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Global attractors for wave equations with nonlinear interior damping and critical exponents

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Abstract

In this paper we study the global attractors for wave equations with nonlinear interior damping. We prove the existence, regularity and finite dimensionality of the global attractors without assuming a large value for the damping parameter, when the growth of the nonlinear terms is critical.

Keywords: Attractors; Wave equations; Regularity of attractors; Finite dimensionality of attractors

1. Introduction

Let $\Omega \subset R^3$ be a bounded regular domain. We consider the following problem:

$$w_{tt} - \Delta w + g(w_t) + f(w) = h(x) \quad \text{in } (0,+\infty)\times\Omega, \tag{1.1}$$

$$w = 0 \quad \text{on } (0,+\infty)\times\partial\Omega, \tag{1.2}$$

$$w(0,\cdot) = w_0, \quad w_t(0,\cdot) = w_1 \quad \text{in } \Omega, \tag{1.3}$$

where $h \in L_2(\Omega)$ and the functions f and g satisfy the following conditions:

$$f \in C^1(R), \qquad \left|f'(s)\right| \leqslant c\left(1+|s|^2\right), \tag{1.4}$$

$$\liminf_{|s|\to\infty} \frac{f(s)}{s} > -\lambda_1, \tag{1.5}$$

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where λ_1 is the first eigenvalue of $-\Delta$ with zero Dirichlet data,

$$g \in C^1(R), \quad g(0)=0, \quad g \text{ is strictly increasing}, \quad \text{and} \quad \liminf_{|s|\to\infty} g'(s) > 0, \tag{1.6}$$

$$\left|g(s)\right| \leqslant c\left(1+|s|^5\right). \tag{1.7}$$

We denote the spaces $L_2(\Omega)$, $W_2^1(\Omega)$ and $\mathring{W}_2^1(\Omega)$, by H, H^1 and H_0^1, respectively. The scalar product and the norm in H are denoted by $\langle,\rangle$ and $\|\cdot\|$. We also denote the norm in H^1 by $\|\cdot\|_1$.

The well-posedness of problem (1.1)–(1.3) was discussed in [1,2]. We recall definitions of strong and generalized (weak) solutions.

Definition 1.1. [1,2] A function $w \in C([0,T]; H_0^1) \cap C^1([0,T]; H)$ possessing the properties $w(0,\cdot) = w_0$ and $w_t(0,\cdot) = w_1$ is said to be

(S) strong solution to problem (1.1)–(1.3) on $[0,T]\times\Omega$, iff
- $w_t \in L_1([a,b]; H_0^1)$ and $w_{tt} \in L_1([a,b]; H)$ for any $0 < a < b < T$;
- $-\Delta w + g(w_t) \in H$ for almost all $t \in [0,T]$;
- Eq. (1.1) is satisfied for almost all $t \in [0,T]$ and $x \in \Omega$;

(G) generalized (weak) solution to problem (1.1)–(1.3) on $[0,T]\times\Omega$, iff there exists a sequence of strong solutions $\{w^n(t,x)\}$ to problem (1.1)–(1.3) with initial data (w_0^n, w_1^n) instead of (w_0, w_1) such that

$$\lim_{n\to\infty}\left(\left\|w-w^n\right\|_{C([0,T];H_0^1)} + \left\|w_t - w_t^n\right\|_{C([0,T];H)}\right) = 0.$$

As mentioned in [2] applying Faedo–Galerkin method (see [3,4]) one can show the existence and uniqueness of a generalized solution, that is:

Theorem 1.1. [1,2] *Assume conditions* (1.4)–(1.7) *hold. Then for every $T > 0$ and every $(w_0, w_1) \in H_0^1 \times H$ there exists a unique generalized solution $(w, w_t) \in C([0,T]; H_0^1 \times H)$ to problem* (1.1)–(1.3)*, which continuously depends on the initial data and satisfies the energy inequality*

$$E\left(w(t)\right) + \int_s^t\int_\Omega g\left(w_t(\tau,x)\right)w_t(\tau,x)\,dx\,d\tau + \int_\Omega F\left(w(t,x)\right)dx - \int_\Omega h(x)w(t,x)\,dx$$
$$\leqslant E\left(w(s)\right) + \int_\Omega F\left(w(s,x)\right)dx - \int_\Omega h(x)w(s,x)\,dx, \quad \forall t \geqslant s \geqslant 0, \tag{1.8}$$

where $E(w(t)) = \frac{1}{2}(\|\nabla w(t)\|^2 + \|w_t(t)\|^2)$ and $F(w) = \int_0^w f(u)\,du$.

Thus problem (1.1)–(1.3) generates a continuous semigroup $\{S(t)\}_{t\geqslant 0}$ in $H_0^1 \times H$ by the formula $S(t)(w_0, w_1) = (w(t), w_t(t))$, where $w(t)$ is the unique generalized solution to problem (1.1)–(1.3) with initial data (w_0, w_1).

The global attractors for wave equations with linear interior damping were studied in [5–10] and references therein. In the case of nonlinear interior damping the first pioneering papers include [11–14]. In [12] a global attractor was established assuming a large value of the damping parameter in the case when the exponent for the growth of the damping function g is $1\frac{2}{3}$. In [13], the existence of the global attractor was claimed assuming a large value of the damping parameter in the case when growth exponents are critical. But, as mentioned in [1], the detailed proof of this result was not published. The existence of the global attractor for the wave equation with supercritical source term was proved in [3], imposing a condition relating the growth of the damping function g to the growth of the source term f. In [14], in the one-dimensional case the finite dimensionality of the attractors for wave equations with nonlinear damping was studied. In [15] the existence, regularity and finite dimensionality of attractors for wave equations with nonlinear interior damping is shown for the two-dimensional case. In [16] an upper bound of the Hausdorff dimension of the global attractor for the wave equation was obtained by imposing linear growth conditions on damping and source terms. Finite dimensionality of the global attractor for problem (1.1)–(1.3) is shown in [17] in the subcritical case. In [1] the authors established the existence and finite dimensionality of the global attractors for the wave equations with nonlinear damping and source terms with critical growth exponents assuming a large value for the damping parameter. Recently in [2] the existence of a global attractor for the wave equation with subcritical damping and critical source term has been proved without assuming a large value for the damping parameter.

In this paper using ideas developed in [1,15,18] we prove the existence of a global attractor for the problem (1.1)–(1.3) under conditions (1.4)–(1.7) and improve the result of [2]. Moreover under an additional condition (see (3.7)) we prove regularity and finite dimensionality of the global attractor and improve the previous results.

2. Existence of a global attractor

We first recall definitions of a global attractor and asymptotically compact semigroup.

Definition 2.1. [19] Let $\{S(t)\}_{t\geqslant 0}$ be a semigroup on a metric space (X, d). A smallest, nonempty, bounded closed set $\mathcal{A} \subset X$ that satisfies

$$\lim_{t\to\infty} \sup_{v\in B} \inf_{u\in\mathcal{A}} d\big(S(t)v, u\big) = 0$$

for each bounded set $B \subset X$, is called a global attractor of $\{S(t)\}_{t\geqslant 0}$.

Definition 2.2. [19] A semigroup $\{S(t)\}_{t\geqslant 0}$ defined on a metric space (X, d), is called asymptotically compact iff for each bounded set $B \subset X$ such that $\bigcup_{t\geqslant 0} S(t)B$ is bounded in (X, d), a sequence of the form $\{S(t_k)v_k\}_{k=1}^{\infty}$, $t_k \to \infty$, $v_k \in B$, has a convergent subsequence.

Taking into account conditions (1.4)–(1.6) in (1.8) we obtain that the problem (1.1)–(1.3) admits a nonincreasing Lyapunov function

$$L\big(w(t)\big) := E\big(w(t)\big) + \int_\Omega F\big(w(t,x)\big)\,dx - \int_\Omega h(x)w(t,x)\,dx$$

and for every bounded set B in $H_0^1 \times H$, the set $\bigcup_{t\geqslant 0} S(t)B$ is bounded in $H_0^1 \times H$. On the other hand, under conditions (1.4)–(1.5) the set of stationary solutions is bounded in H_0^1. Thus according to Theorem 3.2 of [19], in order to prove the existence of a global attractor it is sufficient to show that $\{S(t)\}_{t\geqslant 0}$ is asymptotically compact. To prove the asymptotic compactness of $\{S(t)\}_{t\geqslant 0}$ we need the following lemmas:

Lemma 2.1. *Assume $f(\cdot)$ satisfies condition* (1.4) *and $(w, w_t) \in L_\infty(0, T; H_0^1 \times H)$. Then $F(w) \in C([0, T]; L_1(\Omega))$ and*

$$\int_\Omega F\bigl(w(t,x)\bigr)\,dx = \int_s^t \bigl\langle f\bigl(w(\tau)\bigr), w_t(\tau)\bigr\rangle\,d\tau + \int_\Omega F\bigl(w(s,x)\bigr)\,dx \tag{2.1}$$

for every $t, s \in [0, T]$.

Proof. By the second condition of the lemma it follows that $w \in C([0, T]; H)$and according to [20, Lemma 8.1, p. 275] we have $w \in C_s(0, T; H_0^1)$. This means that if $t_n \to t_0$, then $w(t_n) \to w(t_0)$ weakly in H_0^1. So by the compact embedding theorem we obtain

$$F\bigl(w(t_n)\bigr) \to F\bigl(w(t_0)\bigr) \quad \text{strongly in } L_1(\Omega).$$

Hence $F(w) \in C([0, T]; L_1(\Omega))$.

Let the sequence $w^n \in C_0^\infty((0, T) \times \Omega)$ be such that

$$w^n \to w \quad \text{strongly in } L_4\bigl(0, T; H_0^1\bigr),$$

and

$$w_t^n \to w_t \quad \text{strongly in } L_4(0, T; H).$$

Then we have

$$F\bigl(w^n\bigr) \to F(w) \quad \text{strongly in } L_1\bigl((0, T) \times \Omega\bigr), \tag{2.2}$$

and

$$\bigl\langle f\bigl(w^n\bigr), w_t^n\bigr\rangle \to \bigl\langle f(w), w_t\bigr\rangle \quad \text{strongly in } L_1(0, T). \tag{2.3}$$

On the other hand, since

$$\frac{\partial}{\partial t}\int_\Omega F\bigl(w^n(t,x)\bigr)\,dx = \bigl\langle f\bigl(w^n(t)\bigr), w_t^n(t)\bigr\rangle$$

by (2.2)–(2.3) we find that

$$\frac{\partial}{\partial t}\int_\Omega F\bigl(w(t,x)\bigr)\,dx = \bigl\langle f(w), w_t\bigr\rangle \in L_\infty(0, T)$$

which yields (2.1). □

Lemma 2.2. *Assume $f(\cdot)$ satisfies condition* (1.4) *and the sequence $\{(w^n(t), w_t^n(t))\}$ is weakly star convergent in $L_\infty(s,T;H_0^1\times H)$. Then*

$$\lim_{n\to\infty}\lim_{m\to\infty}\int_s^T \big\langle f\big(w^n(t,x)\big)-f\big(w^m(t,x)\big), w_t^n(t,x)-w_t^m(t,x)\big\rangle\,dt=0. \tag{2.4}$$

Proof. Let

$$\begin{cases} w^n\to w & \text{weakly star in } L_\infty\big(s,T;H_0^1\big),\\ w_t^n\to w_t & \text{weakly star in } L_\infty(s,T;H).\end{cases} \tag{2.5}$$

Then by the embedding theorem we have

$$w^n\to w \quad \text{strongly in } C\big([s,T];H\big). \tag{2.6}$$

On the other hand, according to [20, Lemma 8.1, p. 275] and (2.5) we have that the sequence $\{w^n\}$ is bounded in $C_s(s,T;H_0^1)$ and consequently the sequence $\{w^n(t)\}$ is bounded in H_0^1 for every $t\in[s,T]$. So by (2.6) we obtain

$$w^n(t)\to w(t) \quad \text{weakly in } H_0^1,$$

which according to the compact embedding theorem yields that

$$F\big(w^n(t)\big)\to F\big(w(t)\big) \quad \text{strongly in } L_1(\Omega),\ \forall t\in[s,T].$$

Now using Lemma 2.1 we find that

$$\begin{aligned}
&\lim_{n\to\infty}\lim_{m\to\infty}\int_s^T \big\langle f\big(w^n(t,x)\big)-f\big(w^m(t,x)\big), w_t^n(t,x)-w_t^m(t,x)\big\rangle\,dt\\
&\quad=\lim_{n\to\infty}\int_\Omega F\big(w^n(T,x)\big)\,dx+\lim_{m\to\infty}\int_\Omega F\big(w^m(T,x)\big)\,dx-\lim_{n\to\infty}\int_\Omega F\big(w^n(s,x)\big)\,dx\\
&\qquad-\lim_{m\to\infty}\int_\Omega F\big(w^m(s,x)\big)\,dx-\lim_{n\to\infty}\lim_{m\to\infty}\int_s^T\int_\Omega f\big(w^n(t,x)\big)w_t^m(t,x)\,dx\,dt\\
&\qquad-\lim_{n\to\infty}\lim_{m\to\infty}\int_s^T\int_\Omega f\big(w^m(t,x)\big)w_t^n(t,x)\,dx\,dt\\
&\quad=2\int_\Omega F\big(w(T,x)\big)\,dx\\
&\qquad-2\int_\Omega F\big(w(s,x)\big)\,dx-2\int_s^T\int_\Omega f\big(w(t,x)\big)w_t(t,x)\,dx\,dt=0. \qquad \square
\end{aligned}$$

Lemma 2.3. *Assume the conditions* (1.4)–(1.7) *are satisfied, and B is a bounded subset of $H_0^1 \times H$. Then for any $\varepsilon > 0$ there exists $T = T(\varepsilon, B)$ such that*

$$\limsup_{n\to\infty} \sup_{p\in\mathbb{N}} \left\| S(T)\theta_{n+p} - S(T)\theta_n \right\|_{H_0^1\times H} \leqslant \varepsilon, \tag{2.7}$$

where $\{\theta_n\}$ is a sequence in B and $\{S(t)\theta_n\}$ is weakly star convergent in $L_\infty(0,\infty; H_0^1 \times H)$.

Proof. We will use the techniques used in [1,15,18]. Let $(w^n(t), w_t^n(t)) = S(t)\theta_n$. Taking into account conditions (1.4)–(1.7) in (1.8), we find that

$$\int_0^T\!\!\int_\Omega g\big(w_t^n(t,x)\big)w_t^n(t,x)\,dx\,dt \leqslant c_1\big(\|B\|_{H_0^1\times H}\big), \quad \forall T \geqslant 0, \tag{2.8}$$

and

$$E\big(w^n(T)\big) \leqslant c_1\big(\|B\|_{H_0^1\times H}\big), \quad \forall T \geqslant 0, \tag{2.9}$$

where $\|B\|_{H_0^1\times H} = \sup_{v\in B}\|v\|_{H_0^1\times H}$. Multiplying both sides of

$$\big(w^n - w^m\big)_{tt} - \Delta\big(w^n - w^m\big) + g\big(w_t^n\big) - g\big(w_t^m\big) + f\big(w^n\big) - f\big(w^m\big) = 0$$

by $(w_t^n - w_t^m)$, integrating over $[s,T]\times\Omega$ we have

$$\begin{aligned} &E\big(w^n(T) - w^m(T)\big) + \int_s^T\!\!\int_\Omega \big(g\big(w_t^n(t,x)\big) - g\big(w_t^m(t,x)\big)\big)\big(w_t^n(t,x) - w_t^m(t,x)\big)\,dx\,dt \\ &\quad + \int_s^T\!\!\int_\Omega \big(f\big(w^n(t,x)\big) - f\big(w^m(t,x)\big)\big)\big(w_t^n(t,x) - w_t^m(t,x)\big)\,dx\,dt \\ &\leqslant E\big(w^n(s) - w^m(s)\big), \end{aligned} \tag{2.10}$$

and consequently

$$\begin{aligned} &\int_0^T\!\!\int_\Omega \big(g\big(w_t^n(t,x)\big) - g\big(w_t^m(t,x)\big)\big)\big(w_t^n(t,x) - w_t^m(t,x)\big)\,dx\,dt \\ &\quad + \int_0^T\!\!\int_\Omega \big(f\big(w^n(t,x)\big) - f\big(w^m(t,x)\big)\big)\big(w_t^n(t,x) - w_t^m(t,x)\big)\,dx\,dt \leqslant c\|B\|_{H_0^1\times H}^2. \end{aligned}$$

It is easy to see [18] that for any $\delta > 0$ there exists $c_2(\delta) > 0$, such that

$$|u - v|^2 \leqslant \delta + c_2(\delta)\big(g(u) - g(v)\big)(u - v), \quad \forall u, v \in R.$$

So by the last two inequalities we obtain

$$\int_0^T \left\| w_t^n(t) - w_t^m(t) \right\|^2 dt$$
$$\leqslant \delta T \operatorname{mes} \Omega + c c_2(\delta) \|B\|^2_{H_0^1 \times H}$$
$$+ c_2(\delta) \int_0^T \int_\Omega \big(f\big(w^m(t,x)\big) - f\big(w^n(t,x)\big)\big)\big(w_t^n(t,x) - w_t^m(t,x)\big)\, dx\, dt \tag{2.11}$$

for every $\delta > 0$. On the other hand, multiplying both sides of

$$\big(w^n - w^m\big)_{tt} - \Delta\big(w^n - w^m\big) + g\big(w_t^n\big) - g\big(w_t^m\big) + f\big(w^n\big) - f\big(w^m\big) = 0$$

by $(w^n - w^m)$, integrating over $[0, T] \times \Omega$ and taking into account (2.9) we have

$$\int_0^T \left\| \nabla\big(w^n(t) - w^m(t)\big) \right\|^2 dt$$
$$\leqslant c_3\big(\|B\|_{H_0^1 \times H}\big) + \int_0^T \left\| w_t^n(t) - w_t^m(t) \right\|^2 dt$$
$$+ \int_0^T \int_\Omega \big(f\big(w^m(t,x)\big) - f\big(w^n(t,x)\big)\big)\big(w^n(t,x) - w^m(t,x)\big)\, dx\, dt$$
$$+ \int_0^T \int_\Omega \big(g\big(w_t^m(t,x)\big) - g\big(w_t^n(t,x)\big)\big)\big(w^n(t,x) - w^m(t,x)\big)\, dx\, dt, \quad \forall T \geqslant 0. \tag{2.12}$$

Thus by (2.11) and (2.12) we obtain

$$\int_0^T E\big(w^n(t) - w^m(t)\big)\, dt$$
$$\leqslant \delta T \operatorname{mes} \Omega + \tilde{c}\big(\|B\|_{H_0^1 \times H}, \delta\big)$$
$$+ c_2(\delta) \int_0^T \int_\Omega \big(f\big(w^m(t,x)\big) - f\big(w^n(t,x)\big)\big)\big(w_t^n(t,x) - w_t^m(t,x)\big)\, dx\, dt$$
$$+ \frac{1}{2} \int_0^T \int_\Omega \big(f\big(w^m(t,x)\big) - f\big(w^n(t,x)\big)\big)\big(w^n(t,x) - w^m(t,x)\big)\, dx\, dt$$

$$+\frac{1}{2}\int_0^T\int_\Omega \big(g\big(w_t^m(t,x)\big)-g\big(w_t^n(t,x)\big)\big)\big(w^n(t,x)-w^m(t,x)\big)\,dx\,dt, \quad \forall T\geqslant 0.$$

Integrating (2.10) with respect to s from 0 to T and taking into account the last inequality we find that

$$\begin{aligned}
&E\big(w^n(T)-w^m(T)\big)\\
&\quad\leqslant \delta \operatorname{mes}\Omega+\frac{1}{T}\tilde{c}\big(\|B\|_{H_0^1\times H},\delta\big)\\
&\qquad+\frac{1}{T}c_2(\delta)\int_0^T\int_\Omega \big(f\big(w^m(t,x)\big)-f\big(w^n(t,x)\big)\big)\big(w_t^n(t,x)-w_t^m(t,x)\big)\,dx\,dt\\
&\qquad+\frac{1}{T}\int_0^T\int_t^T\int_\Omega \big(f\big(w^m(s,x)\big)-f\big(w^n(s,x)\big)\big)\big(w_t^n(s,x)-w_t^m(s,x)\big)\,dx\,ds\,dt\\
&\qquad+\frac{1}{2T}\int_0^T\int_\Omega \big(f\big(w^m(t,x)\big)-f\big(w^n(t,x)\big)\big)\big(w^n(t,x)-w^m(t,x)\big)\,dx\,dt\\
&\qquad+\frac{1}{2T}\int_0^T\int_\Omega \big(g\big(w_t^m(t,x)\big)-g\big(w_t^n(t,x)\big)\big)\big(w^n(t,x)-w^m(t,x)\big)\,dx\,dt\\
&\quad\equiv \delta \operatorname{mes}\Omega+\frac{1}{T}\tilde{c}\big(\|B\|_{H_0^1\times H},\delta\big)+K_1+K_2+K_3+K_4. \qquad (2.13)
\end{aligned}$$

From Lemma 2.2 it follows that

$$\lim_{n\to\infty}\lim_{m\to\infty}K_1=0 \quad \text{and} \quad \lim_{n\to\infty}\lim_{m\to\infty}K_2=0. \qquad (2.14)$$

On the other hand, since

$$w^n \text{ converges weakly star in } L_\infty\big(0,T;H_0^1\big),$$
$$w_t^n \text{ converges weakly star in } L_\infty(0,T;H),$$

by the compact embedding theorem (see [4, Theorem 5.1, p. 58]) we have

$$w^n \text{ converges strongly in } L_2\big(0,T;H_0^{1-\varepsilon}\big),\ \forall\varepsilon\in(0,1],$$

which yields

$$\lim_{n\to\infty}\lim_{m\to\infty}K_3=0. \qquad (2.15)$$

Now let us estimate K_4. Applying Hölder inequality we have

$$|K_4| \leqslant \frac{1}{2T} T^{\frac{1}{6}} c_4\big(\|B\|_{H_0^1\times H}\big)\big\|g\big(w_t^m\big)-g\big(w_t^n\big)\big\|_{L_{\frac{6}{5}}((0,T)\times\Omega)}. \tag{2.16}$$

Using the same techniques as in [15], by (1.7) and (2.8) we find that

$$\begin{aligned}
&c^{-\frac{1}{5}} \int_0^T\!\!\int_\Omega \big|g\big(w_t^n(t,x)\big)\big|^{\frac{6}{5}}\,dx\,dt \\
&\quad\leqslant \int_0^T\!\!\int_\Omega \big|g\big(w_t^n(t,x)\big)\big|\big(1+\big|w_t^n(t,x)\big|\big)\,dx\,dt \\
&\quad\leqslant c_1\big(\|B\|_{H_0^1\times H}\big) + \int_0^T\!\!\int_\Omega \big|g\big(w_t^n(t,x)\big)\big|\,dx\,dt \\
&\quad= c_1\big(\|B\|_{H_0^1\times H}\big) + \int_0^T \int_{\{x:\ x\in\Omega,\ |w_t^n(t,x)|\geqslant\delta\}} \big|g\big(w_t^n(t,x)\big)\big|\,dx\,dt \\
&\qquad+ \int_0^T \int_{\{x:\ x\in\Omega,\ |w_t^n(t,x)|<\delta\}} \big|g\big(w_t^n(t,x)\big)\big|\,dx\,dt \\
&\quad\leqslant c_1\big(\|B\|_{H_0^1\times H}\big) \\
&\qquad+ \frac{1}{\delta}\int_0^T \int_{\{x:\ x\in\Omega,\ |w_t^n(t,x)|\geqslant\delta\}} g\big(w_t^n(t,x)\big)w_t^n(t,x)\,dx\,dt \\
&\qquad+ T\,\mathrm{mes}\,\Omega\big(\big|g(-\delta)\big|+\big|g(\delta)\big|\big) \leqslant \left(1+\frac{1}{\delta}\right)c_1\big(\|B\|_{H_0^1\times H}\big) \\
&\qquad+ T\,\mathrm{mes}\,\Omega\big(\big|g(-\delta)\big|+\big|g(\delta)\big|\big), \quad \forall T>0.
\end{aligned} \tag{2.17}$$

From (2.16) and (2.17) it follows that

$$\begin{aligned}
|K_4| &\leqslant \frac{c^{\frac{1}{6}}}{T^{\frac{5}{6}}} c_4\big(\|B\|_{H_0^1\times H}\big)\left[\left(1+\frac{1}{\delta}\right)c_1\big(\|B\|_{H_0^1\times H}\big)\right]^{\frac{5}{6}} \\
&\quad+ c^{\frac{1}{6}} c_4\big(\|B\|_{H_0^1\times H}\big)\big[\mathrm{mes}\,\Omega\big(\big|g(-\delta)\big|+\big|g(\delta)\big|\big)\big]^{\frac{5}{6}}.
\end{aligned} \tag{2.18}$$

Thus by (2.13)–(2.15) and (2.18) we get

$$
\begin{aligned}
&\limsup_{n\to\infty}\limsup_{m\to\infty} E\big(w^n(T)-w^m(T)\big)\\
&\quad\leqslant \delta \operatorname{mes}\Omega + \frac{1}{T}\tilde{c}\big(\|B\|_{H_0^1\times H},\delta\big)\\
&\qquad+\frac{c^{\frac{1}{6}}}{T^{\frac{5}{6}}}c_4\big(\|B\|_{H_0^1\times H}\big)\bigg[\bigg(1+\frac{1}{\delta}\bigg)c_1\big(\|B\|_{H_0^1\times H}\big)\bigg]^{\frac{5}{6}}\\
&\qquad+c^{\frac{1}{6}}c_4\big(\|B\|_{H_0^1\times H}\big)\big[\operatorname{mes}\Omega\big(\big|g(-\delta)\big|+\big|g(\delta)\big|\big)\big]^{\frac{5}{6}}.
\end{aligned}
$$

Consequently

$$
\begin{aligned}
&\limsup_{n\to\infty}\sup_{p\in\mathbb{N}} E\big(w^{n+p}(T)-w^n(T)\big)\\
&\quad\leqslant 2\limsup_{n\to\infty}\sup_{p\in\mathbb{N}}\limsup_{m\to\infty} E\big(w^{n+p}(T)-w^m(T)\big)\\
&\qquad+2\limsup_{n\to\infty}\limsup_{m\to\infty} E\big(w^m(T)-w^n(T)\big)\\
&\quad\leqslant 4\bigg(\delta \operatorname{mes}\Omega + \frac{1}{T}\tilde{c}\big(\|B\|_{H_0^1\times H},\delta\big)\bigg)\\
&\qquad+\frac{4c^{\frac{1}{6}}}{T^{\frac{5}{6}}}c_4\big(\|B\|_{H_0^1\times H}\big)\bigg[\bigg(1+\frac{1}{\delta}\bigg)c_1\big(\|B\|_{H_0^1\times H}\big)\bigg]^{\frac{5}{6}}\\
&\qquad+4c^{\frac{1}{6}}c_4\big(\|B\|_{H_0^1\times H}\big)\big[\operatorname{mes}\Omega\big(\big|g(-\delta)\big|+\big|g(\delta)\big|\big)\big]^{\frac{5}{6}}.
\end{aligned}
$$

Since $g \in C(R)$ and $g(0) = 0$, the last inequality yields (2.7). □

Now we can prove asymptotic compactness of $\{S(t)\}_{t\geqslant 0}$.

Theorem 2.1. *Assume the conditions* (1.4)–(1.7) *hold. Then the semigroup* $\{S(t)\}_{t\geqslant 0}$ *generated by* (1.1)–(1.3) *is asymptotically compact in* $H_0^1 \times H$.

Proof. Using Lemma 2.2 and repeating the argument used in the proof of Theorem 2 from [18] we obtain the asymptotic compactness of $\{S(t)\}_{t\geqslant 0}$ in $H_0^1 \times H$. □

Thus using Theorem 3.2 of [19] we have the following result.

Theorem 2.2. *Assume that* (1.4)–(1.7) *hold. Then problem* (1.1)–(1.3) *has a global attractor* $\mathcal{A}$ *in* $H_0^1 \times H$, *which is invariant and compact.*

3. Regularity and finite dimensionality of attractors

By the invariance of $\mathcal{A}$ it follows (see [7, p. 159]) that, for every $\varphi \in \mathcal{A}$ there exists an invariant trajectory $\gamma = \{W(t), t \in R\} \subset \mathcal{A}$ such that

$$W(0) = \varphi. \tag{3.1}$$

Here, by an invariant trajectory we mean a continuous curve $\gamma = \{W(t),\ t \in R\}$ such that $S(t)W(\tau) = W(t+\tau)$ for $\forall t \geqslant 0$ and $\forall \tau \in R$ (see [7, p. 157]).

To prove the regularity and finite dimensionality of the global attractor $\mathcal{A}$ we need the following lemmas:

Lemma 3.1. *Assume K is a relatively compact subset of H^1 and $f(\cdot)$ satisfies condition* (1.4). *Then for any $\varepsilon > 0$ there exists $\delta > 0$ such that $\|u_2 - u_1\|_1 < \delta$ implies*

$$\left\| f'(u_2) - f'(u_1)\right\|_{L_3(\Omega)} \leqslant \varepsilon \tag{3.2}$$

for every $u_1, u_2 \in K$.

Proof. Assume that lemma is not true. Then there exists $\varepsilon_0 > 0$ and the sequences $\{u_n^1\}$, $\{u_n^2\}$ in K, such that

$$\begin{cases} u_n^1 \to v \quad \text{strongly in } H^1, \\ u_n^2 \to v \quad \text{strongly in } H^1, \\ \left\| f'(u_n^2) - f'(u_n^1)\right\|_{L_3(\Omega)} > \varepsilon_0 \quad \text{for every } n. \end{cases} \tag{3.3}$$

From $(3.3)_1$ and $(3.3)_2$ it follows that there exist subsequences $\{u_{n_k}^1\}$ and $\{u_{n_k}^2\}$ such that

$$u_{n_k}^1 \to v \quad \text{and} \quad u_{n_k}^2 \to v \quad \text{a.e. in } \Omega.$$

So applying Egorov's theorem we obtain that for any $\delta > 0$ there exists a set $A_\delta \subset \Omega$ such that $\operatorname{mes} A_\delta < \delta$ and

$$u_{n_k}^1 \to v \quad \text{and} \quad u_{n_k}^2 \to v \quad \text{uniformly in } \Omega \setminus A_\delta,$$

which implies

$$\lim_{k\to\infty} \left\| f'(u_{n_k}^2) - f'(u_{n_k}^1)\right\|_{L_3(\Omega\setminus A_\delta)} = 0. \tag{3.4}$$

On the other hand, from $(3.3)_1$, $(3.3)_2$ and (1.4) it follows that

$$\lim_{\delta\to 0} \limsup_{k\to\infty} \left\| f'(u_{n_k}^2) - f'(u_{n_k}^1)\right\|_{L_3(A_\delta)} = 0,$$

which together with (3.4) yields

$$\begin{aligned} \limsup_{k\to\infty} \left\| f'(u_{n_k}^2) - f'(u_{n_k}^1)\right\|_{L_3(\Omega)} &\leqslant \lim_{\delta\to 0} \lim_{k\to\infty} \left\| f'(u_{n_k}^2) - f'(u_{n_k}^1)\right\|_{L_3(\Omega\setminus A_\delta)} \\ &\quad + \lim_{\delta\to 0} \limsup_{k\to\infty} \left\| f'(u_{n_k}^2) - f'(u_{n_k}^1)\right\|_{L_3(A_\delta)} = 0. \end{aligned}$$

The last relation contradicts $(3.3)_3$. $\square$

Lemma 3.2. *Assume K is a relatively compact subset of H^1 and $f(\cdot)$ satisfies condition* (1.4). *Then for any $\varepsilon > 0$ there exists $c(\varepsilon) > 0$ such that*

$$\left\| f'(v)u \right\| \leqslant \varepsilon \|u\|_1 + c(\varepsilon)\|u\| \tag{3.5}$$

for every $v \in K$ and every $u \in H^1$.

Proof. Let $\delta > 0$ be as in Lemma 3.1 for $\frac{\varepsilon}{2} > 0$. Since K is relatively compact in H^1, there exist $n_\delta \in \mathbb{N}$ and $v_i \in K$, $i = \overline{1, n_\delta}$, such that

$$\min_{1 \leqslant i \leqslant n_\delta} \|v - v_i\|_1 \leqslant \delta$$

for every $v \in K$. Then by Lemma 3.1 we obtain

$$\min_{1 \leqslant i \leqslant n_\delta} \left\| f'(v) - f'(v_i) \right\|_{L_3(\Omega)} \leqslant \frac{\varepsilon}{2}$$

for every $v \in K$. On the other hand, since $C^\infty(\overline{\Omega})$ is dense in H^1, there exist $w_i \in C^\infty(\overline{\Omega})$, $i = \overline{1, n_\delta}$, such that

$$\left\| f'(v_i) - f'(w_i) \right\|_{L_3(\Omega)} \leqslant \frac{\varepsilon}{2} \quad \text{for } i = \overline{1, n_\delta}.$$

So we have

$$\min_{1 \leqslant i \leqslant n_\delta} \left\| f'(v) - f'(w_i) \right\|_{L_3(\Omega)} \leqslant \varepsilon \tag{3.6}$$

for every $v \in K$. Since

$$\max_{1 \leqslant i \leqslant n_\delta} \max_{x \in \overline{\Omega}} \left| f'\big(w_i(x)\big) \right| \leqslant c(\varepsilon),$$

by (3.6) we find (3.5). □

Now we can prove regularity of the global attractor.

Theorem 3.1. *Let the conditions* (1.4)–(1.7) *hold. In addition assume that*

$$0 < m \leqslant g'(s) \leqslant M\big(1 + g(s)s\big)^{\frac{2}{3}}, \quad \forall s \geqslant 0. \tag{3.7}$$

Then there exists $\mathcal{R} > 0$ such that

$$\|\varphi_1\|_1 + \|\varphi_2\|_1 + \left\| \Delta\varphi_1 + g(\varphi_2) \right\| \leqslant \mathcal{R}$$

for every $\varphi = (\varphi_1, \varphi_2) \in \mathcal{A}$.

Proof. Let $\varphi = (\varphi_1, \varphi_2) \in \mathcal{A}$ and $\gamma = \{W(t) \in \mathcal{A},\ t \in R\}$ be an invariant trajectory which satisfies (3.1). From definition of the invariant trajectory it follows that $W(\cdot) = (w(\cdot), w_t(\cdot))$ and $(w(t+s), w_t(t+s)) = S(t)(w(s), w_t(s))$ for $\forall t \geqslant 0$ and $\forall s \in R$. Then $v(t) = w(t+s)$ is the solution of the problem:

$$\begin{cases} v_{tt} - \Delta v + g(v_t) + f(v) = h & \text{in } (0,+\infty) \times \Omega, \\ v = 0 & \text{on } (0,+\infty) \times \partial\Omega, \\ v(0,\cdot) = w(s,\cdot), \quad v_t(0,\cdot) = w_t(s,\cdot) & \text{in } \Omega. \end{cases} \tag{3.8}$$

Let $s < 0$, and $0 < l < -s$. Denoting $z(t,\cdot) = v(t+l,\cdot) - v(t,\cdot)$ from (3.8) we have

$$\begin{cases} z_{tt} - \Delta z + \tilde{g}\big(v_t(t), l\big) z_t + \tilde{f}\big(v(t), l\big) z = 0 & \text{in } (0,+\infty) \times \Omega, \\ z = 0 & \text{on } (0,+\infty) \times \partial\Omega, \\ z(0) = z_0, \quad z_t(0) = z_1 & \text{in } \Omega, \end{cases} \tag{3.9}$$

where

$$\tilde{g}\big(v_t(t), l\big) = \int_0^1 g'\big(\tau v_t(t+l) + (1-\tau) v_t(t)\big)\, d\tau,$$

$$\tilde{f}\big(v(t), l\big) = \int_0^1 f'\big(\tau v(t+l) + (1-\tau) v(t)\big)\, d\tau,$$

$$z_0 = w(s+l) - w(s) \quad \text{and} \quad z_1 = w_t(s+l) - w_t(s).$$

Multiplying Eq. $(3.9)_1$ by z_t and by z, and integrating over $(\sigma, t) \times \Omega$ we obtain

$$E\big(z(t)\big) + \int_\sigma^t \int_\Omega \tilde{g}\big(v_t(\tau), l\big) z_t^2(\tau)\, dx\, d\tau \leqslant E\big(z(\sigma)\big) - \int_\sigma^t \big\langle \tilde{f}\big(v(t), l\big) z, z_t(\tau)\big\rangle\, d\tau \tag{3.10}$$

and

$$\big\langle z_t(t), z(t)\big\rangle + \int_\sigma^t \big\|\nabla z(\tau)\big\|^2\, d\tau + \int_\sigma^t \int_\Omega \tilde{g}\big(v_t(\tau), l\big) z_t(\tau) z(\tau)\, dx\, d\tau$$
$$= \int_\sigma^t \big\|z_t(\tau)\big\|^2\, d\tau + \big\langle z_t(\sigma), z(\sigma)\big\rangle - \int_\sigma^t \big\langle \tilde{f}\big(v(t), l\big) z, z(\tau)\big\rangle\, d\tau. \tag{3.11}$$

On the other hand, by (3.7) we have

$$0 < m \leqslant \int_0^1 g'\big(\tau u_1 + (1-\tau) u_2\big)\, d\tau \leqslant M\big(1 + g(u_1) u_1 + g(u_2) u_2\big)^{\frac{2}{3}}, \quad \forall u_1, u_2 \in R,$$

which together with (3.10) yields

$$\int_\sigma^t \|z_t(\tau)\|^2\,d\tau \leqslant \frac{1}{m}\left(E\big(z(\sigma)\big) - \int_\sigma^t \big\langle \tilde f\big(v(t),l\big)z, z_t(\tau)\big\rangle\,d\tau\right) \tag{3.12}$$

and

$$\begin{aligned}
&\left|\int_\sigma^t\int_\Omega \tilde g\big(v_t(\tau),l\big)z_t(\tau)z(\tau)\,dx\,d\tau\right| \\
&\quad\leqslant \lambda\int_\sigma^t\int_\Omega \tilde g\big(v_t(\tau),l\big)z^2(\tau)\,dx\,d\tau \\
&\qquad + \frac{1}{4\lambda}\int_\sigma^t\int_\Omega \tilde g\big(v_t(\tau),l\big)z_t^2(\tau)\,dx\,d\tau \\
&\quad\leqslant \frac{1}{4\lambda}\left(E\big(z(\sigma)\big) - \int_\sigma^t \big\langle \tilde f\big(v(t),l\big)z, z_t(\tau)\big\rangle\,d\tau\right) \\
&\qquad + \lambda\left(\int_\sigma^t\int_\Omega \big|\tilde g\big(v_t(\tau),l\big)\big|^{\frac{3}{2}}\,dx\,d\tau\right)^{\frac{2}{3}} \sup_{\sigma\leqslant\tau\leqslant t}\|z(\tau)\|^2_{L^6(\Omega)} \\
&\quad\leqslant \frac{1}{4\lambda}\left(E\big(z(\sigma)\big) - \int_\sigma^t \big\langle \tilde f\big(v(t),l\big)z, z_t(\tau)\big\rangle\,d\tau\right) \\
&\qquad + \lambda c_1(\mathcal{A}) \sup_{\sigma\leqslant\tau\leqslant t} E\big(z(\tau)\big), \quad \forall\lambda > 0.
\end{aligned} \tag{3.13}$$

By (3.10)–(3.13) we find that

$$\begin{aligned}
E\big(z(t)\big) + \int_\sigma^t E\big(z(\tau)\big)\,d\tau &\leqslant \mu \sup_{\sigma\leqslant\tau\leqslant t} E\big(z(\tau)\big) + c_2(\mathcal{A})E\big(z(\sigma)\big) \\
&\quad + c_2(\mathcal{A})\left(\left|\int_\sigma^t \big\langle \tilde f\big(v(t),l\big)z, z_t(\tau)\big\rangle\,d\tau\right| \right.\\
&\quad \left.+ \left|\int_\sigma^t \big\langle \tilde f\big(v(t),l\big)z, z(\tau)\big\rangle\,d\tau\right|\right), \quad 0\leqslant\sigma\leqslant t,
\end{aligned} \tag{3.14}$$

for some $\mu \in (0, 1)$.

We note that the argument above is of a formal character. We can justify it by considering strong solutions. Since (3.14) is true for strong solutions of (3.8), it remains true also for generalized solutions, because they can be approximated by a sequence of strong solutions.

Now let us estimate the right side of (3.14). From compactness of $\mathcal{A}$ it follows that the set $\bigcup_{0\leqslant\tau\leqslant 1}(\tau\mathcal{A}+(1-\tau)\mathcal{A})$ is also compact in $H_0^1\times H$. So by Lemma 3.2 we find that

$$\left\|\tilde f\big(v(t),l\big)z\right\|\leqslant\varepsilon\|z\|_1+c_3(\varepsilon)\|z\|,\quad\forall t\geqslant 0.\tag{3.15}$$

Thus by choosing ε small enough, from (3.14) and (3.15) we obtain

$$E\big(z(t)\big)+\int_\sigma^t E\big(z(\tau)\big)\,d\tau\leqslant c_4(\mathcal{A})\left(E\big(z(\sigma)\big)+\int_\sigma^t\left\|z(\tau)\right\|^2d\tau\right),\quad 0\leqslant\sigma\leqslant t.\tag{3.16}$$

Integrating (3.16) with respect to σ from 0 to t we have

$$\begin{aligned}tE\big(z(t)\big)&\leqslant c_4(\mathcal{A})\int_0^t E\big(z(\sigma)\big)\,d\sigma+c_4(\mathcal{A})\int_0^t\int_\sigma^t\left\|z(\tau)\right\|^2d\tau\,d\sigma\\&\leqslant c_4^2(\mathcal{A})\left(E\big(z(0)\big)+\int_0^t\left\|z(\tau)\right\|^2d\tau\right)+tc_4(\mathcal{A})\int_0^t\left\|z(\tau)\right\|^2d\tau\\&\leqslant c_5(\mathcal{A})E\big(z(0)\big)+c_5(\mathcal{A})(1+t)\int_0^t\left\|z(\tau)\right\|^2d\tau,\quad\forall t\geqslant 0.\end{aligned}\tag{3.17}$$

On the other hand, since $(w(s),w_t(s))\in\mathcal{A}$, for $\forall s\in R$, using by (1.8) and (3.7) we obtain

$$\int_0^t\left\|v_t(\tau)\right\|^2d\tau\leqslant c(\mathcal{A}),\quad\forall t\geqslant 0,$$

and consequently

$$\int_0^t\left\|z(\tau)\right\|^2d\tau\leqslant l^2\int_0^t\int_0^1\left\|v_t(\tau+sl)\right\|^2ds\,d\tau\leqslant l^2c(\mathcal{A}),\quad\forall t\geqslant 0.$$

Taking into account the last inequality in (3.17) we find

$$tE\big(z(t)\big)\leqslant cE\big(z(0)\big)+cl^2(1+t),\quad\forall t\geqslant 0,\tag{3.18}$$

where c depends on $\mathcal{A}$, but is independent of $z(t)$. Taking $t=\tau-s-l$ in (3.18), we have

$$E\big(w(\tau)-w(\tau-l)\big)\leqslant\frac{cE(z(0))+cl^2(1+\tau-s-l)}{\tau-s-l},\quad\forall\tau\geqslant 0,$$

passing to limit as $s \to -\infty$,

$$E\big(w(\tau) - w(\tau - l)\big) \leqslant cl^2, \quad \forall \tau, l \geqslant 0. \tag{3.19}$$

Inequality (3.19) together with $w_t \in C([0,\infty); H)$ yields

$$w_t \in C_s\big(0,\infty; H_0^1\big) \quad \text{and} \quad \big\|w_t(\tau)\big\|_1 \leqslant r_1, \quad \forall \tau \geqslant 0, \tag{3.20}$$

and consequently

$$g(w_t) \in C_s\big(0,\infty; L_{\frac{6}{5}}(\Omega)\big) \quad \text{and} \quad \big\|g\big(w_t(\tau)\big)\big\|_{L_{\frac{6}{5}}(\Omega)} \leqslant r_2, \quad \forall \tau \geqslant 0, \tag{3.21}$$

where r_i $(i = 1, 2)$ are independent of $w(t)$. Since $w(\tau) = v(\tau - s)$, taking into account (3.21) in $(3.8)_1$ we obtain $w_{tt} \in C_s(0,\infty; H^{-1})$, which together with (3.19) implies

$$w_{tt} \in C_s(0,\infty; H) \quad \text{and} \quad \big\|w_{tt}(\tau)\big\| \leqslant r_3, \quad \forall \tau \geqslant 0, \tag{3.22}$$

where r_3 is independent of $w(t)$. Taking into account (3.20)–(3.22) in Eq. $(3.8)_1$ we find that

$$\big\|w_t(\tau)\big\|_1 + \big\|\Delta w(\tau) + g\big(w_t(\tau)\big)\big\| \leqslant r_4, \quad \forall \tau \geqslant 0, \tag{3.23}$$

where r_4 is independent of $w(t)$. Thus for every $\varphi = (\varphi_1, \varphi_2) \in \mathcal{A}$ we obtain

$$\|\varphi_1\|_1 + \|\varphi_2\|_1 + \big\|\Delta\varphi_1 + g(\varphi_2)\big\| \leqslant \mathcal{R},$$

where $\mathcal{R}$ is independent of φ. □

Now let us prove finite dimensionality of $\mathcal{A}$.

Theorem 3.2. *Assume the conditions of Theorem* 3.1 *are satisfied. Then the fractal dimension of the global attractor $\mathcal{A}$ is finite.*

Proof. Let $\varphi_1 = (w_0, w_1) \in \mathcal{A}$, $\varphi_2 = (v_0, v_1) \in \mathcal{A}$, $(w, w_t) = S(t)\varphi_1$, $(v, v_t) = S(t)\varphi_2$ and $u = w - v$. Then $u(t,x)$ is the solution of the problem

$$\begin{cases} u_{tt} - \Delta u + g(u_t + v_t) - g(v_t) + f(u+v) - f(v) = 0 & \text{in } (0,+\infty) \times \Omega, \\ u = 0 \quad \text{on } (0,+\infty) \times \partial\Omega, \\ u(0) = w_0 - v_0, \quad u_t(0) = w_1 - v_1 & \text{in } \Omega. \end{cases} \tag{3.24}$$

Formally multiplying Eq. $(3.24)_1$ by u_t and by u, and integrating over $(\sigma, t) \times \Omega$ we obtain

$$\begin{aligned} &E\big(u(t)\big) + \int_\sigma^t \int_\Omega \big(g(u_t + v_t) - g(v_t)\big)u_t(\tau)\,dx\,d\tau \\ &\quad \leqslant E\big(z(\sigma)\big) - \int_\sigma^t \big\langle f(u+v) - f(v), u_t(\tau)\big\rangle\,d\tau \end{aligned} \tag{3.25}$$

and

$$\langle u_t(t), u(t)\rangle + \int_\sigma^t \|\nabla u(\tau)\|^2\,d\tau + \int_\sigma^t\int_\Omega \big(g(u_t+v_t)-g(v_t)\big)u(\tau)\,dx\,d\tau$$
$$= \int_\sigma^t \|u_t(\tau)\|^2\,d\tau + \langle u_t(\sigma), u(\sigma)\rangle - \int_\sigma^t \langle f(u+v)-f(v), z(\tau)\rangle\,d\tau. \tag{3.26}$$

As mentioned in the proof of Theorem 3.1 we can justify (3.25) and (3.26) using a density argument. Now using Gronwall's lemma, from (3.25) we obtain

$$\|S(t)\varphi_1 - S(t)\varphi_2\|_{H_0^1\times H} \leqslant c_1 e^{\omega t}\|\varphi_1-\varphi_2\|_{H_0^1\times H}, \tag{3.27}$$

where constants c_1 and ω depend on $\mathcal{A}$, but are independent of φ_i $(i=1,2)$.

On the other hand, taking into account Lemma 3.2 in (3.25)–(3.26) and repeating the argument which has been done in the proof of Theorem 3.1, we find that

$$E\big(u(t)\big) + \int_s^t E\big(u(\tau)\big)\,d\tau \leqslant c_2\left(E\big(u(s)\big) + \int_s^t \|u(\tau)\|^2\,d\tau\right), \quad \forall t \geqslant s \geqslant 0.$$

Integrating the last inequality with respect to s from 0 to t we have

$$E\big(u(t)\big) \leqslant \frac{c_3}{t}E\big(u(0)\big) + c_3(1+t)\sup_{0\leqslant\tau\leqslant t}\|u(\tau)\|^2, \quad \forall t>0, \tag{3.28}$$

where c_3 depends on $\mathcal{A}$, but is independent of φ_i $(i=1,2)$.

Thus according to [1, Theorem 3.11] by (3.27) and (3.28) it follows that the fractal dimension of $\mathcal{A}$ is finite. □

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