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Fractional Calculus and Certain Starlike Functions with Negative Coefficients

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Abstract—A certain subclass $\mathcal{T}(n,p,\lambda,\alpha)$ of starlike functions in the unit disk is introduced. The object of the present paper is to derive several interesting properties of functions belonging to the class $\mathcal{T}(n,p,\lambda,\alpha)$. Various distortion inequalities for fractional calculus of functions in the class $\mathcal{T}(n,p,\lambda,\alpha)$ are also given.

Keywords—Fractional calculus, Starlike functions, Analytic functions, Extremal function, Hadamard product, Cauchy-Schwarz inequality, Simply-connected region, Fractional integrals, Fractional derivatives.

1. INTRODUCTION

Let $\mathcal{T}(n,p)$ denote the class of functions $f(z)$ of the form:

$$f(z) = z^p - \sum_{k=n}^{\infty} a_{k+p}\, z^{k+p} \qquad (a_{k+p} \geq 0;\ p \in \mathbb{N} := \{1,2,3,\dots\};\ n \in \mathbb{N}), \tag{1.1}$$

which are analytic in the *open* unit disk

$$\mathcal{U} = \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}.$$

A function $f(z) \in \mathcal{T}(n,p)$ is said to be in the class $\mathcal{T}(n,p,\lambda,\alpha)$ if it satisfies the inequality:

$$\Re\left\{\frac{zf'(z) + \lambda z^2 f''(z)}{(1-\lambda)\, f(z) + \lambda z f'(z)}\right\} > \alpha \tag{1.2}$$

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for some α $(0 \leq \alpha < 1)$ and λ $(0 \leq \lambda \leq 1)$, and for all $z \in \mathcal{U}$. We note that

$$\mathcal{T}(n,1,0,\alpha) \equiv \mathcal{T}_\alpha(n), \tag{1.3}$$
$$\mathcal{T}(n,1,1,\alpha) \equiv \mathcal{C}_\alpha(n), \tag{1.4}$$
$$\mathcal{T}(1,1,0,\alpha) \equiv \mathcal{T}^*(\alpha), \tag{1.5}$$
$$\mathcal{T}(1,1,1,\alpha) \equiv \mathcal{C}(\alpha), \tag{1.6}$$

and

$$\mathcal{T}(n,1,\lambda,\alpha) \equiv \mathcal{P}(n,\lambda,\alpha). \tag{1.7}$$

The classes $\mathcal{T}_\alpha(n)$ and $\mathcal{C}_\alpha(n)$ were studied earlier by Srivastava *et al.* [1], the classes

$$\mathcal{T}^*(\alpha) = \mathcal{T}_\alpha(1) \quad \text{and} \quad \mathcal{C}(\alpha) = \mathcal{C}_\alpha(1)$$

were studied by Silverman [2], and the class $\mathcal{P}(n,\lambda,\alpha)$ was studied by Altintaş [3].

The object of the present paper is to give various basic properties of functions belonging to the general class $\mathcal{T}(n,p,\lambda,\alpha)$. We also prove (in Section 3) several distortion theorems (involving certain operators of fractional calculus) for functions in the class $\mathcal{T}(n,p,\lambda,\alpha)$.

2. A THEOREM ON COEFFICIENT BOUNDS

We begin by proving some sharp coefficient inequalities contained in the following theorem.

THEOREM 1. *A function $f(z) \in \mathcal{T}(n,p)$ is in the class $\mathcal{T}(n,p,\lambda,\alpha)$ if and only if*

$$\sum_{k=n}^{\infty} (k+p-\alpha)(\lambda k + \lambda p - \lambda + 1)\, a_{k+p} \leq (p-\alpha)(1+\lambda p - \lambda) \tag{2.1}$$

$$(0 \leq \alpha < 1;\ 0 \leq \lambda \leq 1;\ \lambda(p-1)(p-\alpha) \geq \alpha\ (p \neq 1);\ p \in \mathbb{N};\ n \in \mathbb{N}).$$

The result is sharp.

PROOF. Suppose that $f(z) \in \mathcal{T}(n,p,\lambda,\alpha)$. Then we find from (1.2) that

$$\Re\left\{\frac{p(1+\lambda p-\lambda)\, z^p - \sum_{k=n}^{\infty}(k+p)(\lambda k+\lambda p-\lambda+1)\, a_{k+p}\, z^{k+p}}{(1+\lambda p-\lambda)\, z^p - \sum_{k=n}^{\infty}(\lambda k+\lambda p-\lambda+1)\, a_{k+p}\, z^{k+p}}\right\} > \alpha$$

$$(0 \leq \alpha < 1;\ 0 \leq \lambda \leq 1;\ \lambda(p-1)(p-\alpha) \geq \alpha\ (p \neq 1);\ p \in \mathbb{N};\ n \in \mathbb{N};\ z \in \mathcal{U}).$$

If we choose z to be real and let $z \to 1-$, we get

$$\frac{p(1+\lambda p-\lambda) - \sum_{k=n}^{\infty}(k+p)(\lambda k+\lambda p-\lambda+1)\, a_{k+p}}{1+\lambda p-\lambda - \sum_{k=n}^{\infty}(\lambda k+\lambda p-\lambda+1)\, a_{k+p}} \geq \alpha$$

$$(0 \leq \alpha < 1;\ 0 \leq \lambda \leq 1;\ \lambda(p-1)(p-\alpha) \geq \alpha\ (p \neq 1);\ p \in \mathbb{N};\ n \in \mathbb{N})$$

or, equivalently,

$$\sum_{k=n}^{\infty} (k+p-\alpha)(\lambda k + \lambda p - \lambda + 1)\, a_{k+p} \leq (p-\alpha)(1+\lambda p - \lambda)$$

$$(0 \leq \alpha < 1;\ 0 \leq \lambda \leq 1;\ \lambda(p-1)(p-\alpha) \geq \alpha\ (p \neq 1);\ p \in \mathbb{N};\ n \in \mathbb{N}),$$

which is precisely the assertion (2.1) of Theorem 1.

Conversely, suppose that the inequality (2.1) holds true and let

$$z \in \partial\mathcal{U} = \{z : z \in \mathbb{C} \quad \text{and} \quad |z| = 1\}.$$

Then we find from the definition (1.1) that

$$
\begin{aligned}
&\left|\frac{zf'(z)+\lambda z^2 f''(z)}{(1-\lambda)f(z)+\lambda zf'(z)}-(p-\alpha)(1+\lambda p-\lambda)\right| \\
&\quad = \frac{\left|-(1+\lambda p-\lambda)[\lambda(p-1)(p-\alpha)-\alpha]z^p-\sum_{k=n}^{\infty}(\lambda k+\lambda p-\lambda+1)[k+\alpha-\lambda(p-1)(p-\alpha)]a_{k+p}z^{k+p}\right|}{\left|(1+\lambda p-\lambda)z^{-p}-\sum_{k=n}^{\infty}(\lambda k+\lambda p-\lambda+1)a_{k+p}z^{k+p}\right|} \\
&\quad \leq \frac{(1+\lambda p-\lambda)[\lambda(p-1)(p-\alpha)-\alpha]\,|z|^p+\sum_{k=n}^{\infty}(\lambda k+\lambda p-\lambda+1)[k+\alpha-\lambda(p-1)(p-\alpha)]\,a_{k+p}\,|z|^{k+p}}{(1+\lambda p-\lambda)\,|z|^p-\sum_{k=n}^{\infty}(\lambda k+\lambda p-\lambda+1)\,a_{k+p}\,|z|^{k+p}} \\
&\quad \leq (p-\alpha)(1+\lambda p-\lambda)-\alpha
\end{aligned}
$$

$$(0\leq\alpha<1;\ 0\leq\lambda\leq 1;\ \lambda(p-1)(p-\alpha)\geq\alpha\ (p\neq 1);\ z\in\partial\mathcal{U};\ p\in\mathbb{N};\ n\in\mathbb{N}),$$

provided that the inequality (2.1) is satisfied. Hence, by the maximum modulus theorem, we have

$$f(z)\in\mathcal{T}(n,p,\lambda,\alpha).$$

Finally, we note that the assertion (2.1) of Theorem 1 is sharp, the extremal function being

$$f(z)=z^p-\frac{(p-\alpha)(1+\lambda p-\lambda)}{(n+p-\alpha)(\lambda n+\lambda p-\lambda+1)}\,z^{n+p}\qquad(p\in\mathbb{N};\ n\in\mathbb{N}).\tag{2.2}$$

THEOREM 2. *Let the function $f(z)$ defined by (1.1) and the function $g(z)$ defined by*

$$g(z)=z^p-\sum_{k=n}^{\infty}b_{k+p}\,z^{k+p}\qquad(b_{k+p}\geq 0;\ p\in\mathbb{N};\ n\in\mathbb{N})\tag{2.3}$$

be in the same class $\mathcal{T}(n,p,\lambda,\alpha)$. Then the function $h(z)$ defined by

$$h(z)=(1-\beta)\,f(z)+\beta g(z)=z^p-\sum_{k=n}^{\infty}c_{k+p}\,z^{k+p}\tag{2.4}$$

$$(c_{k+p}:=(1-\beta)\,a_{k+p}+\beta b_{k+p}\geq 0;\ 0\leq\beta\leq 1;\ p\in\mathbb{N})$$

is also in the class $\mathcal{T}(n,p,\lambda,\alpha)$.

PROOF. Suppose that each of the functions $f(z)$ and $g(z)$ is in the class $\mathcal{T}(n,p,\lambda,\alpha)$. Then, making use of (2.1), we see that

$$
\begin{aligned}
\sum_{k=n}^{\infty}(k+p-\alpha)(\lambda k+\lambda p-\lambda+1)\,c_{k+p} &= (1-\beta)\sum_{k=n}^{\infty}(k+p-\alpha)(\lambda k+\lambda p-\lambda+1)\,a_{k+p} \\
&\quad +\beta\sum_{k=n}^{\infty}(k+p-\alpha)(\lambda k+\lambda p-\lambda+1)\,b_{k+p} \\
&= (1-\beta)(p-\alpha)(1+\lambda p-\lambda)+\beta(p-\alpha)(1+\lambda p-\lambda) \\
&= (p-\alpha)(1+\lambda p-\lambda)
\end{aligned}\tag{2.5}
$$

$$(0\leq\alpha<1;\ 0\leq\lambda\leq 1;\ \lambda(p-1)(p-\alpha)\geq\alpha\ (p\neq 1);\ p\in\mathbb{N};\ ,n\in\mathbb{N}),$$

which completes the proof of Theorem 2.

Next we define the *modified* Hadamard product of the functions $f(z)$ and $g(z)$, which are defined by (1.1) and (2.3), respectively, by

$$f * g(z) = z^p - \sum_{k=n}^{\infty} a_{k+p}\, b_{k+p}\, z^{k+p} \qquad (a_{k+p} \geq 0;\ b_{k+p} \geq 0;\ p \in \mathbb{N}). \tag{2.6}$$

THEOREM 3. *If each of the functions $f(z)$ and $g(z)$ is in the class $\mathcal{T}(n,p,\lambda,\alpha)$, then*

$$f * g(z) \in \mathcal{T}(n,p,\lambda,\delta),$$

where

$$\delta \leq p - \frac{(p-\alpha)^2(1+\lambda p-\lambda)}{(p+n-\alpha)(\lambda p+\lambda n-\lambda+1)} \qquad (p \in \mathbb{N};\ n \in \mathbb{N}). \tag{2.7}$$

The result is sharp for the functions $f(z)$ and $g(z)$ given by

$$f(z) = g(z) = z^p - \frac{(p-\alpha)(1+\lambda p-\lambda)}{(p+n-\alpha)(\lambda p+\lambda n-\lambda+1)}\, z^{p+n} \qquad (p \in \mathbb{N};\ n \in \mathbb{N}). \tag{2.8}$$

PROOF. From Theorem 1, we have

$$\sum_{k=n}^{\infty} \frac{(k+p-\alpha)(\lambda k+\lambda p-\lambda+1)}{(p-\alpha)(1+\lambda p-\lambda)}\, a_{k+p} \leq 1 \qquad (p \in \mathbb{N};\ n \in \mathbb{N}) \tag{2.9}$$

and

$$\sum_{k=n}^{\infty} \frac{(k+p-\alpha)(\lambda k+\lambda p-\lambda+1)}{(p-\alpha)(1+\lambda p-\lambda)}\, b_{k+p} \leq 1 \qquad (p \in \mathbb{N};\ n \in \mathbb{N}). \tag{2.10}$$

We have to find the largest δ such that

$$\sum_{k=n}^{\infty} \frac{(k+p-\delta)(\lambda k+\lambda p-\lambda+1)}{(p-\delta)(1+\lambda p-\lambda)}\, a_{k+p}\, b_{k+p} \leq 1 \qquad (p \in \mathbb{N};\ n \in \mathbb{N}). \tag{2.11}$$

From (2.9) and (2.10) we find, by means of the Cauchy-Schwarz inequality, that

$$\sum_{k=n}^{\infty} \frac{(k+p-\alpha)(\lambda k+\lambda p-\lambda+1)}{(p-\alpha)(1+\lambda p-\lambda)} \sqrt{a_{k+p}\, b_{k+p}} \leq 1 \qquad (p \in \mathbb{N}). \tag{2.12}$$

Therefore, (2.11) holds true if

$$\sqrt{a_{k+p}\, b_{k+p}} \leq \frac{p-\delta}{p-\alpha} \qquad (k \geq n;\, p \in \mathbb{N};\ n \in \mathbb{N}), \tag{2.13}$$

that is, if

$$\frac{(p-\alpha)(1+\lambda p-\lambda)}{(k+p-\alpha)(\lambda k+\lambda p-\lambda+1)} \leq \frac{p-\delta}{p-\alpha} \qquad (k \geq n;\ p \in \mathbb{N};\, n \in \mathbb{N}), \tag{2.14}$$

which readily yields

$$\delta \leq p - \frac{(p-\alpha)^2(1+\lambda p-\lambda)}{(k+p-\alpha)(\lambda k+\lambda p-\lambda+1)} \qquad (k \geq n;\ p \in \mathbb{N};\ n \in \mathbb{N}). \tag{2.15}$$

Finally, letting

$$\Phi(k) = p - \frac{(p-\alpha)^2(1+\lambda p-\lambda)}{(k+p-\alpha)(\lambda k+\lambda p-\lambda+1)} \qquad (k \geq n;\ p \in \mathbb{N};\ n \in \mathbb{N}), \tag{2.16}$$

we see that the function $\Phi(k)$ is increasing in k. This shows that

$$\delta \leq \Phi(n) = p - \frac{(p-\alpha)^2(1+\lambda p-\lambda)}{(p+n-\alpha)(\lambda p+\lambda n-\lambda+1)} \qquad (p \in \mathbb{N}), \tag{2.17}$$

which completes the proof of Theorem 3.

COROLLARY 1. *If* $f(z) \in \mathcal{T}(n,p,\lambda,\alpha)$, *then*

$$a_{p+n} \leq \frac{(p-\alpha)(1+\lambda p-\lambda)}{(p+n-\alpha)(\lambda p+\lambda n-\lambda+1)} \qquad (p \in \mathbb{N};\ n \in \mathbb{N}). \tag{2.18}$$

Numerous consequences of Theorems 1–3 (and of Corollary 1) can indeed be deduced by specializing the various parameters involved. Many of these consequences were proven by earlier workers on the subject (cf., e.g., [1–3]).

3. DISTORTION THEOREMS INVOLVING OPERATORS OF FRACTIONAL CALCULUS

In this section, we shall prove several distortion theorems for functions belonging to the general class $\mathcal{T}(n,p,\lambda,\alpha)$. Each of these theorems would involve certain operators of fractional calculus, which are defined as follows (cf., e.g., [4–6]).

DEFINITION 1. *The fractional integral of order* μ *is defined by*

$$D_z^{-\mu} f(z) = \int_0^z \frac{f(\zeta)}{(z-\zeta)^{1-\mu}}\, d\zeta \qquad (\mu > 0),$$

where $f(z)$ *is an analytic function in a simply-connected region of the* z*-plane containing the origin, and the multiplicity of* $(z-\zeta)^{\mu-1}$ *is removed by requiring* $\log(z-\zeta)$ *to be real when* $z-\zeta > 0$.

DEFINITION 2. *The fractional derivative of order* μ *is defined by*

$$D_z^{\mu} f(z) = \frac{1}{\Gamma(1-\mu)} \frac{d}{dz} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{\mu}}\, d\zeta \qquad (0 \leq \mu < 1),$$

where $f(z)$ *is constrained, and the multiplicity of* $(z-\zeta)^{-\mu}$ *is removed, as in Definition 1.*

DEFINITION 3. *Under the hypotheses of Definition 1, the fractional derivative of order* $n+\mu$ *is defined by*

$$D_z^{n+\mu} f(z) = \frac{d^n}{dz^n} D_z^{\mu} f(z) \qquad (0 \leq \mu < 1;\ n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}).$$

THEOREM 4. *If* $f(z) \in \mathcal{T}(n,p,\lambda,\alpha)$, *then*

$$\left|D_z^{-\mu} f(z)\right| \leq |z|^{p+\mu} \left(\frac{\Gamma(p+1)}{\Gamma(p+\mu+1)} + \frac{(p-\alpha)(1+\lambda p-\lambda)\,\Gamma(n+p+1)}{(n+p-\alpha)(\lambda n+\lambda p-\lambda+1)\,\Gamma(n+p+\mu+1)}\,|z|\right) \tag{3.1}$$

and

$$\left|D_z^{-\mu} f(z)\right| \geq |z|^{p+\mu} \left(\frac{\Gamma(p+1)}{\Gamma(p+\mu+1)} - \frac{(p-\alpha)(1+\lambda p-\lambda)\,\Gamma(n+p+1)}{(n+p-\alpha)(\lambda n+\lambda p-\lambda+1)\,\Gamma(n+p+\mu+1)}\,|z|\right) \tag{3.2}$$

for $\mu > 0$, $n \in \mathbb{N}$, *and* $p \in \mathbb{N}$, *and for all* $z \in \mathcal{U}$.

The result is sharp for the function $f(z)$ *given by*

$$f(z) = z^p - \frac{(p-\alpha)(1+\lambda p-\lambda)}{(n+p-\alpha)(\lambda n+\lambda p-\lambda+1)}\, z^{n+p} \qquad (n \in \mathbb{N};\ p \in \mathbb{N}). \tag{3.3}$$

PROOF. Suppose that $f(z) \in \mathcal{T}(n,p,\lambda,\alpha)$. We then find from (2.1) that

$$\sum_{k=n}^{\infty} a_{k+p} \leq \frac{(p-\alpha)(1+\lambda p-\lambda)}{(n+p-\alpha)(\lambda n+\lambda p-\lambda+1)} \qquad (n \in \mathbb{N};\ p \in \mathbb{N}). \tag{3.4}$$

Making use of (3.4) and Definition 1, we have

$$\begin{aligned} D_z^{-\mu} f(z) &= z^{p+\mu}\left(\frac{\Gamma(p+1)}{\Gamma(p+\mu+1)} - \sum_{k=n}^{\infty} \frac{\Gamma(k+p+1)}{\Gamma(k+p+\mu+1)} a_{k+p} z^k\right) \\ &= z^{p+\mu}\left(\frac{\Gamma(p+1)}{\Gamma(p+\mu+1)} - \sum_{k=n}^{\infty} \Psi(k)\, a_{k+p} z^k\right), \end{aligned} \tag{3.5}$$

where, for convenience,

$$\Psi(k) = \frac{\Gamma(k+p+1)}{\Gamma(k+p+\mu+1)} \qquad (\mu > 0;\ k \geq n;\ n \in \mathbb{N};\ p \in \mathbb{N}).$$

Clearly, the function $\Psi(k)$ is decreasing in k, and we have

$$0 < \Psi(k) \leq \Psi(n) = \frac{\Gamma(n+p+1)}{\Gamma(n+p+\mu+1)}. \tag{3.6}$$

Thus we find from (3.4)–(3.6) that

$$\begin{aligned} \left|D_z^{-\mu} f(z)\right| &\leq |z|^{p+\mu}\left(\frac{\Gamma(p+1)}{\Gamma(p+\mu+1)} + |z|\Psi(n)\sum_{k=n}^{\infty} a_{k+p}\right) \\ &\leq |z|^{p+\mu}\left(\frac{\Gamma(p+1)}{\Gamma(p+\mu+1)} + \frac{(p-\alpha)(1+\lambda p-\lambda)\Gamma(n+p+1)}{(n+p-\alpha)(\lambda n+\lambda p-\lambda+1)\Gamma(n+p+\mu+1)}|z|\right), \end{aligned}$$

which is precisely the assertion (3.1), and that

$$\begin{aligned} \left|D_z^{-\mu} f(z)\right| &\geq |z|^{p+\mu}\left(\frac{\Gamma(p+1)}{\Gamma(p+\mu+1)} - |z|\Psi(n)\sum_{k=n}^{\infty} a_{k+p}\right) \\ &\geq |z|^{p+\mu}\left(\frac{\Gamma(p+1)}{\Gamma(p+\mu+1)} - \frac{(p-\alpha)(1+\lambda p-\lambda)\Gamma(n+p+1)}{(n+p-\alpha)(\lambda n+\lambda p-\lambda+1)\Gamma(n+p+\mu+1)}|z|\right), \end{aligned}$$

which is the same as the assertion (3.2).

In order to complete the proof of Theorem 4, it is easily observed that the equalities in (3.1) and (3.2) are satisfied by the function $f(z)$ given by (3.3).

The proofs of Theorems 5 and 6 below are much akin to that of Theorem 4, which we have detailed above fairly fully. Indeed, instead of Definition 1, we make use of Definitions 2 and 3 to prove Theorems 5 and 6, respectively.

THEOREM 5. *If $f(z) \in \mathcal{T}(n,p,\lambda,\alpha)$, then*

$$|D_z^{\mu} f(z)| \leq |z|^{p-\mu}\left(\frac{\Gamma(p+1)}{\Gamma(p-\mu+1)} + \frac{(p-\alpha)(1+\lambda p-\lambda)\Gamma(n+p+1)}{(n+p-\alpha)(\lambda n+\lambda p-\lambda+1)\Gamma(n+p-\mu+1)}|z|\right) \tag{3.7}$$

and

$$|D_z^{\mu} f(z)| \geq |z|^{p-\mu}\left(\frac{\Gamma(p+1)}{\Gamma(p-\mu+1)} - \frac{(p-\alpha)(1+\lambda p-\lambda)\Gamma(n+p+1)}{(n+p-\alpha)(\lambda n+\lambda p-\lambda+1)\Gamma(n+p-\mu+1)}|z|\right) \tag{3.8}$$

for $0 \leq \mu < 1$, $n \in \mathbb{N}$, and $p \in \mathbb{N}$, and for all $z \in \mathcal{U}$.

The result is sharp for the function $f(z)$ given by (3.3).

THEOREM 6. *If $f(z) \in \mathcal{T}(n,p,\lambda,\alpha)$, then*

$$\left|D_z^{1+\mu} f(z)\right| \le |z|^{p-\mu-1} \left(\frac{\Gamma(p+1)}{\Gamma(p-\mu)} + \frac{(p-\alpha)(1+\lambda p-\lambda)\Gamma(n+p+1)}{(n+p-\alpha)(\lambda n+\lambda p-\lambda+1)\Gamma(n+p-\mu)}|z|\right) \tag{3.9}$$

and

$$\left|D_z^{1+\mu} f(z)\right| \ge |z|^{p-\mu-1} \left(\frac{\Gamma(p+1)}{\Gamma(p-\mu)} - \frac{(p-\alpha)(1+\lambda p-\lambda)\Gamma(n+p+1)}{(n+p-\alpha)(\lambda n+\lambda p-\lambda+1)\Gamma(n+p-\mu)}|z|\right) \tag{3.10}$$

for $0 \le \mu < 1$, $n \in \mathbb{N}$, and $p \in \mathbb{N}$, and for all $z \in \mathcal{U}$.

The result is sharp for the function $f(z)$ given by (3.3).

Setting $\mu = 0$ in Theorem 5, we obtain the following corollary.

COROLLARY 2. *If $f(z) \in \mathcal{T}(n,p,\lambda,\alpha)$, then*

$$|f(z)| \le |z|^p \left(1 + \frac{(p-\alpha)(1+\lambda p-\lambda)}{(n+p-\alpha)(\lambda n+\lambda p-\lambda+1)}|z|\right) \tag{3.11}$$

and

$$|f(z)| \ge |z|^p \left(1 - \frac{(p-\alpha)(1+\lambda p-\lambda)}{(n+p-\alpha)(\lambda n+\lambda p-\lambda+1)}|z|\right) \tag{3.12}$$

for $n \in \mathbb{N}$ and $p \in \mathbb{N}$, and for all $z \in \mathcal{U}$.

The result is sharp for the function $f(z)$ given by (3.3).

If, on the other hand, we set $\mu = 0$ in Theorem 6, we shall arrive at Corollary 3.

COROLLARY 3. *If $f(z) \in \mathcal{T}(n,p,\lambda,\alpha)$, then*

$$|f'(z)| \le |z|^{p-1} \left(p + \frac{(p-\alpha)(1+\lambda p-\lambda)(n+p)}{(n+p-\alpha)(\lambda n+\lambda p-\lambda+1)}|z|\right) \tag{3.13}$$

and

$$|f'(z)| \ge |z|^{p-1} \left(p - \frac{(p-\alpha)(1+\lambda p-\lambda)(n+p)}{(n+p-\alpha)(\lambda n+\lambda p-\lambda+1)}|z|\right) \tag{3.14}$$

for $n \in \mathbb{N}$ and $p \in \mathbb{N}$, and for all $z \in \mathcal{U}$.

The result is sharp for the function $f(z)$ given by (3.3).

Further consequences of the distortion properties (given by Corollary 2 and Corollary 3) can be obtained for each of the function classes studied by earlier workers. The details may be omitted.

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