b_1]} (\alpha+1)u^\alpha(|x|)\nabla u(|x|)w(x)\mathrm{d}x.\end{aligned}$$

Thus, we have

$$\int_{\Omega[a_1,b_1]} C(x)u^{\alpha+1}(|x|)\mathrm{d}x \leq \int_{\Omega[a_1,b_1]} \Theta(x)u^{\alpha+1}(|x|)w(x)\mathrm{d}x - \int_{\Omega[a_1,b_1]} \frac{\alpha\lambda_{\min}(x)}{|A(x)|^{\frac{\alpha+1}{\alpha}}}u^{\alpha+1}(|x|)|w(x)|^{\frac{\alpha+1}{\alpha}}\mathrm{d}x$$

where

$$\Theta(x) = (\alpha+1)\frac{\nabla u(|x|)}{u(|x|)} - b^T(x)A^{-1}(x).$$

Employing Young's inequality we can write

$$\Theta(x)w(x) \le \frac{1}{\lambda_{\min}^{\alpha}(x)(\alpha+1)^{\alpha+1}}|A(x)|^{\alpha+1}|\Theta(x)|^{\alpha+1} + \frac{\alpha\lambda_{\min}(x)}{|A(x)|^{\frac{\alpha+1}{\alpha}}}|w(x)|^{\frac{\alpha+1}{\alpha}}.$$

Substituting this into the first integral above yields

$$\int_{\Omega[a_1,b_1]} C(x)u^{\alpha+1}(|x|)\mathrm{d}x \le \int_{\Omega[a_1,b_1]} \frac{1}{(\alpha+1)^{\alpha+1}\lambda_{\min}^{\alpha}(x)}|A(x)|^{\alpha+1}|\Theta(x)|^{\alpha+1}u^{\alpha+1}(|x|)\mathrm{d}x,$$

which means that

$$\int_{\Omega[a_1,b_1]}\left[C(x)u^{\alpha+1}(|x|) - \frac{1}{(\alpha+1)^{\alpha+1}\lambda_{\min}^{\alpha}(x)}|A(x)|^{\alpha+1}|H(x)|^{\alpha+1}\right]\mathrm{d}x \le 0.$$

But this inequality contradicts (3.3), completing the proof when $y(x)$ is eventually positive.

The proof when $y(x)$ is eventually negative is analogous by repeating the arguments on the annulus $\Omega[a_2, b_2]$ instead of $\Omega[a_1, b_1]$. □

Theorem 3.1 fails to apply when $e(x) \equiv 0$. Fortunately, we have the following theorem in that case.

Theorem 3.2. *Let*

$$\eta_1 = \frac{\alpha-\gamma}{\beta-\gamma}, \qquad \eta_2 = \frac{\beta-\alpha}{\beta-\gamma}.$$

Suppose that for any given $r \ge r_0$, there exist a, b such that $r \le a < b$ and that

$$c_i(x) \ge 0 \quad \text{for } x \in \Omega[a,b],\ (i=1,2). \tag{3.7}$$

If there exists a function $u \in D(a, b)$ such that

$$\int_{\Omega[a,b]}\left[\tilde{C}(x)u^{\alpha+1}(|x|) - \frac{1}{(\alpha+1)^{\alpha+1}\lambda_{\min}^{\alpha}(x)}|A(x)|^{\alpha+1}|H(x)|^{\alpha+1}\right]\mathrm{d}x > 0 \tag{3.8}$$

where

$$H(x) = (\alpha+1)\nabla u(|x|) - b^T(x)A^{-1}(x)u(|x|)$$

and

$$\tilde{C}(x) = c(x) + (c_1(x)/\eta_1)^{\eta_1}(c_2(x)/\eta_2)^{\eta_2},$$

then Eq. (1.1) with $e(x) \equiv 0$ is oscillatory.

Proof. The proof is in fact a simpler version of the proof of Theorem 3.1. It suffices to take $e(x) \equiv 0$ and $\eta_0 = 0$. □

In the next theorem we remove the sign condition on $c_2(x)$ by requiring that $e(x)$ never vanishes in the domain of interest.

Theorem 3.3. *Suppose that for any given $r \ge r_0$, there exist a_1, b_1, a_2, b_2 such that $r \le a_1 < b_1, r \le a_2 < b_2$ and that*

$$c_1(x) \ge 0 \quad \text{for } x \in \Omega[a_1,b_1] \cup \Omega[a_2,b_2] \tag{3.9}$$

and

$$(-1)^k e(x) > 0 \quad \text{for } x \in \Omega[a_k,b_k],\ (k=1,2). \tag{3.10}$$

If there exist a function $u \in D(a_k, b_k)$, positive numbers δ and ϵ with $\delta + \epsilon = 1$ such that

$$\int_{\Omega[a_k,b_k]}\left[\hat{C}(x)u^{\alpha+1}(|x|) - \frac{1}{(\alpha+1)^{\alpha+1}\lambda_{\min}^{\alpha}(x)}|A(x)|^{\alpha+1}|H(x)|^{\alpha+1}\right]\mathrm{d}x > 0 \tag{3.11}$$

for $k = 1, 2$, where

$$\begin{aligned}\hat{C}(x) &= c(x) + \beta(\beta-\alpha)^{\alpha/\beta-1}\alpha^{-\alpha/\beta}\delta^{1-\alpha/\beta}c_1^{\alpha/\beta}(x)|e(x)|^{1-\alpha/\beta}\\ &\quad - (\gamma/\alpha)(1-\gamma/\alpha)^{\alpha/\gamma-1}\epsilon^{1-\alpha/\gamma}(-c_2)_+^{\alpha/\gamma}(x)|e(x)|^{1-\alpha/\gamma}\end{aligned}$$

with

$$(-c_2)_+(x) = \max\{-c_2(x), 0\},$$

and

$$H(x) = (\alpha + 1)\nabla u(|x|) - b^T(x)A^{-1}(x)u(|x|)$$

then Eq. (1.1) is oscillatory.

Proof. Suppose that Eq. (1.1) has a nonoscillatory solution. We may assume that $y(x)$ is positive on $\Omega(a_0)$ for some $a_0 \geq r_0$. Let $y \in \Omega[a_1, b_1]$, where $a_1 \geq a_0$ is sufficiently large. If $y(x)$ is negative, then one can repeat the proof on the interval $\Omega[a_2, b_2]$. We rewrite Eq. (1.1) as follows:

$$\nabla \cdot (A(x)|\nabla y|^{\alpha-1}\nabla y) + \langle b(x), |\nabla y|^{\alpha-1}\nabla y\rangle + c(x)y^\alpha + g(x, y) = 0, \tag{3.12}$$

where

$$\begin{aligned} g(x, y) &= [c_1(x)y^\beta - \delta e(x)] + [c_2(x)y^\gamma - \epsilon e(x)] \\ &\geq [c_1(x)y^\beta + \delta|e(x)|] - [(-c_2)_+(x)y^\gamma - \epsilon|e(x)|]. \end{aligned}$$

Applying the inequalities (2.2) and (2.3) to each summation on the right side with

$$A = c_1(x), \qquad B = \delta|e(x)|, \qquad C = (-c_2)_+(x), \qquad D = \epsilon|e(x)|$$

we see that

$$g(x, y) \geq [P(x) - R(x)]y^\alpha, \tag{3.13}$$

where

$$P(x) = \beta(\beta - \alpha)^{\alpha/\beta-1}\alpha^{-\alpha/\beta}\delta^{1-\alpha/\beta}c_1^{\alpha/\beta}(x)|e(x)|^{1-\alpha/\beta}$$

and

$$R(x) = (\gamma/\alpha)(1 - \gamma/\alpha)^{\alpha/\gamma-1}\epsilon^{1-\alpha/\gamma}(-c_2)_+^{\alpha/\gamma}(x)|e(x)|^{1-\alpha/\gamma}.$$

From (3.12) and (3.13) we obtain

$$\nabla \cdot (A(x)|\nabla y|^{\alpha-1}\nabla y) + \langle b(x), |\nabla y|^{\alpha-1}\nabla y\rangle + C(x)y^\alpha \leq 0. \tag{3.14}$$

The remainder of the proof is the same as that of Theorem 3.1, hence it is omitted. $\square$

In case $0 < \alpha \leq 1$ we have the following theorems. Since steps of the proofs are almost same as the corresponding one above, we just give an outline for the proof of the first theorem.

Theorem 3.4. *Let $0 < \alpha \leq 1$. Suppose that for any given $r \geq r_0$, there exist a_1, b_1, a_2, b_2 such that $r \leq a_1 < b_1$, $r \leq a_2 < b_2$ and that (3.1) and (3.2) are satisfied. Then Eq. (1.1) is oscillatory, if there exists a function $u \in D(a_k, b_k)(k = 1, 2)$ such that*

$$\int_{\Omega[a_k,b_k]} \left[C(x)u^{\alpha+1}(|x|) - \frac{\lambda_{\max}(x)}{(\alpha + 1)^{\alpha+1}}|H(x)|^{\alpha+1}\right] dx > 0, \tag{3.15}$$

where the functions C and H are as given in Theorem 3.1.

Proof. Proceeding as in the proof of Theorem 3.1, we arrive at

$$\nabla \cdot w(x) \leq -C(x) - \langle A^{-1}(x)b(x), w(x)\rangle - \frac{\alpha}{y^{\alpha+1}}\langle A(x)|\nabla y|^{\alpha-1}\nabla y, \nabla y\rangle.$$

Using the estimates

$$\frac{|w(x)|^{1/\alpha}}{|A(x)|^{1/\alpha}} \leq \frac{|\nabla y|}{y}, \quad w^T(x)A^{-1}(x)w(x) \geq \frac{|w(x)|^2}{\lambda_{\max}(x)},$$

we have

$$\frac{\alpha}{y^{\alpha+1}}\langle A(x)|\nabla y|^{\alpha-1}\nabla y, \nabla y\rangle \geq \alpha\frac{|w(x)|^{(1+\alpha)/\alpha}}{\lambda_{\max}^{1/\alpha}(x)}.$$

It follows that

$$\nabla \cdot w(x) \leq -C(x) - \langle A^{-1}(x)b(x), w(x)\rangle - \alpha\frac{|w(x)|^{(1+\alpha)/\alpha}}{\lambda_{\max}^{1/\alpha}(x)}.$$

The remainder of the proof is similar to that of Theorem 3.1. $\square$

Theorem 3.5. *Let $0 < \alpha \le 1$. Suppose that for any given $r \ge r_0$, there exist a, b such that $r \le a < b$ and that (3.7) holds. Then Eq. (1.1) with $e(x) \equiv 0$ is oscillatory, if there exists a function $u \in D(a, b)$ such that*

$$\int_{\Omega[a,b]} \left[\tilde{C}(x) u^{\alpha+1}(|x|) - \frac{\lambda_{\max}(x)}{(\alpha+1)^{\alpha+1}} |H(x)|^{\alpha+1} \right] \mathrm{d}x > 0 \tag{3.16}$$

where the functions $\tilde{C}(x)$ and H are as given in Theorem 3.2.

Theorem 3.6. *Let $0 < \alpha \le 1$. Suppose that for any given $r \ge r_0$, there exist $a_1, b_1, a_2, b_2 \in [r, \infty)$ such that $r \le a_1 < b_1$, $r \le a_2 < b_2$ and that (3.9) and (3.10) hold. Then Eq. (1.1) is oscillatory, if there exist a function $u \in D(a_k, b_k)(k = 1, 2)$, and positive numbers δ and ϵ with $\delta + \epsilon = 1$ such that*

$$\int_{\Omega[a_k,b_k]} \left[\hat{C}(x) u^{\alpha+1}(|x|) - \frac{\lambda_{\max}(x)}{(\alpha+1)^{\alpha+1}} |H(x)|^{\alpha+1} \right] \mathrm{d}x > 0,$$

where the functions $\hat{C}(x)$ and H are as given in Theorem 3.3.

Remark 1. In view of $|A(x)| = \lambda_{\max}(x)$, we may replace the conditions (3.3), (3.8) and (3.11), respectively, by

$$\int_{\Omega} \left[C(x) u^{\alpha+1}(|x|) - \frac{\lambda_{\max}^{\alpha+1}(x)}{(\alpha+1)^{\alpha+1} \lambda_{\min}^{\alpha}(x)} |H(x)|^{\alpha+1} \right] \mathrm{d}x > 0, \tag{3.17}$$

$$\int_{\Omega} \left[\tilde{C}(x) u^{\alpha+1}(|x|) - \frac{\lambda_{\max}^{\alpha+1}(x)}{(\alpha+1)^{\alpha+1} \lambda_{\min}^{\alpha}(x)} |H(x)|^{\alpha+1} \right] \mathrm{d}x > 0, \tag{3.18}$$

and

$$\int_{\Omega} \left[\hat{C}(x) u^{\alpha+1}(|x|) - \frac{\lambda_{\max}^{\alpha+1}(x)}{(\alpha+1)^{\alpha+1} \lambda_{\min}^{\alpha}(x)} |H(x)|^{\alpha+1} \right] \mathrm{d}x > 0. \tag{3.19}$$

Remark 2. If we take $\alpha = 1$ and replace (3.3) by (3.17) in Theorem 3.1, and (3.8) by (3.18) in Theorem 3.3, then we recover [22, Theorem 2.1] and [22, Theorem 2.2], respectively. The case $e \equiv 0$ is not considered in [22], therefore Theorem 3.2 (even for $\alpha = 1$) and Theorem 3.6 are new.

Remark 3. All of the theorems in this paper can be extended to a class of more general equations of the form

$$\nabla \cdot (A(x)|\nabla y|^{\alpha-1}\nabla y) + \langle b(x), |\nabla y|^{\alpha-1}\nabla y \rangle + f(x, y) = e(x).$$

where

$$f(x, y) = c(x)|y|^{\alpha-1}y + \sum_{i=1}^{m} c_i(x)|y|^{\beta_i - 1}y$$

with

$$\beta_m > \beta_{m-1} > \cdots \beta_{k+1} > \alpha > \beta_k > \cdots \beta_1 > 0, \quad m \ge 2.$$

In this case, one should use [15, Lemma 1], see also [12, Lemma 2.4], to get the numbers $\eta_1, \eta_2, \ldots, \eta_m$.

4. Examples

Two examples are given to illustrate the results. We should note that no oscillation criterion in the literature is applicable for these examples.

Example 4.1. Consider the nonlinear PDE

$$\nabla \cdot (|\nabla y|^2 \nabla y) - \frac{(x_1, x_2)^T}{x_1^2 + x_2^2} |\nabla y|^2 \nabla y + m_1 \sin^{1/7} \sqrt{x_1^2 + x_2^2}\, |y|^3 y + m_2 \sin^9 \sqrt{x_1^2 + x_2^2}\, |y| y = \cos^5 \sqrt{x_1^2 + x_2^2}. \tag{4.1}$$

We see that $x = (x_1, x_2) \in \mathbb{R}^2$ and $A(x) = I$ (Identity matrix), $c(x) = 0$, $\alpha = 3$, $\beta = 4$ and $\gamma = 2$. With the choice of $\eta_0 = 1/5$, we have $\eta_1 = 7/10$, $\eta_2 = 1/10$, and

$$C(x) = K|\cos\sqrt{x_1^2+x_2^2}|\sin\sqrt{x_1^2+x_2^2}, \qquad K = \left(\frac{10^8}{7^7}{m_1}^7 m_2\right)^{1/10}.$$

Let $a_1 = 2i\pi + \pi/2$, $b_1 = 2i\pi + \pi$, $a_2 = 2(i+1)\pi$, $b_2 = 2(i+1)\pi + \pi/2$ for $i \in \mathbb{N}$ setting

$$u(x) = \frac{\sin 2\sqrt{x_1^2+x_2^2}}{(x_1^2+x_2^2)^{1/8}},$$

we have

$$H(x) = \frac{8\cos 2\sqrt{x_1^2+x_2^2}}{(x_1^2+x_2^2)^{5/8}}(x_1, x_2).$$

A simple calculation yields for $k = 1, 2$ that

$$\int_{\Omega[a_k,b_k]}\left[C(x)u^4\left(\sqrt{x_1^2+x_2^2}\right) - \frac{1}{4^3}|H(x)|^4\right]\mathrm{d}x = 2\pi\int_{a_k}^{b_k}[K|\cos r|\sin r\sin^4 2r - 64\cos^4 2r]dr = \frac{2^5\pi}{60}(K - 45\pi).$$

Clearly, if

$$m_1^7 m_2 > \frac{7^7(9\pi)^{10}}{2^8},$$

then the condition (3.3) is satisfied, and so Eq. (4.1) is oscillatory by Theorem 3.1.

Example 4.2. Consider the nonlinear PDE

$$\nabla\cdot(|\nabla y|^2\nabla y) - \frac{(x_1, x_2)^T}{x_1^2+x_2^2}|\nabla y|^2\nabla y + m\sin\sqrt{x_1^2+x_2^2}|y|^3 y + m\sin\sqrt{x_1^2+x_2^2}|y|y = 0. \tag{4.2}$$

In this case we have $e(x) \equiv 0$, so $\eta_1 = \eta_2 = 1/2$.

Taking $a = 2i\pi$ and $b = (2i+1)\pi$ for $i \in \mathbb{N}$, and using the same H and u given in the previous example, we have $\tilde{C}(x) = 2m\sin\sqrt{x_1^2+x_2^2}$.

$$\int_{\Omega[a,b]}\left[\tilde{C}(x)u^4\left(\sqrt{x_1^2+x_2^2}\right) - \frac{1}{4^3}|H(x)|^4\right]\mathrm{d}x = 2\pi\int_a^b[2m\sin r\sin^4 2r - 64\cos^4 2r]dr = 2^5\pi\left(\frac{16m}{315} - \frac{3\pi}{4}\right).$$

Note that the condition (3.8) is satisfied when $m > 35\pi(\frac{3}{4})^3$. Applying Theorem 3.2, we may conclude in this case that Eq. (4.2) is oscillatory.

References

[1] D. Gilbarg, N.S. Trudinger, Elliptic Partial Differential Equations of Second Order, Springer-Verlag, New York, 1983.
[2] N. Yoshida, Oscillation Theory of Partial Differential Equations, World Scientific Publishing Co. Pte. Ltd, 2008.
[3] E.S. Noussair, C.A. Swanson, Oscillation of semilinear elliptic inequalities by Riccati transformation, Canad. J. Math. 4 (32) (1980) 908–923.
[4] C.A. Swanson, Semilinear second order elliptic oscillation, Canad. Math. Bull. 2 (22) (1979) 139–157.
[5] R.K. Zhuang, Q.R. Wong, Z.A. Yao, Some new oscillation theorems for second order nonlinear elliptic equations with damping, J. Math. Anal. Appl. 330 (2007) 622–632.
[6] Z. Xu, Oscillation of second order damped elliptic differential equations, Math. Comput. Modelling 47 (2008) 341–351.
[7] Z. Xu, On second order damped elliptic oscillation, J. Math. Pures Appl. 89 (2008) 134–144.
[8] R. Marik, Ordinary differential equation in the oscillation theory of partial differential equations, J. Math. Anal. Appl. 338 (2008) 194–208.
[9] N. Yoshida, Forced oscillation criteria for superlinear–sublinear elliptic equations via picone-type inequality, J. Math. Anal. Appl. 363 (2010) 711–717.
[10] W.T. Li, X.H. Li, Oscillation criteria for second order nonlinear differential equations with integrable coefficient, Appl. Math. Lett. 13 (2000) 1–6.
[11] J. Jaros, T. Kusano, N. Yoshida, Picone-type inequalities for half-linear elliptic equations and their applications, Adv. Math. Sci. Appl. 12 (2002) 709–724.
[12] R.P. Agarwal, A. Zafer, Interval oscillation criteria for second-order forced dynamic equations with mixed nonlinearities, Comput. Math. Appl. 59 (2010) 977–993.
[13] M.A. El-Sayed, An oscillation criterion for a forced second order linear differential equation, Proc. Amer. Math. Soc. 118 (1993) 813–817.
[14] A.H. Nasr, Sufficient conditions for the oscillation of forced superlinear second order differential equations with oscillatory potential, Proc. Amer. Math. Soc. 126 (1998) 123–125.
[15] Y.G. Sun, J.S.W. Wong, Oscillation criteria for second order forced ordinary differential equations with mixed nonlinearities, J. Math. Anal. Appl. 334 (2007) 549–560.
[16] J.S.W. Wong, Oscillation criteria for a forced second order linear differential equation, J. Math. Anal. Appl. 231 (1999) 235–240.
[17] A. Zafer, Interval oscillation criteria for second order super-half-linear functional differential equations with delay and advanced arguments, Math. Nachr. 282 (9) (2009) 1334–1341.
[18] R.K. Zhuang, Annual oscillation criteria for second-order nonlinear elliptic differential equations, J. Comput. Appl. Math. 217 (2008) 268–276.

[19] R.K. Zhuang, Annulus oscillation criteria for second-order nonlinear elliptic differential equations with damping, Electron. J. Differential Equations 4 (2009) 14.
[20] Q. Yang, Interval oscillation criteria for a forced second order nonlinear ordinary differential equations with oscillatory potential, Appl. Math. Comput. 135 (2003) 49–64.
[21] Z. Xu, Oscillation criteria for second order forced elliptic differential equations with mixed nonlinearities, Comput. Math. Appl. 56 (2008) 1225–1235.
[22] Z. Xu, Sun-Wong type theorems for second order damped elliptic equations, J. Math. Anal. Appl. 359 (2009) 322–332.
[23] N. Yoshida, Picone-type inequalities for a class of quasilinear elliptic equations and their applications, in: Proceedings of the Conference on Differential and Difference Equations and Applications, Florida, New York 2006, 2005, pp. 1177–1185.