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J. Differential Equations 225 (2006) 528–548
Journal of Differential Equations
www.elsevier.com/locate/jde

Global attractors for the plate equation with a localized damping and a critical exponent in an unbounded domain

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Received 6 December 2004
Available online 18 January 2006

Abstract

In this paper, we study the asymptotic behavior of solutions for the plate equation with a localized damping and a critical exponent. We prove the existence, regularity and finite dimensionality of a global attractor in $W_2^2(R^n) \times L_2(R^n)$.

Keywords: Global attractors; Regularity of attractors; Finite dimensionality of attractors

1. Introduction

The paper is devoted to the investigation of a global attractor for the following plate equation in R^n:

$$u_{tt} + \alpha(x)u_t + \Delta^2 u + \lambda u + f(u) = g(x), \tag{1.1}$$

where λ is a positive constant, $\alpha(\cdot)$ and $g(\cdot)$ are given functions and $f(\cdot)$ is a nonlinear function satisfying certain growth conditions.

The global attractors for the hyperbolic equations with interior dissipation were investigated in [1–6] and references therein. The long-time behavior of solutions for the semilinear wave equations with localized damping was investigated in [7,8], where the author established the exponential decay of solutions using a unique continuation result for wave equations from [9].

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0022-0396/$ – see front matter
doi:10.1016/j.jde.2005.12.001

In [10], using finite speed of propagation and a unique continuation result for the wave equations, the existence of a global attractor for the semilinear wave equations with localized damping has been established. The existence of a global attractor for Eq. (1.1), when $\alpha(x) \equiv \alpha_0 > 0$, was studied in [11].

The main objective of this paper is to study the existence, regularity and finite dimensionality of a global attractor for the plate equation with a localized damping and a critical exponent in an unbounded domain. The paper is organized as follows: In the next section we establish the asymptotic compactness of the semigroup generated by the Cauchy problem for Eq. (1.1), in Section 3 we present the proof of point dissipativity and then applying the result from [12], we establish the existence of a global attractor, and finally in Section 4 we prove regularity and finite dimensionality of the global attractor.

2. Preliminaries

We consider the problem

$$u_{tt} + \alpha(x)u_t + \Delta^2 u + \lambda u + f(u) = g(x), \quad (t,x) \in R \times R^n, \tag{2.1}$$

$$u(0,x) = u_0(x), \qquad u_t(0,x) = u_1(x), \quad x \in R^n, \tag{2.2}$$

where $\lambda > 0$, $g \in L_2(R^n)$ and the functions $\alpha(\cdot)$ and $f(\cdot)$ satisfy the following conditions:

$$\alpha \in L_\infty\big(R^n\big), \quad \alpha(\cdot) \geqslant 0, \tag{2.3}$$

$$\alpha(x) \geqslant \alpha_0 > 0 \quad \text{for every } |x| \geqslant r_0 > 0, \tag{2.4}$$

$$f \in C^1(R), \quad \big|f'(u)\big| \leqslant c\big(1 + |u|^p\big), \quad p > 0, \quad (n-4)p \leqslant 4, \tag{2.5}$$

$$f(u) \cdot u \geqslant 0 \quad \text{for every } u \in R. \tag{2.6}$$

We denote the spaces $W_2^s(R^n)$ by H^s, and the norm in H^s by $\|\cdot\|_s$. We introduce the spaces $\mathcal{H}_s = H^{2+2s} \times H^{2s}$.

It is well known that under conditions (2.3)–(2.6) the solution operator $S(t)(u_0,u_1) = (u(t), u_t(t))$, $t \in R$, of problem (2.1), (2.2) generates a C_0-group on the space $\mathcal{H}_0$, which satisfies the following equation

$$\frac{d}{dt} S(t)\theta_0 = AS(t)\theta_0 + F\big(S(t)\theta_0\big), \tag{2.7}$$

and consequently

$$S(t)\theta_0 = e^{tA}\theta_0 + \int_0^t e^{(t-\tau)A} F\big(S(\tau)\theta_0\big)\, d\tau \quad \forall t \geqslant 0, \tag{2.8}$$

where $\theta_0 = (u_0, u_1)$, $F(S(\tau)\theta_0) = (0, -f(u(\tau)) + g)$ and $A\, w = (w_2, -\Delta^2 w_1 - \lambda w_1 - \alpha(\cdot) w_2)$ for $w = (w_1, w_2)$. Using techniques of [8] it is easy to show that

$$\big\|e^{tA}\big\|_{L(\mathcal{H}_s, \mathcal{H}_s)} \leqslant Me^{-\omega t} \quad \forall t \geqslant 0,\ \forall s \in [-1, 1], \tag{2.9}$$

holds for some $M > 0$, $\omega > 0$.

To prove the existence of a global attractor we need the following lemmas:

Lemma 1. *Let us assume that conditions* (2.3)–(2.6) *are satisfied. If* $\theta_m \to \theta_0$ *weakly in* $\mathcal{H}_0$ *as* $m \to \infty$, *then*

$$\begin{aligned} &S(t)\theta_m \to S(t)\theta_0 && \text{weakly in } L_2(0,T;\mathcal{H}_0),\\ &\frac{\partial}{\partial t}S(t)\theta_m \to \frac{\partial}{\partial t}S(t)\theta_0 && \text{weakly in } L_2(0,T;\mathcal{H}_{-1}),\\ &S(t)\theta_m \to S(t)\theta_0 && \text{weakly in } \mathcal{H}_0 \text{ for every } t \geqslant 0. \end{aligned}$$

Proof. The proof of this lemma is the same as the proof of [11, Lemma 1]. □

Lemma 2. *Let us assume the conditions* (2.3)–(2.6) *are satisfied and* B *is a bounded subset of* $\mathcal{H}_0$. *Then for any* $\varepsilon > 0$ *there exist* $t_1 = t_1(\varepsilon, B) > 0$ *and* $r_1 = r_1(\varepsilon, B)$ *such that for every* $t \geqslant t_1$, $r \geqslant r_1$ *and every* $\theta \in B$ *we have*

$$\frac{1}{t}\int_0^t \big\| S(\tau)\theta \big\|^2_{W_2^2(R^n\setminus B(0,r))\times L_2(R^n\setminus B(0,r))}\, d\tau \leqslant \varepsilon, \tag{2.10}$$

where $B(0,r) = \{x\colon x \in R^n,\ |x| \leqslant r\}$.

Proof. Multiplying (2.1) by u_t and integrating over $[\tau, t] \times R^n$ we obtain

$$\begin{aligned} &E\big(u(t), u_t(t)\big) + \int_{R^n} \Phi\big(u(t,x)\big)\, dx - \int_{R^n} g(x)u(t,x)\, dx + \int_\tau^t \int_{R^n} \alpha(x)u_t^2(s,x)\, dx\, ds \\ &\quad = E\big(u(\tau), u_t(\tau)\big) + \int_{R^n} \Phi\big(u(\tau,x)\big)\, dx - \int_{R^n} g(x)u(\tau,x)\, dx, \end{aligned} \tag{2.11}$$

where $E(u(t), u_t(t)) = \frac{1}{2}\|u_t(t)\|^2 + \frac{1}{2}\|\Delta u(t)\|^2 + \frac{\lambda}{2}\|u(t)\|^2$ and $\Phi(s) = \int_0^s f(\tau)\, d\tau$.

Let $\varphi(\cdot) \in C^\infty(R^n)$ be such that $0 \leqslant \varphi(x) \leqslant 1$ and

$$\varphi(x) = \begin{cases} 1, & |x| \geqslant 2, \\ 0, & |x| \leqslant 1, \end{cases}$$

furthermore define $\varphi_r(x) = \varphi(\frac{x}{r})$. Multiplying (2.1) by $\varphi_r u(t,x)$, integrating over $[0,t] \times R^n$ and taking into account (2.5), (2.6), (2.11) we obtain that for every $r \geqslant r_1$ and $t > 0$

$$\int_0^t \big(\|\Delta u\|^2_{L_2(R^n\setminus B(0,2r))} + \|u\|^2_{L_2(R^n\setminus B(0,2r))}\big)\, ds \leqslant c_1\left(1 + \frac{t}{r} + t\|g\|^2_{L_2(R^n\setminus B(0,r))}\right),$$

which together with (2.11) yields (2.10). □

Lemma 3. *Assume the conditions* (2.3)–(2.6) *are satisfied, and B is a bounded subset of $\mathcal{H}_0$. If $\{\theta_m\}$ is a sequence in B weakly converging to θ in $\mathcal{H}_0$, then for any $\varepsilon > 0$ there exists a $T_0 = T_0(\varepsilon, B) > 0$ such that whenever $T \geqslant T_0$*

$$\limsup_{m\to\infty} \left\| S(T)\theta_m - S(T)\theta \right\|_{\mathcal{H}_0} \leqslant \varepsilon \tag{2.12}$$

holds.

Proof. Let $\theta_m = (u_{0m}, u_{1m})$, then $S(t)\theta_m = (u^{(m)}(t), u_t^{(m)}(t))$, where $u^m(t,\cdot)$ is the solution of Eq. (2.1) subject to the conditions $u^m(0,x) = u_{0m}(x)$ and $u_t^m(0,x) = u_{1m}(x)$. Multiplying (2.1) by $(u_t + \frac{\alpha_0}{2}\varphi_{r_0} u)$, integrating over $[0,T]\times R^n$ and taking into account (2.5), (2.6) and (2.11) we obtain that for every $T > 0$

$$\begin{aligned} &\Bigg| \int_0^T \Bigg[\int_{R^n} \varphi_{r_0} \big[|u_t(t)|^2 + |\Delta u(t)|^2 + \lambda |u(t)|^2 \big]\, dx + \int_{R^n} \varphi_{r_0} f\big(u(t,x)\big) u(t,x)\, dx \Bigg] dt \\ &\quad + \int_0^T \Bigg[\int_{B(0,2r_0)} \Delta u(t) u(t) \Delta\varphi_{r_0}\, dx + 2\sum_{i=1}^n \int_{B(0,2r_0)} \Delta u(t) \frac{\partial}{\partial x_i} u(t) \frac{\partial}{\partial x_i}\varphi_{r_0}\, dx \Bigg] dt \\ &\quad - \int_0^T \int_{R^n} \varphi_{r_0} g(x) u(t,x)\, dx\, dt \Bigg| \leqslant c_2. \end{aligned} \tag{2.13}$$

Similar to (2.13), since B is bounded in $\mathcal{H}_0$ and $\theta_m \in B$, for every $T > 0$

$$\begin{aligned} &\Bigg| \int_0^T \Bigg[\int_{R^n} \varphi_{r_0} \big[|u_t^{(m)}|^2 + |\Delta u^{(m)}|^2 + \lambda |u^{(m)}|^2 \big]\, dx + \int_{R^n} \varphi_{r_0} f\big(u^{(m)}\big) u^{(m)}\, dx \Bigg] dt \\ &\quad + \int_0^T \Bigg[\int_{B(0,2r_0)} u^{(m)} \Delta u^{(m)} \Delta\varphi_{r_0}\, dx + 2\sum_{i=1}^n \int_{B(0,2r_0)} \Delta u^{(m)} \frac{\partial u^{(m)}}{\partial x_i} \frac{\partial \varphi_{r_0}}{\partial x_i}\, dx \Bigg] dt \\ &\quad - \int_0^T \int_{R^n} \varphi_{r_0} g(x) u^{(m)}(t,x)\, dx\, dt \Bigg| \leqslant c_2 \end{aligned} \tag{2.14}$$

holds.

By (2.6), Lemma 1 and compact embedding theorems we have

$$\liminf_{m\to\infty} \int_0^T \int_{R^n} \varphi_{r_0} f\big(u^{(m)}\big) u^{(m)}\, dx\, dt \geqslant \int_0^T \int_{R^n} \varphi_{r_0} f(u) u\, dx\, dt. \tag{2.15}$$

By Lemma 1 and inequalities (2.13)–(2.15) we obtain

$$\limsup_{m\to\infty}\int_0^T\int_{R^n}\varphi_{r_0}\left[\left|u_t^{(m)}\right|^2+\left|\Delta u^{(m)}\right|^2+\lambda\left|u^{(m)}\right|^2\right]dx\,dt$$
$$\leqslant 2c_2+\int_0^T\int_{R^n}\varphi_{r_0}\left[|u_t|^2+|\Delta u|^2+\lambda|u|^2\right]dx\,dt,\quad \forall T>0,$$

or

$$\limsup_{m\to\infty}\int_0^T\int_{R^n\setminus B(0,2r_0)}\left[\left|u_t^{(m)}-u_t\right|^2+\left|\Delta(u^{(m)}-u)\right|^2+\lambda\left|u^{(m)}-u\right|^2\right]dx\,dt\leqslant 2c_2,$$
$$\forall T>0. \tag{2.16}$$

Now let $p_i(\cdot)\in C^\infty(R^n)$ be such that

$$p_i(x)=\begin{cases}0, & |x|\geqslant 3r_0,\\ x_i, & |x|\leqslant 2r_0,\end{cases}\quad i=\overline{1,n}.$$

Multiplying (2.1) by $\sum_{i=1}^n(p_i\frac{\partial}{\partial x_i}u+(\frac{1}{2}-\mu)\frac{\partial p_i}{\partial x_i}u)$, integrating over $[0,T]\times R^n$ and taking into account (2.11) we obtain that for every $T>0$

$$\Bigg|\int_0^T\left[\int_{B(0,2r_0)}\left[n\mu\left|u_t(t)\right|^2+(2-n\mu)\left|\Delta u(t)\right|^2-\lambda n\mu\left|u(t)\right|^2\right]dx\right]dt$$
$$+\mu\int_0^T\left[\sum_{i=1}^n\int_{R^n\setminus B(0,2r_0)}\frac{\partial p_i}{\partial x_i}\left[\left|u_t(t)\right|^2-\left|\Delta u(t)\right|^2-\lambda\left|u(t)\right|^2\right]dx\right]dt$$
$$-\int_0^T\sum_{i=1}^n\int_{R^n}\frac{\partial p_i}{\partial x_i}\Phi(u)\,dx\,dt+2\int_0^T\sum_{i=1}^n\sum_{k=1}^n\int_{R^n\setminus B(0,2r_0)}\frac{\partial p_i}{\partial x_k}\Delta u\frac{\partial^2 u}{\partial x_i\partial x_k}\,dx\,dt$$
$$+\left(\frac{1}{2}-\mu\right)\int_0^T\sum_{i=1}^n\int_{R^n}\Delta u\left[\frac{\partial(\Delta p_i)}{\partial x_i}u+2\sum_{k=1}^n\frac{\partial u}{\partial x_k}\frac{\partial^2 p_i}{\partial x_i\partial x_k}\right]dx\,dt$$
$$+\int_0^T\sum_{i=1}^n\int_{R^n}\Delta u\Delta p_i\frac{\partial u}{\partial x_i}\,dx\,dt-\int_0^T\sum_{i=1}^n\int_{R^n}g\left[p_i\frac{\partial u}{\partial x_i}+\left(\frac{1}{2}-\mu\right)\frac{\partial p_i}{\partial x_i}u\right]dx\,dt\Bigg|\leqslant c_3. \tag{2.17}$$

Similar to (2.17)—since B is bounded in $\mathcal{H}_0$ and $\theta_m\in B$—for every $T>0$ we can say that

$$\Bigg|\int_0^T\left[\int_{B(0,2r_0)}\left[n\mu\left|u_t^{(m)}(t)\right|^2+(2-n\mu)\left|\Delta u^{(m)}(t)\right|^2-\lambda n\mu\left|u^{(m)}(t)\right|^2\right]dx\right]dt$$

$$+\mu\int_0^T\Biggl[\sum_{i=1}^n\int_{R^n\setminus B(0,2r_0)}\frac{\partial p_i}{\partial x_i}\bigl[\bigl|u_t^{(m)}(t)\bigr|^2-\bigl|\Delta u^{(m)}(t)\bigr|^2-\lambda\bigl|u^{(m)}(t)\bigr|^2\bigr]\,dx\Biggr]\,dt$$

$$-\int_0^T\sum_{i=1}^n\int_{R^n}\frac{\partial p_i}{\partial x_i}\Phi\bigl(u^{(m)}\bigr)\,dx\,dt+2\int_0^T\sum_{i=1}^n\sum_{k=1}^n\int_{R^n\setminus B(0,2r_0)}\frac{\partial p_i}{\partial x_k}\Delta u^{(m)}\frac{\partial^2u^{(m)}}{\partial x_i\partial x_k}\,dx\,dt$$

$$+\left(\frac{1}{2}-\mu\right)\int_0^T\sum_{i=1}^n\int_{R^n}\Delta u^{(m)}\left[\frac{\partial(\Delta p_i)}{\partial x_i}u^{(m)}+2\sum_{k=1}^n\frac{\partial u^{(m)}}{\partial x_k}\frac{\partial^2 p_i}{\partial x_i\partial x_k}\right]dx\,dt$$

$$+\int_0^T\sum_{i=1}^n\int_{R^n}\Delta u^{(m)}\Delta p_i\frac{\partial u^{(m)}}{\partial x_i}\,dx\,dt-\int_0^T\sum_{i=1}^n\int_{R^n}p_i g\frac{\partial u^{(m)}}{\partial x_i}\,dx\,dt$$

$$-\left(\frac{1}{2}-\mu\right)\int_0^T\sum_{i=1}^n\int_{R^n}g\frac{\partial p_i}{\partial x_i}u^{(m)}\,dx\,dt\Biggr|\leqslant c_3. \tag{2.18}$$

If we take $\mu\in(0,\frac{2}{n})$, then by Lemma 1 and from (2.16)–(2.18) we have

$$\limsup_{m\to\infty}\int_0^T\int_{B(0,2r_0)}\bigl[\bigl|u_t^{(m)}-u_t\bigr|^2+\bigl|\Delta\bigl(u^{(m)}-u\bigr)\bigr|^2\bigr]\,dx\,dt\leqslant c_4(1+\sqrt{T}\,) \tag{2.19}$$

for every $T>0$.

On the other hand, by (2.5), (2.6), (2.10) and (2.11), for any $\delta>0$ there exist $t_0=t_0(\delta,B)>0$ and $r_1=r_1(\delta,B)$ such that for every $T\geqslant t_0$, $r\geqslant r_1$

$$\frac{1}{T}\int_0^T\int_{R^n\setminus B(0,r)}\Phi\bigl(u^{(m)}(t,x)\bigr)\,dx\,dt\leqslant\delta. \tag{2.20}$$

Thus (2.16), (2.19) and (2.20) give us

$$\limsup_{m\to\infty}\frac{1}{T}\int_0^T\biggl[E\bigl(u^{(m)}(t),u_t^{(m)}(t)\bigr)+\int_{R^n}\Phi\bigl(u^{(m)}(t,x)\bigr)\,dx-\int_{R^n}g(x)u^{(m)}(t,x)\,dx\biggr]\,dt$$

$$\leqslant\frac{1}{T}\int_0^T\biggl[E\bigl(u(t),u_t(t)\bigr)+\int_{R^n}\Phi\bigl(u(t,x)\bigr)\,dx-\int_{R^n}g(x)u(t,x)\,dx\biggr]\,dt+\frac{c_5(1+\sqrt{T}\,)}{T}+\delta,$$

which together with (2.11) yields

$$\limsup_{m\to\infty}\left[E\big(u^{(m)}(T),u_t^{(m)}(T)\big)+\int\limits_{R^n}\Phi\big(u^{(m)}(T,x)\big)\,dx+\frac{1}{T}\int\limits_0^T\int\limits_t^T\int\limits_{R^n}\alpha(x)\big|u_t^{(m)}(s,x)\big|^2\,dx\,ds\,dt\right]$$

$$\leqslant E\big(u(T),u_t(T)\big)+\int\limits_{R^n}\Phi\big(u(T,x)\big)\,dx+\frac{1}{T}\int\limits_0^T\int\limits_t^T\int\limits_{R^n}\alpha(x)\big|u_t(s,x)\big|^2\,dx\,ds\,dt$$

$$+\frac{c_5(1+\sqrt{T})}{T}+\delta \tag{2.21}$$

for every $T\geqslant t_0$.

Since by Lemma 1

$$\liminf_{m\to\infty}\int\limits_{R^n}\Phi\big(u^{(m)}(T,x)\big)\,dx\geqslant\int\limits_{R^n}\Phi\big(u(T,x)\big)\,dx,$$

$$\liminf_{m\to\infty}\int\limits_0^T\int\limits_t^T\int\limits_{R^n}\alpha(x)\big|u_t^{(m)}(s,x)\big|^2\,dx\,ds\,dt\geqslant\int\limits_0^T\int\limits_t^T\int\limits_{R^n}\alpha(x)\big|u_t(s,x)\big|^2\,dx\,ds\,dt,$$

inequality (2.21) gives (2.12). □

Now Lemmas 1–3 give us asymptotic compactness of $S(t)$, which is included in the following theorem:

Theorem 1. *Assume conditions* (2.3)–(2.6) *hold. Then for any bounded subset B of $\mathcal{H}_0$, the set $\{S(t_m)\theta_m\}_{m=1}^{\infty}$ is relatively compact in $\mathcal{H}_0$, where $t_m\to\infty$ and $\{\theta_m\}_{m=1}^{\infty}\subset B$.*

Proof. Since B is bounded, taking into account (2.5) and (2.6) in (2.11) we find that $\sup_{t\geqslant 0}\sup_{\theta\in B}\|S(t)\theta\|_{\mathcal{H}_0}<\infty$. Therefore there exists a bounded subset B_0 of $\mathcal{H}_0$ such that $S(t)\theta\in B_0$ for every $t\geqslant 0$ and $\theta\in B$. Thus $\{S(t_m)\theta_m\}_{m=1}^{\infty}$ has a subsequence $\{S(t_{m_k})\theta_{m_k}\}_{k=1}^{\infty}$ weakly converging in $\mathcal{H}_0$ to some $\psi\in\mathcal{H}_0$. From Lemma 3 we know that, if $\{\xi_\nu\}_{\nu=1}^{\infty}\subset B_0$ and $\xi_\nu\to\xi$ weakly in $\mathcal{H}_0$, then for any $\varepsilon>0$ there exists a $T_0=T_0(\varepsilon,B_0)>0$ such that

$$\limsup_{\nu\to\infty}\big\|S(T_0)\xi_\nu-S(T_0)\xi\big\|_{\mathcal{H}}\leqslant\varepsilon. \tag{2.22}$$

For $t_{m_k}\geqslant T_0$—since $S(t_{m_k}-T_0)\theta_{m_k}\in B_0$—there is a subsequence $\{S(t_{m_{k_\nu}}-T_0)\theta_{m_{k_\nu}}\}_{\nu=1}^{\infty}$ weakly converging to some ξ in $\mathcal{H}_0$. Then by Lemma 1, the sequence $\{S(t_{m_{k_\nu}})\theta_{m_{k_\nu}}\}_{\nu=1}^{\infty}$ weakly converges to $S(T_0)\xi$ in $\mathcal{H}_0$. Hence from the uniqueness of the limit we get $\psi=S(T_0)\xi$. Taking $\xi_\nu=S(t_{m_{k_\nu}}-T_0)\theta_{m_{k_\nu}}$ in (2.22) we obtain

$$\limsup_{\nu\to\infty}\big\|S(t_{m_{k_\nu}})\theta_{m_{k_\nu}}-\psi\big\|_{\mathcal{H}_0}\leqslant\varepsilon$$

and consequently $\liminf_{m\to\infty}\|S(t_m)\theta_m-\psi\|_{\mathcal{H}_0}=0$. In other words the sequence $\{S(t_m)\theta_m\}_{m=1}^{\infty}$ has a subsequence strongly convergent in $\mathcal{H}_0$. It can be seen in a similar way that every subsequence of $\{S(t_m)\theta_m\}_{m=1}^{\infty}$ has a subsequence strongly convergent in $\mathcal{H}_0$. Thus the set $\{S(t_m)\theta_m\}_{m=1}^{\infty}$ is relatively compact in $\mathcal{H}_0$. □

3. Existence of a global attractor

In this section, we shall show the existence of the global attractor. To this end, we first prove the point dissipativity of $S(t)$.

Let $Z = \{\varphi\colon \varphi \in \mathcal{H}_0,\ S(t)\varphi = \varphi,\ \forall t \in R\}$. From condition (2.6) it follows that Z is bounded in $\mathcal{H}_0$ (even in $\mathcal{H}_1$).

Theorem 2. *Assume conditions* (2.3)–(2.6) *hold. Then*

$$\lim_{t\to\infty} \operatorname{dist}\big(S(t)\theta, Z\big) = 0, \tag{3.1}$$

for every $\theta \in \mathcal{H}_0$.

Proof. Let $\theta \in \mathcal{H}_0$. From Theorem 1 it follows that the ω-limit set of θ namely

$$\omega(\theta) = \bigcap_{\tau \geqslant 0} \overline{\bigcup_{t \geqslant \tau} S(t)\theta}^{cl}$$

is compact in $\mathcal{H}_0$, invariant with respect to $S(t)$ and

$$\lim_{t\to\infty} \operatorname{dist}\big(S(t)\theta, \omega(\theta)\big) = 0. \tag{3.2}$$

Let $(u, u_t) = S(t)\theta$. Since by (2.3) and (2.11) the Lyapunov function $L(S(t)\theta) := E(u(t), u_t(t)) + \int_{R^n} \Phi(u(t,x))\,dx - \int_{R^n} g(x)u(t,x)\,dx$ is nonincreasing and bounded below, it has a limit at positive infinity, i.e.

$$\lim_{t\to\infty} L\big(S(t)\theta\big) = l.$$

This means that

$$L(\varphi) = l$$

for every $\varphi \in \omega(\theta)$. Consequently if $\varphi \in \omega(\theta)$ and $(v, v_t) = S(t)\varphi$, then $v(t,x)$ satisfies the following equations:

$$\begin{cases} v_{tt} + \Delta^2 v + \lambda v + f(v) = g(x), & (t,x) \in R \times R^n, \\ \alpha v_t = 0, & (t,x) \in R \times R^n. \end{cases} \tag{3.3}$$

Let $w(t,\cdot) = v(t+t_0,\cdot) - v(t,\cdot)$ and $h(t,x) = \int_0^1 f'(v + \tau w)\,d\tau$. Then from (3.3) we obtain

$$\begin{cases} w_{tt} + \Delta^2 w + \lambda w + hw = 0, & (t,x) \in R \times R^n, \\ \alpha w = 0, & (t,x) \in R \times R^n. \end{cases} \tag{3.4}$$

Now let us show that $\varphi \in Z$. (Note that when $n \leqslant 3$, since $h \in L_\infty(R \times R^n)$ we see $w = 0$ in $R \times R^n$ from the result of [13], consequently $\varphi \in Z$.) Multiplying both sides of $(3.4)_1$ by $e^{k\sum_{j=1}^n x_j}$ and applying the Fourier transform we have

$$\frac{\partial^2}{\partial t^2}\mathcal{F}\big(we^{k\sum_{j=1}^n x_j}\big)(y) + \left[\sum_{j=1}^n (k - iy_j)^2\right]^2 \mathcal{F}\big(we^{k\sum_{j=1}^n x_j}\big)(y) + \lambda \mathcal{F}\big(we^{k\sum_{j=1}^n x_j}\big)(y)$$
$$+ \mathcal{F}\big(hwe^{k\sum_{j=1}^n x_j}\big)(y) = 0, \quad (t, y) \in R \times R^n, \tag{3.5}$$

where $k > 1$ and $\mathcal{F}(\varphi)(y) = \frac{1}{(2\pi)^{n/2}} \int_{R^n} \varphi(x) e^{-i\langle x, y\rangle}\, dx$, $\langle x, y\rangle = \sum_{j=1}^n x_j y_j$.

Multiplying both sides of (3.5) by

$$\frac{i(y_1 + y_2 + \cdots + y_n)}{\sum_{j=1}^n (k - iy_j)^2} \overline{\mathcal{F}}\big(we^{k\sum_{j=1}^n x_j}\big)(y)$$

and integrating the real parts over $(0, t) \times R^n$ we obtain

$$\int_0^t \int_{R^n} \frac{2k(y_1 + y_2 + \cdots + y_n)^2}{\left|\sum_{j=1}^n (k - iy_j)^2\right|^2} \left|\mathcal{F}\big(w_t e^{k\sum_{j=1}^n x_j}\big)\right|^2 dy\, d\tau$$
$$+ \int_0^t \int_{R^n} \big[2k(y_1 + y_2 + \cdots + y_n)^2 - \lambda\big] \left|\mathcal{F}\big(we^{k\sum_{j=1}^n x_j}\big)\right|^2 dy\, d\tau$$
$$+ \operatorname{Re} \int_0^t \int_{R^n} \frac{i(y_1 + y_2 + \cdots + y_n)}{\sum_{j=1}^n (k - iy_j)^2} \mathcal{F}\big(hwe^{k\sum_{j=1}^n x_j}\big) \overline{\mathcal{F}}\big(we^{k\sum_{j=1}^n x_j}\big)\, dy\, d\tau$$
$$\leqslant c_1\big(\|\omega(\theta)\|_{\mathcal{H}_0}\big), \quad \forall t > 0,$$

which together with Plancherel's theorem gives us

$$\operatorname{Re} \int_0^t \int_{R^n} \frac{i(y_1 + y_2 + \cdots + y_n)}{\sum_{j=1}^n (k - iy_j)^2} \mathcal{F}\big(hwe^{k\sum_{j=1}^n x_j}\big) \overline{\mathcal{F}}\big(we^{k\sum_{j=1}^n x_j}\big)\, dy\, d\tau$$
$$+ 2k \int_0^t \int_{R^n} \left[\sum_{j=1}^n \frac{\partial}{\partial x_j}\big(we^{k\sum_{j=1}^n x_j}\big)\right]^2 dx\, d\tau$$
$$\leqslant \lambda \int_0^t \int_{R^n} \big[we^{k\sum_{j=1}^n x_j}\big]^2 dx\, dt + c_1\big(\|\omega(\theta)\|_{\mathcal{H}_0}\big), \quad \forall t > 0, \tag{3.6}$$

where $\|\omega(\theta)\|_{\mathcal{H}_0} = \sup_{\varphi \in \omega(\theta)} \|\varphi\|_{\mathcal{H}_0}$.

Now in order to estimate the first term on the left side of (3.6) let us show that if $u, v \in W_2^2(R^n)$ and $v = 0$ in $R^n \backslash B(0, r_0)$ then

$$\left\| f'(u)v \right\|_{L_q(R^n)} \leqslant c\big(r_0, \|u\|_{W_2^2(R^n)}\big) \left\| \sum_{j=1}^{n} \frac{\partial}{\partial x_j} v \right\|_{L_2(R^n)} \tag{3.7}$$

where

$$q = \min\left\{ \frac{2n}{n+3}, \frac{2(n-1)(n-4)}{n(n-1)-16} \right\} \quad \text{for } n \geqslant 5$$

and $1 < q < 2$ for $n < 5$. We will prove (3.7) in the case $n \geqslant 5$ (in the case $n < 5$ the proof is simpler). Let $x_i = \sum_{j=1}^{i} y_j$, $i = \overline{1,n}$ and $\tilde{u}(y) = u(x)$, $\tilde{v}(y) = v(x)$. Since $\tilde{v}(y) = 0$ for $|y_1| \geqslant r_0$ we have

$$\sup_{y_1 \in R} \left\| \tilde{v}(y_1, \cdot) \right\|_{L_2(R^{n-1})} \leqslant c_2(\tilde{r}_0) \left\| \frac{\partial}{\partial y_1} \tilde{v} \right\|_{L_2(R^n)} \qquad \|\tilde{v}\|_{L_2(R^n)} \leqslant c_2(\tilde{r}_0) \left\| \frac{\partial}{\partial y_1} \tilde{v} \right\|_{L_2(R^n)}. \tag{3.8}$$

Let

$$\beta = \max\left\{ 1, \frac{4n}{(n-4)(n+3)} \right\}.$$

Since $W_2^2(R^n) \subset L_{2\beta}(R_{y_1}; L_{2(n-1)/(n-4-1/\beta)}(R^{n-1}))$ using (1.3) and Holder inequality we find that

$$\begin{aligned}
\int_{R^n} \left| f'(u)v \right|^q dx &= \int_{-r_0}^{r_0} dy_1 \int_{\{y' :\, y' \in R^{n-1},\ |y'| \leqslant \tilde{r}_0\}} \left| f'(\tilde{u})\tilde{v} \right|^q dy' \\
&\leqslant c_3(\tilde{r}_0) \int_{-r_0}^{r_0} \big(1 + \big\| |\tilde{u}|^p \big\|^q_{L_{2q/(2-q)}(R^{n-1})}\big) \|\tilde{v}\|^q_{L_2(R^{n-1})}\, dy_1 \\
&\leqslant c_4(r_0) \int_{-r_0}^{r_0} \big(1 + \|\tilde{u}\|^{2\beta}_{L_{2(n-1)/(n-3-2/\beta)}(R^{n-1})}\big) \|\tilde{v}\|^q_{L_2(R^{n-1})}\, dy_1 \\
&\leqslant \check{c}\big(r_0, \|u\|_{W_2^2(R^n)}\big) \sup_{y_1 \in R} \left\| \tilde{v}(y_1, \cdot) \right\|^q_{L_2(R^{n-1})}
\end{aligned}$$

which together with $(3.8)_1$ yields (3.7).

It is easy to verify that

$$\left| \frac{(1 + \sum_{j=1}^{n} y_j^2)^{\varepsilon/2} \sum_{j=1}^{n} y_j}{\sum_{j=1}^{n} (k - i y_j)^2} \right| \leqslant c_4(n) \left| \sum_{j=1}^{n} y_j \right| + k^{\varepsilon - 1} c_5(n), \quad \text{for } \varepsilon \in (1, 2).$$

Then we have

$$\left|\int_0^t\int_{R^n}\frac{i(y_1+y_2+\cdots+y_n)}{\sum_{j=1}^n(k-iy_j)^2}\mathcal{F}\big(hwe^{k\sum_{j=1}^n x_j}\big)\overline{\mathcal{F}}\big(we^{k\sum_{j=1}^n x_j}\big)\,dy\,d\tau\right|$$

$$\leqslant\int_0^t\int_{R^n}\left(c_4(n)\left|\sum_{j=1}^n y_j\right|+c_5(n)k^{\varepsilon-1}\right)\left(1+\sum_{j=1}^n y_j^2\right)^{-\varepsilon/2}\left|\mathcal{F}\big(hwe^{k\sum_{j=1}^n x_j}\big)\right|$$

$$\times\left|\overline{\mathcal{F}}\big(we^{k\sum_{j=1}^n x_j}\big)\right|dy\,d\tau$$

$$\leqslant c_4(n)\int_0^t\big\|hwe^{k\sum_{j=1}^n x_j}\big\|_{W_2^{-\varepsilon}(R^n)}\left\|\sum_{j=1}^n\frac{\partial}{\partial x_j}\big(we^{k\sum_{j=1}^n x_j}\big)\right\|_{L_2(R^n)}d\tau$$

$$+c_5(n)k^{\varepsilon-1}\int_0^t\big\|hwe^{k\sum_{j=1}^n x_j}\big\|_{W_2^{-\varepsilon}(R^n)}\big\|we^{k\sum_{j=1}^n x_j}\big\|_{L_2(R^n)}d\tau$$

$$\leqslant c_6(n)\int_0^t\big\|hwe^{k\sum_{j=1}^n x_j}\big\|_{L_\alpha(R^n)}\left\|\sum_{j=1}^n\frac{\partial}{\partial x_j}\big(we^{k\sum_{j=1}^n x_j}\big)\right\|_{L_2(R^n)}d\tau$$

$$+c_7(n)k^{\varepsilon-1}\int_0^t\big\|hwe^{k\sum_{j=1}^n x_j}\big\|_{L_\alpha(R^n)}\big\|we^{k\sum_{j=1}^n x_j}\big\|_{L_2(R^n)}d\tau, \tag{3.9}$$

where $\alpha=\frac{2n}{n+2\varepsilon}$ when $n\geqslant 5$, and $1<\alpha<2$ when $n<5$. From the definition of q follows that $1<\frac{n(2-q)}{2q}<2$, when $n\geqslant 5$. So taking $\varepsilon=\frac{n(2-q)}{2q}$ we have $\alpha=q$ when $n\geqslant 5$. Hence taking into account (3.7) in (3.9) we obtain

$$\left|\int_0^t\int_{R^n}\frac{i(y_1+y_2+\cdots+y_n)}{\sum_{j=1}^n(k-iy_j)^2}\mathcal{F}\big(hwe^{k\sum_{j=1}^n x_j}\big)\overline{\mathcal{F}}\big(we^{k\sum_{j=1}^n x_j}\big)\,dy\,d\tau\right|$$

$$\leqslant c_8\big(\|\omega(\theta)\|_{\mathcal{H}_0}\big)\int_0^t\left\|\sum_{j=1}^n\frac{\partial}{\partial x_j}\big(we^{k\sum_{j=1}^n x_j}\big)\right\|_{L_2(R^n)}^2d\tau$$

$$+c_9\big(\|\omega(\theta)\|_{\mathcal{H}_0}\big)k^{\varepsilon-1}\int_0^t\left\|\sum_{j=1}^n\frac{\partial}{\partial x_j}\big(we^{k\sum_{j=1}^n x_j}\big)\right\|_{L_2(R^n)}\big\|we^{k\sum_{j=1}^n x_j}\big\|_{L_2(R^n)}d\tau. \tag{3.10}$$

Since by $(3.4)_2$ we have $w=0$ in $(0,t)\times R^n\backslash B(0,r_0)$, from $(3.8)_2$ it follows that

$$\int_0^t\int_{R^n}\left[\sum_{j=1}^n\frac{\partial}{\partial x_j}\big(we^{k\sum_{j=1}^n x_j}\big)\right]^2dx\,d\tau\geqslant\hat{c}\int_0^t\int_{B(0,r_0)}\big[we^{k\sum_{j=1}^n x_j}\big]^2dx\,d\tau,\quad\forall t>0. \tag{3.11}$$

Thus taking into account (3.10) and (3.11) in (3.6) for large enough k we find

$$\int_0^t \left\| w(\tau)e^{k\sum_{j=1}^n x_j} \right\|^2_{L_2(B(0,r_0))} d\tau \leqslant c_{10}\left(\left\|\omega(\theta)\right\|_{\mathcal{H}_0}\right), \quad \forall t > 0, \tag{3.12}$$

which yields

$$\int_0^t \left\| w(\tau) \right\|^2_{L_2(B(0,r_0))} d\tau \leqslant c_{11}\left(\left\|\omega(\theta)\right\|_{\mathcal{H}_0}\right), \quad \forall t > 0, \tag{3.13}$$

Since $w(t,x)$ is the solution of the equation

$$w_{tt} + \alpha w_t + \Delta^2 w + \lambda w + hw = 0, \quad (t,x) \in R \times R^n,$$

taking into account (2.5), (2.9) and (3.13) we obtain

$$\int_0^t \left(\left\|w_t\right\|^2_{-2} + \left\|w\right\|^2_0\right) d\tau \leqslant c_{12}\left(\left\|\omega(\theta)\right\|_{\mathcal{H}_0}\right), \quad \forall t > 0. \tag{3.14}$$

Now let $T > 0$ and define $\widehat{w} = t(T-t)w$. Then from (3.4) we have

$$\begin{cases} \widehat{w}_{tt} + \Delta^2\widehat{w} + \lambda\widehat{w} + h\widehat{w} = 2(T-2t)w_t - 2w, & (t,x) \in R \times R^n, \\ \alpha\widehat{w} = 0, & (t,x) \in R \times R^n. \end{cases} \tag{3.15}$$

As above, multiplying both sides of $(3.15)_1$ by $e^{k\sum_{j=1}^n x_j}$ and applying the Fourier transform, similar to (3.5) we find

$$\begin{aligned} &\frac{\partial^2}{\partial t^2}\mathcal{F}\left(\widehat{w}e^{k\sum_{j=1}^n x_j}\right)(y) + \left[\sum_{j=1}^n (k - iy_j)^2\right]^2 \mathcal{F}\left(\widehat{w}e^{k\sum_{j=1}^n x_j}\right)(y) + \lambda\mathcal{F}\left(\widehat{w}e^{k\sum_{j=1}^n x_j}\right)(y) \\ &\quad + \mathcal{F}\left(h\widehat{w}e^{k\sum_{j=1}^n x_j}\right)(y) \\ &= 2(T-2t)\frac{\partial}{\partial t}\mathcal{F}\left(we^{k\sum_{j=1}^n x_j}\right)(y) - 2\mathcal{F}\left(we^{k\sum_{j=1}^n x_j}\right)(y) \end{aligned}$$

from which by multiplying by

$$\frac{i(y_1 + y_2 + \cdots + y_n)}{\sum_{j=1}^n (k - iy_j)^2}\overline{\mathcal{F}}\left(\widehat{w}e^{k\sum_{j=1}^n x_j}\right)(y),$$

integrating real parts over $(0,T) \times R^n$ and taking into account that $\widehat{w}(0,\cdot) = \widehat{w}(T,\cdot) = 0$ we obtain

$$\int_0^T \int_{R^n} \frac{2k(y_1+y_2+\cdots+y_n)^2}{\left|\sum_{j=1}^n (k-iy_j)^2\right|^2} \left|\mathcal{F}\left(\widehat{w}_t e^{k\sum_{j=1}^n x_j}\right)\right|^2 dy\,dt$$

$$+\int_0^T \int_{R^n} \left[2k(y_1+y_2+\cdots+y_n)^2-\lambda\right] \left|\mathcal{F}\left(\widehat{w} e^{k\sum_{j=1}^n x_j}\right)\right|^2 dy\,dt$$

$$+\operatorname{Re}\int_0^T \int_{R^n} \frac{i(y_1+y_2+\cdots+y_n)}{\sum_{j=1}^n (k-iy_j)^2} \mathcal{F}\left(h\widehat{w} e^{k\sum_{j=1}^n x_j}\right)\overline{\mathcal{F}}\left(\widehat{w} e^{k\sum_{j=1}^n x_j}\right) dy\,dt$$

$$=2\operatorname{Re}\int_0^T \int_{R^n} \frac{i(y_1+y_2+\cdots+y_n)}{\sum_{j=1}^n (k-iy_j)^2} \mathcal{F}\left(w e^{k\sum_{j=1}^n x_j}\right)\overline{\mathcal{F}}\left(\widehat{w} e^{k\sum_{j=1}^n x_j}\right) dy\,dt$$

$$+2\operatorname{Re}\int_0^T \int_{R^n} \frac{i(y_1+y_2+\cdots+y_n)}{\sum_{j=1}^n (k-iy_j)^2} (2t-T)\mathcal{F}\left(w e^{k\sum_{j=1}^n x_j}\right)\overline{\mathcal{F}}\left(\widehat{w}_t e^{k\sum_{j=1}^n x_j}\right) dy\,dt$$

which yields

$$\int_0^T \int_{R^n} \left[2k(y_1+y_2+\cdots+y_n)^2-\lambda-1\right] \left|\mathcal{F}\left(\widehat{w} e^{k\sum_{j=1}^n x_j}\right)\right|^2 dy\,dt$$

$$+\operatorname{Re}\int_0^T \int_{R^n} \frac{i(y_1+y_2+\cdots+y_n)}{\sum_{j=1}^n (k-iy_j)^2} \mathcal{F}\left(h\widehat{w} e^{k\sum_{j=1}^n x_j}\right)\overline{\mathcal{F}}\left(\widehat{w} e^{k\sum_{j=1}^n x_j}\right) dy\,dt$$

$$\leqslant \left(T^2+1\right)\int_0^T \int_{R^n} \left|\mathcal{F}\left(w e^{k\sum_{j=1}^n x_j}\right)\right|^2 dy\,dt. \tag{3.16}$$

Taking into account (3.10), (3.11) and (3.12) in (3.16) for large enough k we find that

$$\int_0^T \left\|\widehat{w} e^{k\sum_{j=1}^n x_j}\right\|^2_{L_2(B(0,r_0))} dt \leqslant \left(T^2+1\right)c_{13}\left(\left\|\omega(\theta)\right\|_{\mathcal{H}_0}\right), \quad \forall T>0,$$

from which follows

$$\int_0^T \|\widehat{w}\|^2_{L_2(B(0,r_0))} dt \leqslant \left(T^2+1\right)c_14\left(\left\|\omega(\theta)\right\|_{\mathcal{H}_0}\right), \quad \forall T>0. \tag{3.17}$$

Since $\widehat{w}(t,x)$ is the solution of the equation

$$\widehat{w}_{tt}+\alpha\widehat{w}_t+\Delta^2\widehat{w}+\lambda\widehat{w}+h\widehat{w}=2(T-2t)w_t-2w, \quad (t,x)\in R\times R^n,$$

taking into account (2.5), (2.9), (3.14) and (3.17) we obtain

$$\int_0^T \left(\|\widehat{w}_t\|_{-2}^2 + \|\widehat{w}\|_0^2\right) dt \leqslant \left(T^2+1\right) c_{15}\left(\|\omega(\theta)\|_{\mathcal{H}_0}\right), \quad \forall T > 0$$

consequently

$$\int_{T/3}^{2T/3} \|S(t+t_0)\varphi - S(t)\varphi\|_{\mathcal{H}_{-1}}^2 \, dt \leqslant \frac{1}{T^2} c_{16}\left(\|\omega(\theta)\|_{\mathcal{H}_0}\right), \quad \forall T > 0,\ \forall \varphi \in \omega(\theta). \tag{3.18}$$

Since

$$\left\| \frac{\partial}{\partial t} S(t)\varphi \right\|_{\mathcal{H}_{-1}} \leqslant c_{17}\left(\|\omega(\theta)\|_{\mathcal{H}_0}\right), \quad \forall t > 0,\ \forall \varphi \in \omega(\theta),$$

using integration by parts on (3.18) we obtain

$$\left\| S\left(\frac{2}{3}T + t_0\right)\varphi - S\left(\frac{2}{3}T\right)\varphi \right\|_{\mathcal{H}_{-1}}^2 \leqslant \frac{1}{T^{1/2}} c_{18}\left(\|\omega(\theta)\|_{\mathcal{H}_0}\right), \quad \forall T > 0,\ \forall \varphi \in \omega(\theta). \tag{3.19}$$

Let $\varphi_n = S(-n)\varphi$. Then from (3.19) we find that

$$\begin{aligned} \|S(t+t_0)\varphi - S(t)\varphi\|_{\mathcal{H}_{-1}}^2 &= \|S(t+t_0+n)\varphi_n - S(t+n)\varphi_n\|_{\mathcal{H}_{-1}}^2 \\ &\leqslant \frac{1}{(t+n)^{1/2}} c_{19}\left(\|\omega(\theta)\|_{\mathcal{H}_0}\right). \end{aligned}$$

The last inequality means that $\varphi \in Z$ and consequently $\omega(\theta) \subset Z$, which together with (3.2) gives (3.1). □

Now Theorems 1, 2 and [12, Theorem 3.1] give the existence of a global attractor.

Theorem 2. *Assume conditions* (2.3)–(2.6) *hold. Then problem* (2.1), (2.2) *has a global attractor* $\mathcal{A} \subset \mathcal{H}$*, which is invariant and compact.*

4. Regularity and finite dimensionality

To prove the regularity of the global attractor $\mathcal{A}$ we will need the following lemmas.

Lemma 4. *For any* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that if* $|t_2 - t_1| < \delta$ *and* $\theta \in \mathcal{A}$ *then* $\|S(t_2)\theta - S(t_1)\theta\|_{\mathcal{H}_0} < \varepsilon$.

Proof. Let $t_2 > t_1$ and $\Delta t = t_2 - t_1$. From (2.8) we obtain

$$\begin{aligned}\left\|S(t_2)\theta - S(t_1)\theta\right\|_{\mathcal{H}_0} &= \left\|S(\Delta t)S(t_1)\theta - S(t_1)\theta\right\|_{\mathcal{H}_0}\\ &\leqslant \left\|e^{\Delta t A}S(t_1)\theta - S(t_1)\theta\right\|_{\mathcal{H}_0} + \int_0^{\Delta t}\left\|e^{(\Delta t-\tau)A}F\big(S(\tau)S(t_1)\theta\big)\right\|_{\mathcal{H}_0}d\tau\\ &= I_1 + I_2. \end{aligned}\tag{4.1}$$

Taking into account (2.5) and (2.9) we get

$$I_2 \leqslant c\Delta t. \tag{4.2}$$

On the other hand, since $\lim_{t\to 0^+} e^{tA}\varphi = \varphi$ for every $\varphi \in \mathcal{H}_0$ and $\mathcal{A}$ is compact in $\mathcal{H}_0$, for any $\varepsilon > 0$ there exist $\delta_1 > 0$ such that if $0 < t < \delta_1$, then for every $\varphi \in \mathcal{A}$

$$\left\|e^{tA}\varphi - \varphi\right\|_{\mathcal{H}_0} < \frac{\varepsilon}{2}$$

holds. Since $\mathcal{A}$ is invariant, from the last inequality we obtain that if $\Delta t < \delta_1$ then

$$I_1 < \frac{\varepsilon}{2}. \tag{4.3}$$

Choose $\delta = \min\{\delta_1, \frac{\varepsilon}{2c}\}$ from (4.1)–(4.3) we get that if $|t_2 - t_1| < \delta$ then

$$\left\|S(t_2)\theta - S(t_1)\theta\right\|_{\mathcal{H}_0} < \varepsilon$$

for every $\theta \in \mathcal{A}$. □

Lemma 5. *Assume K is a compact subset of H^2 and $f(\cdot)$ satisfies condition* (2.5). *Then for any $\varepsilon > 0$ there exists $\delta > 0$ such that $\|v_2 - v_1\|_2 < \delta$ implies*

$$\left\|\left[f'(v_2) - f'(v_1)\right]u\right\|_{-2+s} \leqslant \varepsilon\|u\|_s \tag{4.4}$$

for every $v_1, v_2 \in K$ and $u \in H^s$ $(s = 0, 2)$.

Proof. Let us perform the proof in the more interesting case $n > 4$. Denote

$$p^* = \frac{2n}{n-4} \quad \text{and} \quad \check{c} = \sup_{\substack{v\in H^2\\ v\neq 0}} \frac{\|v\|_{L_{p^*}(R^n)}}{\|v\|_2}.$$

Since K is compact in H^2 for $\varepsilon > 0$ there exists $\delta_1 > 0$ such that if $\operatorname{mes} E < \delta_1$ then

$$\left\|f'(v)\right\|_{L_{n/2}(E)} < \frac{\varepsilon}{8\check{c}} \tag{4.5}$$

for every $v \in K$. Let $\alpha = \frac{\check{c}}{\delta_1^{1/p^*}} \sup_{v \in K} \|v\|_2$. Define the following subsets of R^n:

$$Q_i = \{x\colon\ x \in R^n,\ |v_i(x)| \leqslant \alpha\}, \quad i = 1, 2.$$

Then

$$\operatorname{mes}(R^n \backslash Q_i) \leqslant \left(\frac{1}{\alpha}\|v_i\|_{L_{p^*}(R^n)}\right)^{p^*} \leqslant \delta_1, \quad i = 1, 2. \tag{4.6}$$

Since $f'(\cdot)$ is uniformly continuous in $[-\alpha, \alpha]$, there exists $\delta_2 > 0$ such that if $s_1, s_2 \in [-\alpha, \alpha]$ and $|s_1 - s_2| < \delta_2$ then

$$|f'(s_2) - f'(s_1)| < \frac{\varepsilon}{4}. \tag{4.7}$$

Let $Q_3 = \{x\colon\ x \in R^n,\ |v_2(x) - v_1(x)| < \delta_2\}$ and $\delta = \frac{1}{\check{c}} \min\{\delta_1^{1+1/p^*}, \delta_2^{1+1/p^*}\}$. Then from $\|v_2 - v_1\|_2 < \delta$ we obtain

$$\operatorname{mes}(R^n \backslash Q_3) \leqslant \left(\frac{1}{\delta_2}\|v_2 - v_1\|_{L_{p^*}(R^n)}\right)^{p^*} < (\check{c}\delta)^{\frac{p^*}{1+p^*}} \leqslant \delta_1. \tag{4.8}$$

Thus if $\|v_2 - v_1\|_2 < \delta$, then from (4.5)–(4.8) we have

$$\begin{aligned}
&\int_{R^n} |f'(v_2) - f'(v_1)||u||\varphi|\,dx \\
&\quad \leqslant \int_{\bigcap_{i=1}^3 Q_i} |f'(v_2) - f'(v_1)||u||\varphi|\,dx + \sum_{i=1}^3 \int_{R^n \backslash Q_i} |f'(v_2) - f'(v_1)||u||\varphi|\,dx \\
&\quad \leqslant \frac{\varepsilon}{4}\|u\|_0\|\varphi\|_0 \\
&\qquad + \sum_{i=1}^3 \|u\|_{L_{2n/n-2s}(R^n)}\left[\|f'(v)\|_{L_{n/2}(R^n \backslash Q_i)} + \|f'(v_2)\|_{L_{n/2}(R^n \backslash Q_i)}\right]\|\varphi\|_{L_{2n/n+2s-4}(R^n)} \\
&\quad \leqslant \frac{\varepsilon}{4}\|u\|_s\|\varphi\|_{2-s} + \sum_{i=1}^3 \frac{\varepsilon}{4\check{c}}\check{c}\|u\|_s\|\varphi\|_{2-s} = \varepsilon\|u\|_s\|\varphi\|_{2-s}
\end{aligned}$$

for every $u \in H^s$ and $\varphi \in H^{2-s}$ $(s = 0, 2)$. The last inequality yields (4.4). □

Now we can prove the regularity of the global attractor.

Theorem 3. *The global attractor $\mathcal{A}$ is a bounded subset of the space $\mathcal{H}_1$.*

Proof. Let $\theta_0 = (u_0, u_1) \in \mathcal{A}$ and choose $\varepsilon \in (0, \frac{\omega}{M})$. According to Lemma 5, there exists $\varepsilon_1 = \varepsilon_1(\varepsilon)$ such that if $\|\theta_2 - \theta_1\|_{\mathcal{H}_0} < \varepsilon_1$ then

$$\left\| \left(\overset{\bullet}{F}(\theta_2) - \overset{\bullet}{F}(\theta_1) \right) \eta \right\|_{\mathcal{H}_s} < \varepsilon \|\eta\|_{\mathcal{H}_s}, \quad \forall \theta_2, \theta_1 \in \mathcal{A} \text{ and } \forall \eta \in \mathcal{H}_s,\ s = 0, -1, \tag{4.9}$$

where $F^\bullet(\theta)\eta = (0, -f'(\theta^1)\eta^1)$ for $\theta = (\theta^1, \theta^2)$, $\eta = (\eta^1, \eta^2)$. On the other hand, since $S(t)\mathcal{A} = \mathcal{A}$ $(t \in R)$, similar to proof of Theorem 2 it is easy to show that $\lim_{t \to -\infty} \operatorname{dist}(S(t)\theta, \mathcal{Z}) = 0$ for every $\theta \in \mathcal{A}$. Thus there exists $T_0 = T_0(\varepsilon_1, \theta_0) < 0$ such that

$$\operatorname{dist}\left(S(t)\theta_0, \mathcal{Z}\right) < \frac{\varepsilon_1}{2}, \quad \forall t \leqslant T_0. \tag{4.10}$$

For $\frac{\varepsilon_1}{2}$ choose $\delta = \delta(\varepsilon_1)$ as in Lemma 4 and denote $T_m = T_0 - m\delta$ $(m = 1, 2, \ldots)$. Then according to (4.10) there exists a sequence $\{a_m\}$ such that $a_m \in \mathcal{Z}$ and $\|S(T_m)\theta_0 - a_m\|_{\mathcal{H}_0} < \frac{\varepsilon_1}{2}$ for every m. Introduce the function $w(t) = \{a_m, t \in (T_m, T_{m-1}],\ m = 1, 2, \ldots\}$. Then from Lemma 4 we have

$$\left\| S(t)\theta_0 - w(t) \right\|_{\mathcal{H}_0} < \varepsilon_1, \quad \forall t \leqslant T_0. \tag{4.11}$$

Let $s < t \leqslant T_0$. According to (2.8) we obtain

$$\begin{aligned} S(t)\theta_0 &= S(t-s)S(s)\theta_0 \\ &= e^{(t-s)A} S(s)\theta_0 + \int_0^{t-s} e^{(t-s-\tau)A} F\left(S(\tau)S(s)\theta_0\right) d\tau \\ &= e^{(t-s)A} S(s)\theta_0 + \int_0^{t-s} e^{\sigma A} F\left(S(t-\sigma)\theta_0\right) d\sigma, \end{aligned}$$

from which for $0 < h < t - s$

$$\begin{aligned} &\frac{S(t)\theta_0 - S(t-h)\theta_0}{h} \\ &\quad = \frac{1}{h}\left(e^{(t-s)A} - e^{(t-h-s)A}\right) S(s)\theta_0 + \frac{1}{h} \int_0^{t-s} e^{\sigma A} \left[F\left(S(t-\sigma)\theta_0\right) - F\left(S(t-h-\sigma)\theta_0\right) \right] d\sigma \\ &\qquad + \frac{1}{h} \int_{t-s-h}^{t-s} e^{\sigma A} F\left(S(t-h-\sigma)\theta_0\right) d\sigma. \end{aligned}$$

Letting $D_h S(t)\theta_0 = \frac{S(t)\theta_0 - S(t-h)\theta_0}{h}$, $\Delta S(t)\theta_0 = S(t)\theta_0 - S(t-h)\theta_0$ and $\widetilde{w}(t, \tau) = \tau w(t) + (1 - \tau) w(t - h)$ from the last equality we have

$$
\begin{aligned}
D_h S(t)\theta_0 &= \frac{1}{h}\big(e^{(t-s)A} - e^{(t-h-s)A}\big)S(s)\theta_0 + \frac{1}{h}\int_{t-s-h}^{t-s} e^{\sigma A} F\big(S(t-h-\sigma)\theta_0\big)\,d\sigma \\
&\quad + \frac{1}{h}\int_0^{t-s} e^{\sigma A}\int_0^1 \overset{\bullet}{F}\big(S(t-\sigma-h)\theta_0 + \tau\Delta S(t-\sigma)\theta_0\big)\Delta S(t-\sigma)\theta_0\,d\tau\,d\sigma \\
&= \frac{1}{h}\big(e^{(t-s)A} - e^{(t-h-s)A}\big)S(s)\theta_0 + \frac{1}{h}\int_{t-s-h}^{t-s} e^{\sigma A} F\big(S(t-h-\sigma)\theta_0\big)\,d\sigma \\
&\quad + \int_0^{t-s} e^{\sigma A}\int_0^1 \big[\overset{\bullet}{F}\big(S(t-\sigma-h)\theta_0 + \tau\Delta S(t-\sigma)\theta_0\big) \\
&\quad - \overset{\bullet}{F}\big(\widetilde{w}(t-\sigma,\tau)\big)\big]D_h S(t-\sigma)\theta_0\,d\tau\,d\sigma \\
&\quad + \int_0^{t-s} e^{\sigma A}\int_0^1 \overset{\bullet}{F}\big(\widetilde{w}(t-\sigma,\tau)\big)D_h S(t-\sigma)\theta_0\,d\tau\,d\sigma \\
&= K_1 + K_2 + K_3 + K_4.
\end{aligned}
\tag{4.12}
$$

Let us continue the proof in the case $n > 8$ (suitable embeddings are used when $n \leqslant 8$). Using (2.9) and condition (2.5) we have

$$
\lim_{s\to-\infty}\|K_1\|_{\mathcal{H}_0} = 0 \quad \text{and} \quad \lim_{s\to-\infty}\|K_2\|_{\mathcal{H}_0} = 0.
$$

On the other hand, since $w(\cdot) \in L_\infty(-\infty, T_0; \mathcal{H}_1)$ and $D_h S(t)\theta_0$ is uniformly bounded in $\mathcal{H}_{-1}$ with respect to h and t, we have

$$
\|K_4\|_{\mathcal{H}_{-\mu}} \leqslant c_1,
$$

where $\mu = \frac{n-8}{n-4}$ and c_1 does not depend on t, s, h and θ_0.

Using (4.9) and (4.11) we obtain

$$
\|K_3\|_{\mathcal{H}_{-\mu}} \leqslant \varepsilon M \int_s^t e^{-\omega(t-\sigma)}\big\|D_h S(\sigma)\theta_0\big\|_{\mathcal{H}_{-\mu}}\,d\sigma.
$$

So from (4.12) we get

$$
\big\|D_h S(t)\theta_0\big\|_{\mathcal{H}_{-\mu}} \leqslant c_1 + \varepsilon M \int_{-\infty}^t e^{-\omega(t-\sigma)}\big\|D_h S(\sigma)\theta_0\big\|_{\mathcal{H}_{-\mu}}\,d\sigma
$$

which yields

$$\left\| D_h S(t)\theta_0 \right\|_{\mathcal{H}_{-\mu}} \leqslant c_2, \quad \text{for } \forall t \leqslant T_0,\ \forall h \in (0,+\infty).$$

Taking into account the last inequality we can get a more regular estimate for K_4, and repeating the above procedure a finite number of times we have

$$\left\| D_h S(t)\theta_0 \right\|_{\mathcal{H}_0} \leqslant c_3, \quad \text{for } \forall t \leqslant T_0,\ \forall h \in (0,+\infty)$$

which means

$$\left\| \frac{d}{dt} S(t)\theta_0 \right\|_{\mathcal{H}_0} \leqslant c_3, \quad \text{for } \forall t \leqslant T_0.$$

Since the operator A is an isomorphism from $\mathcal{H}_1$ to $\mathcal{H}$, taking into account the last inequality in (2.7) we obtain

$$\left\| S(t)\theta_0 \right\|_{\mathcal{H}_1} \leqslant c_4, \quad \text{for } \forall t \leqslant T_0, \tag{4.13}$$

where c_4 does not depend on θ_0. It is known that $S(T_0)\theta_0 \in \mathcal{H}_1$ implies $S(t)S(T_0)\theta_0 \in \mathcal{H}_1$ for every $t \in R$. Then $\theta_0 = S(-T_0)S(T_0)\theta_0 \in \mathcal{H}_1$ and since θ_0 is an arbitrary element of $\mathcal{A}$ we get $\mathcal{A} \subset \mathcal{H}_1$. Let $\eta_0 \in \mathcal{A}$. Then from (2.7)–(2.9) we obtain

$$\begin{aligned}\left\| \frac{d}{dt} S(t)\eta_0 \right\|_{\mathcal{H}_0} &\leqslant c_5\big(\|\eta_0\|_{\mathcal{H}_1} + \|\eta_0\|_{\mathcal{H}_0}^{(p+1)}\big)e^{-t\omega} \\ &\quad + M\int_0^t e^{-(t-s)\omega} \left\| \overset{\bullet}{F}\big(S(\tau)\eta_0\big)\frac{d}{dt}S(\tau)\eta_0 \right\|_{\mathcal{H}_0} d\tau, \quad \forall t \geqslant 0.\end{aligned} \tag{4.14}$$

Since $\mathcal{A}$ is compact in $\mathcal{H}_0$ there exists $\xi(\varepsilon) > 0$ such that

$$\left\| \overset{\bullet}{F}(\eta)\varphi \right\|_{\mathcal{H}_0} = \left\| f'\big(\eta^1\big)\varphi^1 \right\|_0 \leqslant \varepsilon \left\|\varphi^1\right\|_2 + \xi(\varepsilon)\left\|\varphi^1\right\|_0 \tag{4.15}$$

for every $\eta = (\eta^1, \eta^2) \in \mathcal{A}$ and every $\varphi = (\varphi^1, \varphi^2) \in \mathcal{H}_0$. Thus from (4.13)–(4.15) we have

$$\left\| \frac{d}{dt} S(t)S(T_0)\theta_0 \right\|_{\mathcal{H}_0} \leqslant c_6, \quad \forall t \geqslant 0, \tag{4.16}$$

where c_6 does not depend on t and θ_0. Consequently from (2.7) and (4.16) we get $\|\theta_0\|_{\mathcal{H}_1} \leqslant c_7$. Since θ_0 is arbitrary element of $\mathcal{A}$ and c_7 does not depend on θ_0, the last inequality means $\mathcal{A}$ is bounded in $\mathcal{H}_1$. □

Now let us prove the finite dimensionality of the global attractor $\mathcal{A}$.

Theorem 4. *The fractal dimension of $\mathcal{A}$ is finite.*

Proof. Again let us present the proof in the more interesting case $n > 8$. By Eq. (2.8) and Theorem 4 we have

$$\begin{aligned}\left\| S(t)\theta_2 - S(t)\theta_1 \right\|_{\mathcal{H}_0} &\leqslant Me^{-\omega t}\|\theta_2-\theta_1\|_{\mathcal{H}_0} + M_1 e^{-\omega t}\int_0^t e^{\omega\tau}\left\|S(\tau)\theta_2 - S(\tau)\theta_1\right\|_{\mathcal{H}_{(8-n)/(n-4)}}\,d\tau \\ &\leqslant Me^{-\omega t}\|\theta_2-\theta_1\|_{\mathcal{H}_0} + M_1 M e^{-\omega t}\|\theta_2-\theta_1\|_{\mathcal{H}_{(8-n)/(n-4)}} \\ &\quad + M_2 e^{-\omega t}\int_0^t\int_0^\tau e^{\omega\sigma}\left\|S(\sigma)\theta_2 - S(\sigma)\theta_1\right\|_{\mathcal{H}_{-1}}\,d\sigma\,d\tau, \qquad (4.17)\end{aligned}$$

for every $t \geqslant 0$ and every $\theta_1, \theta_2 \in \mathcal{A}$.

Let $(u^1(t), u_t^1(t)) = S(t)\theta_1$, $(u^2(t), u_t^2(t)) = S(t)\theta_2$, $u(t) = u^2(t) - u^1(t)$ and $\varphi_r(\cdot)$ be as in the proof of Lemma 2. Furthermore define $\eta_r(t) = (\varphi_r u(t), \varphi_r u_t(t))$. Then from (2.1) (or (2.7)) we have

$$\frac{d}{dt}\eta_r(t) = \breve{A}\eta_r(t) + F_1(t) + F_2(t), \quad \forall t \geqslant 0, \qquad (4.18)$$

where

$$\breve{A} = A + \overset{\bullet}{F}(0), \qquad F_2(t) = \left[\int_0^1 \overset{\bullet}{F}\big(S(t)\theta_2 + \tau\big(S(t)\theta_1 - S(t)\theta_2\big)\big)\,d\tau - \overset{\bullet}{F}(0)\right]\eta_r(t)$$

and

$$\begin{aligned}F_1(t) = \Bigg(0,\ &\Delta u\Delta\varphi_r + \Delta\left(4\sum_{i=1}^n u_{x_i}(\varphi_r)_{x_i} + u\Delta\varphi_r\right) - 2\sum_{i=1}^n \Delta(\varphi_r)_{x_i}u_{x_i} \\ &- 4\sum_{i=1}^n\sum_{j=1}^n u_{x_ix_j}(\varphi_r)_{x_ix_j}\Bigg).\end{aligned}$$

Since $f'(0) \geqslant 0$ (thanks to (2.6)) it is easy to show that

$$\left\|e^{t\breve{A}}\right\|_{L(\mathcal{H}_s,\mathcal{H}_s)} \leqslant M_1 e^{-\omega t}, \quad \forall t \geqslant 0,\ \forall s \in [-1, 1]. \qquad (4.19)$$

On the other hand, for some $k_1 > 0$ we have

$$\left\|F_1(t)\right\|_{\mathcal{H}_{-1}} \leqslant \frac{c_8}{r}\left\|S(t)\theta_2 - S(t)\theta_1\right\|_{\mathcal{H}_0} \leqslant \frac{c_9}{r}e^{k_1 t}\|\theta_2-\theta_1\|_{\mathcal{H}_0}, \quad \forall t \geqslant 0. \qquad (4.20)$$

Choose $\varepsilon \in (0, \frac{\omega}{M_1})$. According to Lemma 5, there exists $r_0 = r_0(\varepsilon)$ such that if $r \geqslant r_0$ then

$$\left\| F_2(t) \right\|_{\mathcal{H}_{-1}} \leqslant \varepsilon \left\| \eta_r(t) \right\|_{\mathcal{H}_{-1}}, \quad \forall t \geqslant 0. \tag{4.21}$$

So from (4.18)–(4.21) we obtain that

$$\left\| \eta_r(t) \right\|_{\mathcal{H}_{-1}} \leqslant M_1 \left(e^{-(\omega - M_1 \varepsilon)t} \left\| \eta_r(0) \right\|_{\mathcal{H}_{-1}} + \frac{c_{10}}{r} e^{k_1 t} \|\theta_2 - \theta_1\|_{\mathcal{H}_0} \right), \quad \forall t \geqslant 0. \tag{4.22}$$

Similarly denoting $\xi_r(t) = ((1 - \varphi_r)u(t), (1 - \varphi_r)u_t(t))$ we have

$$\left\| \xi_r(t) \right\|_{\mathcal{H}_{-1}} \leqslant M_1 e^{k_2 t} \left(\left\| \xi_r(0) \right\|_{\mathcal{H}_{-1}} + \frac{c_{11}}{r} e^{k_1 t} \|\theta_2 - \theta_1\|_{\mathcal{H}_0} \right), \quad \forall t \geqslant 0. \tag{4.23}$$

Since $S(t)\theta_2 - S(t)\theta_1 = \eta_r(t) + \xi_r(t)$ from (4.17), (4.22) and (4.23) we obtain

$$\left\| S(t)\theta_2 - S(t)\theta_1 \right\|_{\mathcal{H}_0} \leqslant m_1(t) \|\theta_2 - \theta_1\|_{\mathcal{H}_0} + m_2(t) \left[\|\theta_2 - \theta_1\|_{\mathcal{H}_{-1}(B(0,r))} + \frac{1}{r} \|\theta_2 - \theta_1\|_{\mathcal{H}_0} \right], \quad \forall t \geqslant 0,$$

where $\mathcal{H}_{-1}(B(0,r)) = L_2(B(0,r)) \times W_2^{-2}(B(0,r))$, $m_2(t)$ is an increasing function and $m_1(t) \to 0$ as $t \to +\infty$.

The last inequality together with [14, Theorem 5.3] yields finiteness of the fractal dimension of the global attractor $\mathcal{A}$. □

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