

## EXISTENCE OF THE GLOBAL ATTRACTOR FOR THE PLATE EQUATION WITH NONLOCAL NONLINEARITY IN $\mathbb{R}^n$

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ABSTRACT. We consider Cauchy problem for the semilinear plate equation with nonlocal nonlinearity. Under mild conditions on the damping coefficient, we prove that the semigroup generated by this problem possesses a global attractor.

1. **Introduction.** In this paper, we study the long-time behavior of the solutions for the following plate equation with localized damping and nonlocal nonlinearity in terms of global attractors:

$$u_{tt} + \Delta^2 u + \alpha(x)u_t + \lambda u + f(\|u(t)\|_{L^p(\mathbb{R}^n)}) |u|^{p-2} u = h(x), \quad (t, x) \in (0, \infty) \times \mathbb{R}^n. \quad (1.1)$$

Plate equations have been investigated for many years due to their importance in some physical areas such as vibration and elasticity theories of solid mechanics. For instance, in the case when  $f(\cdot)$  is identically constant, equation (1.1) becomes an equation with local polynomial nonlinearity which arises in aeroelasticity modeling (see, for example, [7], [8]), whereas in the case when  $p = 2$ , the nonlinearity  $f(\|u\|_{L^p(\mathbb{R}^n)}) |u|^{p-2} u$  appears in the models of Kerr-like medium (see [15], [21]).

The study of the long-time dynamics of evolution equations has become an outstanding area during the recent decades. As it is well known, the attractors can be used as a tool to describe the long-time dynamics of these equations. In particular, there have been many works on the investigation of the attractors for the plate equations over the last few years. For the attractors of the plate equations with local and nonlocal nonlinearities in bounded domains, we refer to [3], [5], [14], [16-19] and [22]. In the case of unbounded domains, owing to the lack of Sobolev compact embedding theorems, there are difficulties in applying the methods given for bounded domains. In order to overcome these difficulties, the authors of [9-10], [13] and [23] established the uniform tail estimates for the plate equations with local nonlinearities and then used the weak continuity of the nonlinear source operators.

In the case when the domain is unbounded and the equation includes nonlocal nonlinearity, an additional obstacle occurs. For equation (1.1), this obstacle is caused by the operator defined by  $\mathcal{F}(u) := f(\|u\|_{L^p(\mathbb{R}^n)}) |u|^{p-2} u$ , because the operator  $\mathcal{F}$ , besides being not compact, is not also weakly continuous from  $H^2(\mathbb{R}^n)$

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to  $L^2(\mathbb{R}^n)$ . This situation does not allow us to apply the standard splitting method and the energy method devised in [2]. Recently in [1], the obstacle mentioned above is handled for the nonlinearity  $f(\|\nabla u\|_{L^2(\mathbb{R}^n)})\Delta u$  by using compensated compactness method introduced in [11]. In that paper, the strictly positivity condition on the damping coefficient  $\alpha(\cdot)$  is critically used. In the present paper, we replace this condition with the weaker conditions (see (2.3), (2.4)), and by using effectiveness of the dissipation for large enough  $x$ , we prove the existence of the global attractor which equals the unstable manifold of the set of stationary points.

The paper is organized as follows: In Section 2, we give the statement of the problem and the main result. In Section 3, we firstly prove two auxiliary lemmas and then establish the asymptotic compactness of the solution, which together with the presence of the strict Lyapunov function leads to the existence of the global attractor.

**2. Statement of the problem and the main result.** We consider the following initial value problem

$$u_{tt} + \Delta^2 u + \alpha(x)u_t + \lambda u + f(\|u(t)\|_{L^p(\mathbb{R}^n)})|u|^{p-2}u = h(x), \quad (t, x) \in (0, \infty) \times \mathbb{R}^n, \quad (2.1)$$

$$u(0, x) = u_0(x), \quad u_t(0, x) = u_1(x), \quad x \in \mathbb{R}^n, \quad (2.2)$$

where  $\lambda > 0$ ,  $h \in L^2(\mathbb{R}^n)$ ,  $p \geq 2$ ,  $p(n-4) \leq 2n-4$  and the functions  $\alpha(\cdot)$ ,  $f(\cdot)$  satisfy the following conditions:

$$\alpha \in L^\infty(\mathbb{R}^n), \quad \alpha(\cdot) > 0 \quad \text{a.e. in } \mathbb{R}^n, \quad (2.3)$$

$$\alpha(\cdot) \geq \alpha_0 > 0 \quad \text{a.e. in } \{x \in \mathbb{R}^n : |x| \geq r_0\}, \quad \text{for some } r_0, \quad (2.4)$$

$$f \in C^1(\mathbb{R}^+), \quad f(\cdot) \geq 0. \quad (2.5)$$

Applying the semigroup theory (see [4, p.56-58]) and repeating the arguments done in the introduction of [1], one can prove the following well-posedness result.

**Theorem 2.1.** *Assume that the conditions (2.3)-(2.5) hold. Then for every  $T > 0$  and  $(u_0, u_1) \in H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ , problem (2.1)-(2.2) has a unique weak solution  $u \in C([0, T]; H^2(\mathbb{R}^n)) \cap C^1([0, T]; L^2(\mathbb{R}^n))$  which satisfies the energy equality*

$$\begin{aligned} E_{\mathbb{R}^n}(u(t)) + \frac{1}{p}F\left(\|u(t)\|_{L^p(\mathbb{R}^n)}^p\right) - \int_{\mathbb{R}^n} h(x)u(t, x)dx + \int_s^t \int_{\mathbb{R}^n} \alpha(x)|u_t(\tau, x)|^2 dx d\tau \\ = E_{\mathbb{R}^n}(u(s)) + \frac{1}{p}F\left(\|u(s)\|_{L^p(\mathbb{R}^n)}^p\right) - \int_{\mathbb{R}^n} h(x)u(s, x)dx, \quad \forall t \geq s \geq 0, \end{aligned} \quad (2.6)$$

where  $F(z) = \int_0^z f(\sqrt{s})ds$ , for all  $z \in \mathbb{R}^+$  and  $E_\Omega(u(t)) = \frac{1}{2} \int_\Omega (|u_t(t, x)|^2 + |\Delta u(t, x)|^2 + \lambda|u(t, x)|^2)dx$ , for subset  $\Omega \subset \mathbb{R}^n$ . Moreover, if  $(u_0, u_1) \in H^4(\mathbb{R}^n) \times H^2(\mathbb{R}^n)$ , then  $u \in C([0, T]; H^4(\mathbb{R}^n)) \cap C^1([0, T]; H^2(\mathbb{R}^n))$ .

In addition, if  $v, w \in C([0, T]; H^2(\mathbb{R}^n)) \cap C^1([0, T]; L^2(\mathbb{R}^n))$  are the weak solutions to (2.1)-(2.2) with initial data  $(v_0, v_1) \in H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$  and  $(w_0, w_1) \in H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ , then

$$\begin{aligned} & \|v(t) - w(t)\|_{H^2(\mathbb{R}^n)} + \|v_t(t) - w_t(t)\|_{L^2(\mathbb{R}^n)} \\ & \leq c(T, \tilde{r}) \left( \|v_0 - w_0\|_{H^2(\mathbb{R}^n)} + \|v_1 - w_1\|_{L^2(\mathbb{R}^n)} \right), \quad \forall t \in [0, T], \end{aligned}$$

where  $c : R_+ \times R_+ \rightarrow R_+$  is a nondecreasing function with respect to each variable and  $\tilde{r} = \max \left\{ \|(v_0, v_1)\|_{H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)}, \|(w_0, w_1)\|_{H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)} \right\}$ .

Thus, according to Theorem 2.1, by the formula  $(u(t), u_t(t)) = S(t)(u_0, u_1)$ , problem (2.1)-(2.2) generates a strongly continuous semigroup  $\{S(t)\}_{t \geq 0}$  in  $H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ , where  $u(t, \cdot)$  is the weak solution of (2.1)-(2.2), determined by Theorem 2.1, with initial data  $(u_0, u_1)$ .

Now, we are in a position to state our main result.

**Theorem 2.2.** *Under the conditions (2.3)-(2.5), the semigroup  $\{S(t)\}_{t \geq 0}$  generated by the problem (2.1)-(2.2) possesses a global attractor  $\mathcal{A}$  in  $H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$  and  $\mathcal{A} = \mathcal{M}^u(\mathcal{N})$ . Here  $\mathcal{M}^u(\mathcal{N})$  is unstable manifold emanating from the set of stationary points  $\mathcal{N}$  (for definition, see [6, p.359]).*

**Remark 2.1.** We note that by using the method of this paper, one can prove the existence of the global attractors for the initial boundary value problems

$$\begin{cases} u_{itt} + (-\Delta)^i u_i + \alpha(x)u_{it} + \lambda u_i \\ + f(\|u_i(t)\|_{L^p(\mathbb{R}^n)}) |u_i|^{p_i-2} u_i = h(x), & (t, x) \in (0, \infty) \times \Omega, \\ u_i(t, x) = \left(\frac{\partial}{\partial \nu}\right)^{i-1} u_i(t, x) = 0, & (t, x) \in (0, \infty) \times \partial\Omega, \\ u_i(0, x) = u_{0i}(x), \quad u_{it}(0, x) = u_{1i}(x), & x \in \Omega, \end{cases}$$

where  $\Omega \subset \mathbb{R}^n$  is an unbounded domain with smooth boundary,  $\nu$  is outer unit normal vector,  $\lambda > 0$ ,  $h \in L^2(\Omega)$ ,  $p_i \geq 2$ ,  $p_i(n - 2i) \leq 2n - 2i$ ,  $i = 1, 2$ , the function  $f(\cdot)$  satisfies the condition (2.5) and the damping coefficient  $\alpha(\cdot)$  satisfies the following conditions

$$\alpha \in L^\infty(\Omega), \quad \alpha(\cdot) > 0, \quad \text{a.e. in } \Omega,$$

$$\alpha(\cdot) \geq \alpha_0 > 0, \quad \text{a.e. in } \omega, \quad \text{for some } \omega \subset \Omega, \text{ such that}$$

$\omega$  is the union of a neighbourhood of the boundary  $\partial\Omega$  and  $\{x \in \Omega : |x| \geq r_0\}$ , for some  $r_0$ .

**3. Proof of Theorem 2.2.** We start with the following lemmas.

**Lemma 3.1.** *Assume that the condition (2.5) holds. Also, assume that the sequence  $\{v_m\}_{m=1}^\infty$  is weakly star convergent in  $L^\infty(0, \infty; H^2(\mathbb{R}^n))$ , the sequence  $\{v_{mt}\}_{m=1}^\infty$  is bounded in  $L^\infty(0, \infty; L^2(\mathbb{R}^n))$  and the sequence  $\left\{ \|v_m(t)\|_{L^p(\mathbb{R}^n)} \right\}_{m=1}^\infty$  is convergent, for all  $t \geq 0$ . Then, for all  $r > 0$*

$$\lim_{m \rightarrow \infty} \limsup_{l \rightarrow \infty} \left| \int_0^t \int_{B(0,r)} \tau \left( f \left( \|v_m(\tau)\|_{L^p(\mathbb{R}^n)} \right) |v_m(\tau, x)|^{p-2} v_m(\tau, x) \right. \right.$$

$$\left. \left. - f \left( \|v_l(\tau)\|_{L^p(\mathbb{R}^n)} \right) |v_l(\tau, x)|^{p-2} v_l(\tau, x) \right) (v_{mt}(\tau, x) - v_{lt}(\tau, x)) dx d\tau \right| = 0, \quad \forall t \geq 0,$$

where  $B(0, r) = \{x \in \mathbb{R}^n : |x| < r\}$ .

*Proof.* Denote  $f_\varepsilon(u) = \begin{cases} f(u), & u \geq \varepsilon, \\ f(\varepsilon), & 0 \leq u < \varepsilon, \end{cases}$  for  $\varepsilon > 0$ . Then, we have

$$\left| f \left( \|v_m(\tau)\|_{L^p(\mathbb{R}^n)} \right) - f_\varepsilon \left( \|v_m(\tau)\|_{L^p(\mathbb{R}^n)} \right) \right| \leq \max_{0 \leq s_1, s_2 \leq \varepsilon} |f(s_1) - f(s_2)|,$$

and consequently

$$\begin{aligned}
& \left| \int_0^t \int_{B(0,r)} \tau (f \left( \|v_m(\tau)\|_{L^p(\mathbb{R}^n)} \right) |v_m(\tau, x)|^{p-2} v_m(\tau, x) \right. \\
& \left. - f \left( \|v_l(\tau)\|_{L^p(\mathbb{R}^n)} \right) |v_l(\tau, x)|^{p-2} v_l(\tau, x) \right) (v_{mt}(\tau, x) - v_{lt}(\tau, x)) dx d\tau \right| \\
& \leq \left| \int_0^t \int_{B(0,r)} \tau (f_\varepsilon \left( \|v_m(\tau)\|_{L^p(\mathbb{R}^n)} \right) |v_m(\tau, x)|^{p-2} v_m(\tau, x) \right. \\
& \left. - f_\varepsilon \left( \|v_l(\tau)\|_{L^p(\mathbb{R}^n)} \right) |v_l(\tau, x)|^{p-2} v_l(\tau, x) \right) (v_{mt}(\tau, x) - v_{lt}(\tau, x)) dx d\tau \right| \\
& \quad + c t \max_{0 \leq s_1, s_2 \leq \varepsilon} |f(s_1) - f(s_2)|, \quad \forall t \geq 0. \tag{3.1}
\end{aligned}$$

Let us estimate the first term on the right hand side of (3.1).

$$\begin{aligned}
& \int_0^t \int_{B(0,r)} \tau (f_\varepsilon \left( \|v_m(\tau)\|_{L^p(\mathbb{R}^n)} \right) |v_m(\tau, x)|^{p-2} v_m(\tau, x) \\
& - f_\varepsilon \left( \|v_l(\tau)\|_{L^p(\mathbb{R}^n)} \right) |v_l(\tau, x)|^{p-2} v_l(\tau, x) \right) (v_{mt}(\tau, x) - v_{lt}(\tau, x)) dx d\tau \\
& = \int_0^t \tau f_\varepsilon \left( \|v_m(\tau)\|_{L^p(\mathbb{R}^n)} \right) \int_{B(0,r)} |v_m(\tau, x)|^{p-2} v_m(\tau, x) v_{mt}(\tau, x) dx d\tau \\
& \quad + \int_0^t \tau f_\varepsilon \left( \|v_l(\tau)\|_{L^p(\mathbb{R}^n)} \right) \int_{B(0,r)} |v_l(\tau, x)|^{p-2} v_l(\tau, x) v_{lt}(\tau, x) dx d\tau \\
& \quad - \int_0^t \tau f_\varepsilon \left( \|v_m(\tau)\|_{L^p(\mathbb{R}^n)} \right) \int_{B(0,r)} |v_m(\tau, x)|^{p-2} v_m(\tau, x) v_{lt}(\tau, x) dx d\tau \\
& \quad - \int_0^t \tau f_\varepsilon \left( \|v_l(\tau)\|_{L^p(\mathbb{R}^n)} \right) \int_{B(0,r)} |v_l(\tau, x)|^{p-2} v_l(\tau, x) v_{mt}(\tau, x) dx d\tau. \tag{3.2}
\end{aligned}$$

For the first two terms on the right hand side of (3.2), we have

$$\begin{aligned}
& \int_0^t \tau f_\varepsilon \left( \|v_m(\tau)\|_{L^p(\mathbb{R}^n)} \right) \int_{B(0,r)} |v_m(\tau, x)|^{p-2} v_m(\tau, x) v_{mt}(\tau, x) dx d\tau \\
& \quad + \int_0^t \tau f_\varepsilon \left( \|v_l(\tau)\|_{L^p(\mathbb{R}^n)} \right) \int_{B(0,r)} |v_l(\tau, x)|^{p-2} v_l(\tau, x) v_{lt}(\tau, x) dx d\tau \\
& = \frac{1}{p} t f_\varepsilon \left( \|v_m(t)\|_{L^p(\mathbb{R}^n)} \right) \|v_m(t)\|_{L^p(B(0,r))}^p + \frac{1}{p} t f_\varepsilon \left( \|v_l(t)\|_{L^p(\mathbb{R}^n)} \right) \|v_l(t)\|_{L^p(B(0,r))}^p \\
& \quad - \frac{1}{p} \int_0^t f_\varepsilon \left( \|v_m(\tau)\|_{L^p(\mathbb{R}^n)} \right) \|v_m(\tau)\|_{L^p(B(0,r))}^p d\tau
\end{aligned}$$

$$\begin{aligned}
& -\frac{1}{p} \int_0^t f_\varepsilon \left( \|v_l(\tau)\|_{L^p(\mathbb{R}^n)} \right) \|v_l(\tau)\|_{L^p(B(0,r))}^p d\tau \\
& -\frac{1}{p} \int_0^t \tau \frac{d}{dt} \left( f_\varepsilon \left( \|v_m(\tau)\|_{L^p(\mathbb{R}^n)} \right) \right) \|v_m(\tau)\|_{L^p(B(0,r))}^p d\tau \\
& -\frac{1}{p} \int_0^t \tau \frac{d}{dt} \left( f_\varepsilon \left( \|v_l(\tau)\|_{L^p(\mathbb{R}^n)} \right) \right) \|v_l(\tau)\|_{L^p(B(0,r))}^p d\tau.
\end{aligned}$$

Since the sequence  $\left\{ \|v_m(t)\|_{L^p(\mathbb{R}^n)} \right\}_{m=1}^\infty$  is convergent, by continuity of  $f_\varepsilon$ , it follows that the sequence  $\left\{ f_\varepsilon \left( \|v_m(t)\|_{L^p(\mathbb{R}^n)} \right) \right\}_{m=1}^\infty$  also converges for all  $t \in [0, \infty)$ . Moreover, by the conditions of the lemma and the definition of  $f_\varepsilon$ , we obtain that the sequence  $\left\{ f_\varepsilon \left( \|v_m(\cdot)\|_{L^p(\mathbb{R}^n)} \right) \right\}_{m=1}^\infty$  is bounded in  $W^{1,\infty}(0, \infty)$ . So, the sequence  $\left\{ f_\varepsilon \left( \|v_m(\cdot)\|_{L^p(\mathbb{R}^n)} \right) \right\}_{m=1}^\infty$  converges weakly star in  $W^{1,\infty}(0, \infty)$  and we have

$$\begin{cases} f_\varepsilon \left( \|v_m(\cdot)\|_{L^p(\mathbb{R}^n)} \right) \rightarrow Q \text{ weakly star in } W^{1,\infty}(0, \infty), \\ v_m \rightarrow v \text{ weakly star in } L^\infty(0, \infty; H^2(\mathbb{R}^n)), \\ v_{mt} \rightarrow v_t \text{ weakly star in } L^\infty(0, \infty; L^2(\mathbb{R}^n)), \end{cases} \quad (3.3)$$

for some  $Q \in W^{1,\infty}(0, \infty)$  and  $v \in L^\infty(0, \infty; H^2(\mathbb{R}^n)) \cap W^{1,\infty}(0, \infty; L^2(\mathbb{R}^n))$ . Applying Aubin–Lions–Simon lemma (see [20]), by (3.3)<sub>2</sub>–(3.3)<sub>3</sub>, we find

$$v_m \rightarrow v \text{ strongly in } C([0, T]; L^q(B(0, r))), \quad \forall T \geq 0, \quad (3.4)$$

where  $q < \frac{2n}{(n-4)^+}$ . Then, considering (3.3) and (3.4), we get

$$\begin{aligned}
& \lim_{m \rightarrow \infty} \int_0^t \tau f_\varepsilon \left( \|v_m(\tau)\|_{L^p(\mathbb{R}^n)} \right) \int_{B(0,r)} |v_m(\tau, x)|^{p-2} v_m(\tau, x) v_{mt}(\tau, x) dx d\tau \\
& + \lim_{l \rightarrow \infty} \int_0^t \tau f_\varepsilon \left( \|v_l(\tau)\|_{L^p(\mathbb{R}^n)} \right) \int_{B(0,r)} |v_l(\tau, x)|^{p-2} v_l(\tau, x) v_{lt}(\tau, x) dx d\tau \\
& = \frac{2}{p} t Q(t) \|v(t)\|_{L^p(B(0,r))}^p - \frac{2}{p} \int_0^t Q(\tau) \|v(\tau)\|_{L^p(B(0,r))}^p d\tau \\
& \quad - \frac{2}{p} \int_0^t \tau \frac{d}{dt} (Q(\tau)) \|v(\tau)\|_{L^p(B(0,r))}^p d\tau. \quad (3.5)
\end{aligned}$$

For the last two terms on the right hand side of (3.2), by using (3.3), we have

$$\begin{aligned}
& \lim_{m \rightarrow \infty} \lim_{l \rightarrow \infty} \int_0^t \tau f_\varepsilon \left( \|v_m(\tau)\|_{L^p(\mathbb{R}^n)} \right) \int_{B(0,r)} |v_m(\tau, x)|^{p-2} v_m(\tau, x) v_{lt}(\tau, x) dx d\tau \\
& + \lim_{m \rightarrow \infty} \lim_{l \rightarrow \infty} \int_0^t \tau f_\varepsilon \left( \|v_l(\tau)\|_{L^p(\mathbb{R}^n)} \right) \int_{B(0,r)} |v_l(\tau, x)|^{p-2} v_l(\tau, x) v_{mt}(\tau, x) dx d\tau
\end{aligned}$$

$$\begin{aligned}
&= 2 \int_0^t \tau Q(\tau) \int_{B(0,r)} |v(\tau, x)|^{p-2} v(\tau, x) v_t(\tau, x) dx d\tau \\
&= \frac{2}{p} t Q(t) \|v(t)\|_{L^p(B(0,r))}^p - \frac{2}{p} \int_0^t Q(\tau) \|v(\tau)\|_{L^p(B(0,r))}^p d\tau \\
&\quad - \frac{2}{p} \int_0^t \tau \frac{d}{dt} (Q(\tau)) \|v(\tau)\|_{L^p(B(0,r))}^p d\tau. \tag{3.6}
\end{aligned}$$

Hence, taking into account (3.5)-(3.6) and passing to limit in (3.2), we obtain

$$\begin{aligned}
&\lim_{m \rightarrow \infty} \lim_{l \rightarrow \infty} \int_0^t \int_{B(0,r)} \tau f_\varepsilon \left( \|v_m(\tau)\|_{L^p(\mathbb{R}^n)} \right) |v_m(\tau, x)|^{p-2} v_m(\tau, x) \\
&\quad - f_\varepsilon \left( \|v_l(\tau)\|_{L^p(\mathbb{R}^n)} \right) |v_l(\tau, x)|^{p-2} v_l(\tau, x) (v_{mt}(\tau, x) - v_{lt}(\tau, x)) dx d\tau = 0. \tag{3.7}
\end{aligned}$$

Then, by (3.1) and (3.7), for all  $r > 0$ , we have

$$\begin{aligned}
&\limsup_{m \rightarrow \infty} \limsup_{l \rightarrow \infty} \left| \int_0^t \int_{B(0,r)} \tau f \left( \|v_m(\tau)\|_{L^p(\mathbb{R}^n)} \right) |v_m(\tau, x)|^{p-2} v_m(\tau, x) \right. \\
&\quad \left. - f \left( \|v_l(\tau)\|_{L^p(\mathbb{R}^n)} \right) |v_l(\tau, x)|^{p-2} v_l(\tau, x) (v_{mt}(\tau, x) - v_{lt}(\tau, x)) dx d\tau \right| \\
&\quad \leq c t \max_{0 \leq s_1, s_2 \leq \varepsilon} |f(s_1) - f(s_2)|, \quad \forall t \geq 0.
\end{aligned}$$

Thus, passing to the limit in the above inequality as  $\varepsilon \rightarrow 0$ , we obtain the claim of the lemma.  $\square$

**Lemma 3.2.** *Assume that in addition to the conditions of Lemma 3.1, conditions (2.3) and (2.4) also hold. Then, for every  $\gamma > 0$  there exists  $c_\gamma > 0$  such that*

$$\begin{aligned}
&\int_0^t \int_{\mathbb{R}^n} \tau (f \left( \|v_l(\tau)\|_{L^p(\mathbb{R}^n)} \right) |v_l(\tau, x)|^{p-2} v_l(\tau, x) \\
&\quad - f \left( \|v_m(\tau)\|_{L^p(\mathbb{R}^n)} \right) |v_m(\tau, x)|^{p-2} v_m(\tau, x) (v_{mt}(\tau, x) - v_{lt}(\tau, x)) dx d\tau \\
&\leq \gamma \int_0^t \tau E_{\mathbb{R}^n \setminus B(0,r)} (v_m(\tau) - v_l(\tau)) d\tau + c_\gamma \int_0^t E_{\mathbb{R}^n \setminus B(0,r)} (v_m(\tau) - v_l(\tau)) d\tau \\
&\quad + c_\gamma \int_0^t \tau \left( \|\sqrt{a} v_{mt}(\tau)\|_{L^2(\mathbb{R}^n)}^2 + \|\sqrt{a} v_{lt}(\tau)\|_{L^2(\mathbb{R}^n)}^2 \right) E_{\mathbb{R}^n \setminus B(0,r)} (v_m(\tau) - v_l(\tau)) d\tau \\
&\quad \quad \quad + K_r^{m,l}(t), \quad \forall t \geq 0, \quad \forall r \geq r_0,
\end{aligned}$$

where  $K_r^{m,l} \in C[0, t]$ ,  $\sup_{m,l} \|K_r^{m,l}\|_{C[0,t]} < \infty$  and  $\lim_{m \rightarrow \infty} \limsup_{l \rightarrow \infty} |K_r^{m,l}(t)| = 0$ , for all  $t \geq 0$ .

*Proof.* Firstly, we have

$$\begin{aligned}
& \int_0^t \int_{\mathbb{R}^n} \tau (f(\|v_l(\tau)\|_{L^p(\mathbb{R}^n)}) |v_l(\tau, x)|^{p-2} v_l(\tau, x) \\
& - f(\|v_m(\tau)\|_{L^p(\mathbb{R}^n)}) |v_m(\tau, x)|^{p-2} v_m(\tau, x)) (v_{m\tau}(\tau, x) - v_{l\tau}(\tau, x)) dx d\tau \\
& = \int_0^t \int_{\mathbb{R}^n \setminus B(0, r)} \tau (f(\|v_l(\tau)\|_{L^p(\mathbb{R}^n)}) |v_l(\tau, x)|^{p-2} v_l(\tau, x) \\
& - f(\|v_m(\tau)\|_{L^p(\mathbb{R}^n)}) |v_m(\tau, x)|^{p-2} v_m(\tau, x)) (v_{m\tau}(\tau, x) - v_{l\tau}(\tau, x)) dx d\tau \\
& \quad + K_{1,r}^{m,l}(t), \quad \forall r > 0, \tag{3.8}
\end{aligned}$$

where

$$\begin{aligned}
K_{1,r}^{m,l}(t) & := \int_0^t \int_{B(0,r)} \tau (f(\|v_l(\tau)\|_{L^p(\mathbb{R}^n)}) |v_l(\tau, x)|^{p-2} v_l(\tau, x) \\
& - f(\|v_m(\tau)\|_{L^p(\mathbb{R}^n)}) |v_m(\tau, x)|^{p-2} v_m(\tau, x)) (v_{m\tau}(\tau, x) - v_{l\tau}(\tau, x)) dx d\tau,
\end{aligned}$$

and by Lemma 3.1, it follows that

$$\sup_{m,l} \|K_{1,r}^{m,l}\|_{C[0,t]} < \infty \quad \text{and} \quad \lim_{m \rightarrow \infty} \limsup_{l \rightarrow \infty} |K_{1,r}^{m,l}(t)| = 0, \quad \forall t \geq 0.$$

On the other hand, for the first term on the right hand side of (3.8), we get

$$\begin{aligned}
& \int_0^t \int_{\mathbb{R}^n \setminus B(0,r)} \tau (f(\|v_l(\tau)\|_{L^p(\mathbb{R}^n)}) |v_l(\tau, x)|^{p-2} v_l(\tau, x) \\
& - f(\|v_m(\tau)\|_{L^p(\mathbb{R}^n)}) |v_m(\tau, x)|^{p-2} v_m(\tau, x)) (v_{m\tau}(\tau, x) - v_{l\tau}(\tau, x)) dx d\tau \\
& = -\frac{(p-1)}{2} \int_0^t \tau f(\|v_m(\tau)\|_{L^p(\mathbb{R}^n)}) \\
& \quad \times \int_{\mathbb{R}^n \setminus B(0,r)} \int_0^1 |v_m(\tau, x) + \sigma(v_l(\tau, x) - v_m(\tau, x))|^{p-2} d\sigma \\
& \quad \times \frac{d}{d\tau} |v_m(\tau, x) - v_l(\tau, x)|^2 dx d\tau + K_{2,r}^{m,l}(t), \tag{3.9}
\end{aligned}$$

where

$$\begin{aligned}
K_{2,r}^{m,l}(t) & := \int_0^t \tau (f(\|v_l(\tau)\|_{L^p(\mathbb{R}^n)}) - f(\|v_m(\tau)\|_{L^p(\mathbb{R}^n)})) \\
& \quad \times \int_{\mathbb{R}^n \setminus B(0,r)} |v_l(\tau, x)|^{p-2} v_l(\tau, x) (v_{m\tau}(\tau, x) - v_{l\tau}(\tau, x)) dx d\tau.
\end{aligned}$$

By the conditions of the lemma, we find

$$\sup_{m,l} \|K_{2,r}^{m,l}\|_{C[0,t]} < \infty \quad \text{and} \quad \lim_{m \rightarrow \infty} \limsup_{l \rightarrow \infty} |K_{2,r}^{m,l}(t)| = 0, \quad \forall t \geq 0.$$

So, denoting  $K_r^{m,l}(t) := K_{1,r}^{m,l}(t) + K_{2,r}^{m,l}(t)$ , by (3.8) and (3.9), we have

$$\begin{aligned}
& \int_0^t \int_{\mathbb{R}^n} \tau f \left( \|v_l(\tau)\|_{L^p(\mathbb{R}^n)} \right) |v_l(\tau, x)|^{p-2} v_l(\tau, x) \\
& - f \left( \|v_m(\tau)\|_{L^p(\mathbb{R}^n)} \right) |v_m(\tau, x)|^{p-2} v_m(\tau, x) (v_{m\tau}(\tau, x) - v_{l\tau}(\tau, x)) dx d\tau \\
& = -\frac{(p-1)}{2} \int_0^t \tau f \left( \|v_m(\tau)\|_{L^p(\mathbb{R}^n)} \right) \\
& \times \int_{\mathbb{R}^n \setminus B(0,r)} \int_0^1 |v_m(\tau, x) + \sigma(v_l(\tau, x) - v_m(\tau, x))|^{p-2} d\sigma \\
& \times \frac{d}{d\tau} |v_m(\tau, x) - v_l(\tau, x)|^2 dx d\tau + K_r^{m,l}(t). \tag{3.10}
\end{aligned}$$

Now, let us estimate the first term on the right hand side of (3.10). Denote

$$\varphi_M(u) = \begin{cases} u, & |u| \leq M, \\ M, & |u| > M, \end{cases} \quad \text{and } \Psi_\varepsilon(u) = \begin{cases} \varepsilon^{p-2}, & |u| \leq \varepsilon, \\ |u|^{p-2}, & |u| > \varepsilon \end{cases}. \quad \text{Then, we get}$$

$$\begin{aligned}
& \left| \|v_m(\tau)\|_{L^p(\mathbb{R}^n)} - \|\varphi_M(v_m(\tau))\|_{L^p(\mathbb{R}^n)} \right| \leq 2 \left( \int_{\{x \in \mathbb{R}^n : |v_m(\tau, x)| > M\}} |v_m(\tau, x)|^p dx \right)^{\frac{1}{p}} \\
& \leq \frac{2}{M^\beta} \left( \int_{\{x \in \mathbb{R}^n : |v_m(\tau, x)| > M\}} |v_m(\tau, x)|^{p+\beta} dx \right)^{\frac{1}{p}} \leq \frac{2}{M^\beta} \|v(\tau)\|_{H^2(\mathbb{R}^n)}^{\frac{p+\beta}{p}}, \tag{3.11}
\end{aligned}$$

where  $\beta \in \left(0, \frac{2n}{(n-4)^+} - p\right)$ . Also, it is clear that

$$\left| |w|^{p-2} - \Psi_\varepsilon(w) \right| \leq \omega(\varepsilon), \tag{3.12}$$

$$\text{where } \omega(\varepsilon) = \begin{cases} \varepsilon^{p-2}, & p > 2, \\ 0, & p = 2 \end{cases}.$$

By (3.11) and (3.12), it is easy to see that

$$\begin{aligned}
& \int_0^t \tau f \left( \|v_m(\tau)\|_{L^p(\mathbb{R}^n)} \right) \int_{\mathbb{R}^n \setminus B(0,r)} \int_0^1 |v_m(\tau, x) + \sigma(v_l(\tau, x) - v_m(\tau, x))|^{p-2} d\sigma \\
& \times \frac{d}{d\tau} |v_m(\tau, x) - v_l(\tau, x)|^2 dx d\tau \\
& \geq \int_0^t \tau f_\varepsilon \left( \|\varphi_M(v_m(\tau))\|_{L^p(\mathbb{R}^n)} \right) \\
& \times \int_{\mathbb{R}^n \setminus B(0,r)} \int_0^1 \Psi_\varepsilon(v_m(\tau, x) + \sigma(v_l(\tau, x) - v_m(\tau, x))) d\sigma \\
& \times \frac{d}{d\tau} |v_m(\tau, x) - v_l(\tau, x)|^2 dx d\tau
\end{aligned}$$



$$\begin{aligned}
 & -c_1 \left( \max_{0 < s_1, s_2 < \varepsilon} |f(s_1) - f(s_2)| + \frac{1}{M^\beta} + \omega(\varepsilon) \right) \\
 & \quad \times \int_0^t \tau E_{\mathbb{R}^n \setminus B(0,r)}(v_m(\tau) - v_l(\tau)) d\tau, \tag{3.13}
 \end{aligned}$$

where  $f_\varepsilon(\cdot)$  is as in Lemma 3.1.

Now, let us estimate the first term on the right hand side of (3.13).

$$\begin{aligned}
 & \int_0^t \tau f_\varepsilon \left( \|\varphi_M(v_m(\tau))\|_{L^p(\mathbb{R}^n)} \right) \int_{\mathbb{R}^n \setminus B(0,r)} \int_0^1 \Psi_\varepsilon(v_m(\tau, x) + \sigma(v_l(\tau, x) - v_m(\tau, x))) d\sigma \\
 & \quad \times \frac{d}{d\tau} |v_m(\tau, x) - v_l(\tau, x)|^2 dx d\tau \\
 & \geq - \int_0^t f_\varepsilon \left( \|\varphi_M(v_m(\tau))\|_{L^p(\mathbb{R}^n)} \right) \\
 & \times \int_{\mathbb{R}^n \setminus B(0,r)} \int_0^1 \Psi_\varepsilon(v_m(\tau, x) + \sigma(v_l(\tau, x) - v_m(\tau, x))) d\sigma |v_m(\tau, x) - v_l(\tau, x)|^2 dx d\tau \\
 & \quad - \int_0^t \tau \frac{d}{d\tau} \left( f_\varepsilon \left( \|\varphi_M(v_m(\tau))\|_{L^p(\mathbb{R}^n)} \right) \right) \\
 & \times \int_{\mathbb{R}^n \setminus B(0,r)} \int_0^1 \Psi_\varepsilon(v_m(\tau, x) + \sigma(v_l(\tau, x) - v_m(\tau, x))) d\sigma |v_m(\tau, x) - v_l(\tau, x)|^2 dx d\tau \\
 & \quad - \int_0^t \tau f_\varepsilon \left( \|\varphi_M(v_m(\tau))\|_{L^p(\mathbb{R}^n)} \right) \\
 & \quad \times \int_{\mathbb{R}^n \setminus B(0,r)} \int_0^1 \frac{d}{d\tau} (\Psi_\varepsilon(v_m(\tau, x) + \sigma(v_l(\tau, x) - v_m(\tau, x)))) d\sigma \\
 & \quad \times |v_m(\tau, x) - v_l(\tau, x)|^2 dx d\tau. \tag{3.14}
 \end{aligned}$$

Since

$$\begin{aligned}
 & \frac{d}{d\tau} f_\varepsilon \left( \|\varphi_M(v_m(\tau))\|_{L^p(\mathbb{R}^n)} \right) \\
 & = \frac{f'_\varepsilon \left( \|\varphi_M(v_m(\tau))\|_{L^p(\mathbb{R}^n)} \right)}{\|\varphi_M(v_m(\tau))\|_{L^p(\mathbb{R}^n)}^{p-1}} \int_{\mathbb{R}^n} |\varphi_M(v_m(\tau, x))|^{p-1} \varphi'_M(v_m(\tau, x)) v_{m\tau}(\tau, x) dx \\
 & \leq \frac{c_2}{\varepsilon^{p-1}} \left( \int_{\mathbb{R}^n \setminus B(0,r_0)} |v_m(\tau, x)|^{p-1} |v_{m\tau}(\tau, x)| dx + M^{p-1} \int_{B(0,r_0)} |v_{m\tau}(\tau, x)| dx \right) \\
 & \leq \frac{c_3}{\varepsilon^{p-1}} \left( \|v_{m\tau}(\tau)\|_{L^2(\mathbb{R}^n \setminus B(0,r_0))} + M^{p-1} \|v_{m\tau}(\tau)\|_{L^1(B(0,r_0))} \right)
 \end{aligned}$$

and

$$\int_{\mathbb{R}^n \setminus B(0,r)} \int_0^1 \frac{d}{d\tau} (\Psi_\varepsilon(v_l(\tau, x) + \sigma(v_m(\tau, x) - v_l(\tau, x)))) d\sigma |v_m(\tau, x) - v_l(\tau, x)|^2 dx$$

$$\begin{aligned}
&\leq \int_{\mathbb{R}^n \setminus B(0,r)} \int_0^1 \Psi'_\varepsilon (v_l(\tau, x) + \sigma(v_m(\tau, x) - v_l(\tau, x))) d\sigma |v_m(\tau, x) - v_l(\tau, x)|^2 dx \\
&\quad \times \left( \|v_{mt}(\tau)\|_{L^2(\mathbb{R}^n \setminus B(0,r))} + \|v_{lt}(\tau)\|_{L^2(\mathbb{R}^n \setminus B(0,r))} \right) \\
&\quad \leq \frac{c_4}{\varepsilon^{\max\{0, 3-p\}}} \|v_m(\tau) - v_l(\tau)\|_{H^2(\mathbb{R}^n \setminus B(0,r))}^2 \\
&\quad \times \left( \|v_{mt}(\tau)\|_{L^2(\mathbb{R}^n \setminus B(0,r))} + \|v_{lt}(\tau)\|_{L^2(\mathbb{R}^n \setminus B(0,r))} \right)
\end{aligned}$$

by (3.14), we find

$$\begin{aligned}
&\int_0^t \tau f_\varepsilon \left( \|\varphi_M(v_m(\tau))\|_{L^p(\mathbb{R}^n)} \right) \int_{\mathbb{R}^n \setminus B(0,r)} \int_0^1 \Psi_\varepsilon (v_l(\tau, x) + \sigma(v_m(\tau, x) - v_l(\tau, x))) d\sigma \\
&\quad \times \frac{d}{d\tau} |v_m(\tau, x) - v_l(\tau, x)|^2 dx d\tau \\
&\quad \geq -c_5 \int_0^t E_{\mathbb{R}^n \setminus B(0,r)}(v_m(\tau) - v_l(\tau)) d\tau \\
&\quad - \frac{c_5}{\varepsilon^{p-1}} \int_0^t \tau E_{\mathbb{R}^n \setminus B(0,r)}(v_m(\tau) - v_l(\tau)) \|v_{mt}(\tau)\|_{L^2(\mathbb{R}^n \setminus B(0,r_0))} d\tau \\
&\quad - \frac{c_5 M^{p-1}}{\varepsilon^{p-1}} \int_0^t \tau E_{\mathbb{R}^n \setminus B(0,r)}(v_m(\tau) - v_l(\tau)) \|v_{mt}(\tau)\|_{L^1(B(0,r_0))} d\tau \\
&\quad - \frac{c_5}{\varepsilon^{\max\{0, 3-p\}}} \int_0^t \tau E_{\mathbb{R}^n \setminus B(0,r)}(v_m(\tau) - v_l(\tau)) \\
&\quad \times \left( \|v_{mt}(\tau)\|_{L^2(\mathbb{R}^n \setminus B(0,r))} + \|v_{lt}(\tau)\|_{L^2(\mathbb{R}^n \setminus B(0,r))} \right) d\tau.
\end{aligned}$$

After applying Young inequality in the above inequality, we get

$$\begin{aligned}
&\int_0^t \tau f_\varepsilon \left( \|\varphi_M(v_m(\tau))\|_{L^p(\mathbb{R}^n)} \right) \int_{\mathbb{R}^n \setminus B(0,r)} \int_0^1 \Psi_\varepsilon (v_l(\tau, x) + \sigma(v_m(\tau, x) - v_l(\tau, x))) d\sigma \\
&\quad \times \frac{d}{d\tau} |v_m(\tau, x) - v_l(\tau, x)|^2 dx d\tau \\
&\quad \geq -c_5 \int_0^t E_{\mathbb{R}^n \setminus B(0,r)}(v_m(\tau) - v_l(\tau)) d\tau - \frac{c_5}{\varepsilon^{p-1} \mu} \int_0^t \tau E_{\mathbb{R}^n \setminus B(0,r)}(v_m(\tau) - v_l(\tau)) d\tau \\
&\quad - \frac{c_5}{\varepsilon^{p-1} \mu^2} \int_0^t \tau E_{\mathbb{R}^n \setminus B(0,r)}(v_m(\tau) - v_l(\tau)) \|v_{mt}(\tau)\|_{L^2(\mathbb{R}^n \setminus B(0,r_0))}^2 d\tau \\
&\quad - \frac{c_5 M^{p-1}}{\varepsilon^{p-1} \mu} \int_0^t \tau E_{\mathbb{R}^n \setminus B(0,r)}(v_m(\tau) - v_l(\tau)) d\tau
\end{aligned}$$

$$\begin{aligned}
& -\frac{c_5 M^{p-1}}{\varepsilon^{p-1} \mu^2} \int_0^t \tau E_{\mathbb{R}^n \setminus B(0,r)} (v_m(\tau) - v_l(\tau)) \|v_{mt}(\tau)\|_{L^1(B(0,r_0))}^2 d\tau \\
& -\frac{c_5}{\varepsilon^{\max\{0,3-p\}}} \mu \int_0^t \tau E_{\mathbb{R}^n \setminus B(0,r)} (v_m(\tau) - v_l(\tau)) d\tau \\
& -\frac{c_5}{\varepsilon^{\max\{0,3-p\}} \mu^2} \int_0^t \tau E_{\mathbb{R}^n \setminus B(0,r)} (v_m(\tau) - v_l(\tau)) \\
& \quad \times \left( \|v_{mt}(\tau)\|_{L^2(\mathbb{R}^n \setminus B(0,r))}^2 + \|v_{lt}(\tau)\|_{L^2(\mathbb{R}^n \setminus B(0,r))}^2 \right) d\tau. \tag{3.15}
\end{aligned}$$

By (3.13) and (3.15), we obtain

$$\begin{aligned}
& \int_0^t \tau f \left( \|v_m(\tau)\|_{L^p(\mathbb{R}^n)} \right) \int_{\mathbb{R}^n \setminus B(0,r)} \int_0^1 |v_m(\tau, x) + \sigma(v_l(\tau, x) - v_m(\tau, x))|^{p-2} d\sigma \\
& \quad \times \frac{d}{d\tau} |v_m(\tau, x) - v_l(\tau, x)|^2 dx d\tau \\
& \geq -c_6 \int_0^t E_{\mathbb{R}^n \setminus B(0,r)} (v_m(\tau) - v_l(\tau)) d\tau \\
& -c_6 \left( \frac{\mu}{\varepsilon^{p-1}} + \frac{\mu}{\varepsilon^{\max\{0,3-p\}}} + \max_{0 \leq s_1, s_2 \leq \varepsilon} |f(s_1) - f(s_2)| + \frac{M^{p-1}}{\varepsilon^{p-1}} \mu \right. \\
& \quad \left. + \frac{1}{M^\beta} + \omega(\varepsilon) \right) \int_0^t \tau E_{\mathbb{R}^n \setminus B(0,r)} (v_m(\tau) - v_l(\tau)) d\tau \\
& -c_6 \left( \frac{1}{\varepsilon^{p-1} \mu^2} + \frac{1}{\varepsilon^{\max\{0,3-p\}} \mu^2} \right) \int_0^t \tau E_{\mathbb{R}^n \setminus B(0,r)} (v_m(\tau) - v_l(\tau)) \\
& \quad \times \left( \|v_{mt}(\tau)\|_{L^2(\mathbb{R}^n \setminus B(0,r_0))}^2 + \|v_{lt}(\tau)\|_{L^2(\mathbb{R}^n \setminus B(0,r_0))}^2 \right) d\tau \\
& -c_6 \frac{M^{p-1}}{\varepsilon^{p-1} \mu^2} \int_0^t \tau E_{\mathbb{R}^n \setminus B(0,r)} (v_m(\tau) - v_l(\tau)) \|v_{mt}(\tau)\|_{L^1(B(0,r_0))}^2 d\tau, \quad \forall r \geq r_0. \tag{3.16}
\end{aligned}$$

To complete the proof, let us estimate the term  $\|v_{mt}(\tau)\|_{L^1(B(0,r_0))}^2$ . By the conditions of the lemma, we have

$$\begin{aligned}
\|v_{mt}(\tau)\|_{L^1(B(0,r_0))}^2 &= \left( \int_{B(0,r_0)} |v_{mt}(\tau, x)| dx \right)^2 \\
&= \left( \int_{B(0,r_0)} \frac{a(x) + \lambda}{a(x) + \lambda} |v_{mt}(\tau, x)| dx \right)^2
\end{aligned}$$

$$\begin{aligned}
&\leq \left( \frac{1}{\lambda} \int_{B(0,r_0)} a(x) |v_{mt}(\tau, x)| dx + \int_{B(0,r_0)} \frac{\lambda}{a(x) + \lambda} |v_{mt}(\tau, x)| dx \right)^2 \\
&\leq \frac{c_7}{\lambda^2} \int_{\mathbb{R}^n} a(x) |v_{mt}(\tau, x)|^2 dx + c_7 \int_{B(0,r_0)} \left( \frac{\lambda}{a(x) + \lambda} \right)^2 dx, \quad \forall \lambda > 0. \quad (3.17)
\end{aligned}$$

Since, by Lebesgue dominated convergence theorem,  $\lim_{\lambda \rightarrow 0^+} \int_{B(0,r_0)} \left( \frac{\lambda}{a(x) + \lambda} \right)^2 dx = 0$ , we can choose positive parameters  $\varepsilon$ ,  $M$ ,  $\mu$  and  $\lambda$  such that

$$\begin{aligned}
&c_6 \left( \frac{\mu}{\varepsilon^{p-1}} + \frac{\mu}{\varepsilon^{\max\{0, 3-p\}}} + \max_{0 \leq s_1, s_2 \leq \varepsilon} |f(s_1) - f(s_2)| + \frac{M^{p-1}}{\varepsilon^{p-1}} \mu \right. \\
&\quad \left. + \frac{1}{M^\beta} + \omega(\varepsilon) \right) + c_6 c_7 \frac{M^{p-1}}{\varepsilon^{p-1} \mu^2} \int_{B(0,r_0)} \left( \frac{\lambda}{a(x) + \lambda} \right)^2 dx \leq \gamma.
\end{aligned}$$

Thus, by (3.10), (3.16) and (3.17), the proof of the lemma is complete.  $\square$

Now, we prove the following theorem on the asymptotic compactness of  $\{S(t)\}_{t \geq 0}$  in  $H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ , which plays a key role in the existence of the global attractor.

**Theorem 3.1.** *Assume that the conditions (2.3)-(2.5) hold and  $B$  is a bounded subset of  $H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ . Then for every sequence of the form  $\{S(t_k)\varphi_k\}_{k=1}^\infty$ , where  $\{\varphi_k\}_{k=1}^\infty \subset B$ ,  $t_k \rightarrow \infty$ , has a convergent subsequence in  $H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ .*

*Proof.* To get the claim of the theorem, it is sufficient to prove the following sequential limit estimate

$$\liminf_{k \rightarrow \infty} \liminf_{m \rightarrow \infty} \|S(t_k)\varphi_k - S(t_m)\varphi_m\|_{H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)} = 0, \quad (3.18)$$

for every  $\{\varphi_k\}_{k=1}^\infty \subset B$  and  $t_k \rightarrow \infty$ . Indeed, establishing (3.18) and using the argument at the end of the proof of [12, Lemma 3.4], we obtain the desired result.

Now, by (2.3), (2.5) and (2.6), we have

$$\sup_{t \geq 0} \sup_{\varphi \in B} \|S(t)\varphi\|_{H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)} < \infty. \quad (3.19)$$

Since  $\{\varphi_k\}_{k=1}^\infty$  is bounded in  $H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ , by (3.19), the sequence  $\{S(\cdot)\varphi_k\}_{k=1}^\infty$  is bounded in  $C_b(0, \infty; H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n))$ , where  $C_b(0, \infty; H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n))$  is the space of continuously bounded functions from  $[0, \infty)$  to  $H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ . Then for any  $T_0 \geq 0$  there exists a subsequence  $\{k_m\}_{m=1}^\infty$  such that  $t_{k_m} \geq T_0$ , and

$$\begin{cases} v_m \rightarrow v \text{ weakly star in } L^\infty(0, \infty; H^2(\mathbb{R}^n)), \\ v_{mt} \rightarrow v_t \text{ weakly star in } L^\infty(0, \infty; L^2(\mathbb{R}^n)), \\ \|v_m(t)\|_{L^p(\mathbb{R}^n)}^p \rightarrow q(t) \text{ weakly star in } W^{1,\infty}(0, \infty), \\ v_m(t) \rightarrow v(t) \text{ weakly in } H^2(\mathbb{R}^n), \forall t \geq 0, \end{cases} \quad (3.20)$$

for some  $q \in W^{1,\infty}(0, \infty)$  and  $v \in L^\infty(0, \infty; H^2(\mathbb{R}^n)) \cap W^{1,\infty}(0, \infty; L^2(\mathbb{R}^n))$ , where  $(v_m(t), v_{mt}(t)) = S(t + t_{k_m} - T_0)\varphi_{k_m}$ . By (2.1), we also have

$$\begin{aligned}
&v_{mtt}(t, x) - v_{ltt}(t, x) + \Delta^2(v_m(t, x) - v_l(t, x)) + \alpha(x)(v_{mt}(t, x) - v_{lt}(t, x)) \\
&\quad + \lambda(v_m(t, x) - v_l(t, x)) = f(\|v_l(t)\|_{L^p(\mathbb{R}^n)}) |v_l(t, x)|^{p-2} v_l(t, x) \\
&\quad - f(\|v_m(t)\|_{L^p(\mathbb{R}^n)}) |v_m(t, x)|^{p-2} v_m(t, x). \quad (3.21)
\end{aligned}$$

We obtain (3.18) by means of the sequential limit estimate of the energy of  $v_m - v_l$  which is proved in the following three steps. In the first step, we get the tail estimates, by using the effect of the damping term. In the second step, we obtain the interior estimates. Finally, in the last step, we get the sequential limit estimate of the energy in  $\mathbb{R}^n$ , by considering the results obtained in the previous steps. Note that we establish these estimates for the smooth solutions of (2.1)-(2.2) with the initial data in  $H^4(\mathbb{R}^n) \times H^2(\mathbb{R}^n)$ , for which the estimates in the following text are justified. These estimates can be extended to the weak solutions with the initial data in  $H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$  by the standard density arguments.

**Step 1 (Tail estimates).** Taking into account (2.3), (2.4), (2.5) and (2.6) we get

$$\int_0^T \|v_{mt}(t)\|_{L^2(\mathbb{R}^n \setminus B(0, r_0))}^2 dt \leq c_1, \quad \forall T \geq 0, \quad (3.22)$$

where the constant  $c_1$  depends on  $B$  and is independent of  $T$  and  $m$ . Now, putting  $v_m$  instead of  $v$  in (2.1), we have

$$\begin{aligned} & v_{mtt}(t, x) + \Delta^2 v_m(t, x) + \alpha(x)v_{mt}(t, x) + \lambda v_m(t, x) \\ & + f(\|v_m(t)\|_{L^p(\mathbb{R}^n)}) |v_m(t, x)|^{p-2} v_m(t, x) = h(x). \end{aligned}$$

Let  $\eta \in C^\infty(\mathbb{R}^n)$ ,  $0 \leq \eta(x) \leq 1$ ,  $\eta(x) = \begin{cases} 0, & |x| \leq 1 \\ 1, & |x| \geq 2 \end{cases}$  and  $\eta_r(x) = \eta(\frac{x}{r})$ .

Multiplying above equation by  $\eta_r^2 v_m$  and integrating over  $(0, T) \times \mathbb{R}^n$ , we get

$$\begin{aligned} & \int_0^T \left( \|\eta_r \Delta v_m(t)\|_{L^2(\mathbb{R}^n)}^2 + \lambda \|\eta_r v_m(t)\|_{L^2(\mathbb{R}^n)}^2 \right) dt \\ & = \int_0^T \|\eta_r v_{mt}(t)\|_{L^2(\mathbb{R}^n)}^2 dt - \left( \int_{\mathbb{R}^n} \eta_r^2(x) v_{mt}(t, x) v_m(t, x) dx \right) \Big|_0^T \\ & \quad - \frac{4}{r} \sum_{i=1}^n \int_0^T \eta_r(x) \eta_{x_i} \left( \frac{x}{r} \right) \Delta v_m(t, x) v_{mx_i}(t, x) dx dt \\ & \quad - \int_0^T \int_{\mathbb{R}^n} \Delta(\eta_r^2(x)) \Delta v_m(t, x) v_m(t, x) dx dt - \frac{1}{2} \left( \int_{\mathbb{R}^n} \eta_r^2(x) \alpha(x) (v_m(t, x))^2 dx \right) \Big|_0^T \\ & \quad - \int_0^T f(\|v_m(t)\|_{L^p(\mathbb{R}^n)}) \int_{\mathbb{R}^n} |v_m(t, x)|^p \eta_r^2(x) dx dt + \int_0^T \int_{\mathbb{R}^n} h(x) \eta_r^2(x) v_m(t, x) dx dt. \end{aligned}$$

Taking into account (2.3), (2.5), (3.19) and (3.22) we obtain

$$\begin{aligned} & \int_0^T \left( \|\Delta(v_m(t))\|_{L^2(\mathbb{R}^n \setminus B(0, 2r))}^2 + \lambda \|v_m(t)\|_{L^2(\mathbb{R}^n \setminus B(0, 2r))}^2 \right) dt \\ & \leq c_2 \left( 1 + \frac{T}{r} + T \|h\|_{L^2(\mathbb{R}^n \setminus B(0, r))} \right), \quad \forall T \geq 0 \text{ and } \forall r \geq r_0, \quad (3.23) \end{aligned}$$

where the constant  $c_2$  depends on  $B$  and is independent of  $T$ ,  $r$  and  $m$ .

**Step 2 (Interior estimates).** Multiplying (3.21) by  $\sum_{i=1}^n x_i (1 - \eta_{2r})(v_m - v_l)_{x_i} + \frac{1}{2}(n-1)(1 - \eta_{2r})(v_m - v_l)$ , and integrating over  $(0, T) \times \mathbb{R}^n$ , we find

$$\begin{aligned}
& \frac{3}{2} \int_0^T \|\Delta(v_m(t) - v_l(t))\|_{L^2(B(0,2r))}^2 dt + \frac{1}{2} \int_0^T \|v_{mt}(t) - v_{lt}(t)\|_{L^2(B(0,2r))}^2 dt \\
& \leq \left| \sum_{i=1}^n \int_{B(0,4r)} (1 - \eta_{2r}(x)) x_i (v_m(T, x) - v_l(T, x))_{x_i} (v_{mt}(T, x) - v_{lt}(T, x)) dx \right| \\
& + \left| \sum_{i=1}^n \int_{B(0,4r)} (1 - \eta_{2r}(x)) x_i (v_m(0, x) - v_l(0, x))_{x_i} (v_{mt}(0, x) - v_{lt}(0, x)) dx \right| \\
& + \frac{1}{2}(n-1) \left| \int_{B(0,4r)} (1 - \eta_{2r}(x)) (v_{mt}(T, x) - v_{lt}(T, x)) (v_m(T, x) - v_l(T, x)) dx \right| \\
& + \frac{1}{2}(n-1) \left| \int_{B(0,4r)} (1 - \eta_{2r}(x)) (v_{mt}(0, x) - v_{lt}(0, x)) (v_m(0, x) - v_l(0, x)) dx \right| \\
& + \frac{1}{4r} \left| \sum_{i=1}^n \int_0^T \int_{B(0,4r) \setminus B(0,2r)} \eta_{x_i} \left( \frac{x}{2r} \right) x_i (v_{mt}(t, x) - v_{lt}(t, x))^2 dx dt \right| \\
& + \frac{1}{4r} \left| \sum_{i=1}^n \int_0^T \int_{B(0,4r) \setminus B(0,2r)} \eta_{x_i} \left( \frac{x}{2r} \right) x_i (\Delta v_m(t, x) - \Delta v_l(t, x))^2 dx dt \right| \\
& + \left| \sum_{i=1}^n \int_0^T \int_{B(0,4r)} \Delta((1 - \eta_{2r}(x)) x_i) (v_m(t, x) - v_l(t, x))_{x_i} \right. \\
& \quad \left. \times \Delta(v_m(t, x) - v_l(t, x)) dx dt \right| \\
& + \frac{1}{r} \left| \sum_{i,j=1}^n \int_0^T \int_{B(0,4r) \setminus B(0,2r)} \eta_{x_j} \left( \frac{x}{2r} \right) x_i (v_m(t, x) - v_l(t, x))_{x_i x_j} \right. \\
& \quad \left. \times \Delta(v_m(t, x) - v_l(t, x)) dx dt \right| \\
& + \frac{1}{2}(n-1) \left| \int_0^T \int_{B(0,4r) \setminus B(0,2r)} \Delta((1 - \eta_{2r}(x))) (v_m(t, x) - v_l(t, x)) \right. \\
& \quad \left. \times \Delta(v_m(t, x) - v_l(t, x)) dx dt \right| \\
& + \frac{1}{2r}(n-1) \left| \sum_{i=1}^n \int_0^T \int_{B(0,4r) \setminus B(0,2r)} \eta_{x_i} \left( \frac{x}{2r} \right) (v_m(t, x) - v_l(t, x))_{x_i} \right. \\
& \quad \left. \times \Delta(v_m(t, x) - v_l(t, x)) dx dt \right|
\end{aligned}$$

$$\begin{aligned}
 & + \left| \sum_{i=1}^n \int_0^T \int_{B(0,4r)} (1 - \eta_{2r}(x)) x_i (v_m(t, x) - v_l(t, x))_{x_i} \right. \\
 & \quad \times a(x) (v_{mt}(t, x) - v_{lt}(t, x)) dx dt \Big| \\
 & + \frac{1}{2} (n-1) \left| \int_0^T \int_{B(0,4r)} (1 - \eta_{2r}(x)) (v_m(t, x) - v_l(t, x)) \right. \\
 & \quad \times a(x) (v_{mt}(t, x) - v_{lt}(t, x)) dx dt \Big| \\
 & + \lambda \left| \sum_{i=1}^n \int_0^T \int_{B(0,4r)} (1 - \eta_{2r}(x)) x_i (v_m(t, x) - v_l(t, x))_{x_i} (v_m(t, x) - v_l(t, x)) dx dt \right| \\
 & + \left| \sum_{i=1}^n \int_0^T \int_{B(0,4r)} (1 - \eta_{2r}(x)) x_i (v_m(t, x) - v_l(t, x))_{x_i} \right. \\
 & \quad \times (f(\|v_m(t)\|_{L^p(\mathbb{R}^n)}) |v_m(t, x)|^{p-2} v_m(t, x) \\
 & \quad - f(\|v_l(t)\|_{L^p(\mathbb{R}^n)}) |v_l(t, x)|^{p-2} v_l(t, x)) dx dt \Big| \\
 & + \frac{1}{2} (n-1) \left| \int_0^T \int_{B(0,4r)} (1 - \eta_{2r}(x)) (v_m(t, x) - v_l(t, x)) \right. \\
 & \quad \times (f(\|v_m(t)\|_{L^p(\mathbb{R}^n)}) |v_m(t, x)|^{p-2} v_m(t, x) \\
 & \quad - f(\|v_l(t)\|_{L^p(\mathbb{R}^n)}) |v_l(t, x)|^{p-2} v_l(t, x)) dx dt \Big| \\
 & \leq c_3 r \left( \|\nabla v_m(T) - \nabla v_l(T)\|_{L^2(B(0,4r))} + \|\nabla v_m(0) - \nabla v_l(0)\|_{L^2(B(0,4r))} \right) \\
 & + c_3 \|v_{mt} - v_{lt}\|_{L^2(0,T;L^2(B(0,4r)\setminus B(0,2r)))}^2 + c_3 \|v_m - v_l\|_{L^2(0,T;H^2(B(0,4r)\setminus B(0,2r)))}^2 \\
 & \quad + c_3 r \sqrt{T} \|\nabla v_m - \nabla v_l\|_{L^2((0,T)\times B(0,4r))}, \tag{3.24}
 \end{aligned}$$

since, by (2.5) and (3.19),

$$\left\| f(\|v_m(t)\|_{L^p(\mathbb{R}^n)}) |v_m(t)|^{p-2} v_m(t) - f(\|v_l(t)\|_{L^p(\mathbb{R}^n)}) |v_l(t)|^{p-2} v_l(t) \right\|_{L^2(B(0,4r))} \leq \tilde{c}.$$

Since the sequence  $\{v_m\}_{m=1}^\infty$  is bounded in  $C([0, T]; H^2(\mathbb{R}^n))$  and the sequence  $\{v_{mt}\}_{m=1}^\infty$  is bounded in  $C([0, T]; L^2(\mathbb{R}^n))$ , by the generalized Arzela-Ascoli theorem, the sequence  $\{v_m\}_{m=1}^\infty$  is relatively compact in  $C([0, T]; H^1(B(0, r)))$  for every  $r > 0$ . So, according to (3.20)<sub>1</sub>-(3.20)<sub>2</sub>, the sequence  $\{v_m\}_{m=1}^\infty$  strongly converges to  $v$  in  $C([0, T]; H^1(B(0, r)))$ . Then, by using (3.22) and (3.23) in (3.24), we get

$$\begin{aligned}
 & \limsup_{m \rightarrow \infty} \limsup_{l \rightarrow \infty} \int_0^T \left[ \|\Delta(v_m(t) - v_l(t))\|_{L^2(B(0,2r))}^2 + \|v_{mt}(t) - v_{lt}(t)\|_{L^2(B(0,2r))}^2 \right] dt \\
 & \leq c_4 \left( 1 + \frac{T}{r} + T \|h\|_{L^2(\mathbb{R}^n \setminus B(0,r))} \right), \quad \forall T \geq 0 \text{ and } \forall r \geq r_0,
 \end{aligned}$$

where the constant  $c_4$  depends on  $B$  and is independent of  $T$  and  $r$ .

**Step 3 (Estimates in  $\mathbb{R}^n$ ).** By using (3.22), (3.23) and the last estimate of the previous step, we obtain

$$\begin{aligned} & \limsup_{m \rightarrow \infty} \limsup_{l \rightarrow \infty} \int_0^T \left[ \|v_m(t) - v_l(t)\|_{H^2(\mathbb{R}^n)}^2 + \|v_{mt}(t) - v_{lt}(t)\|_{L^2(\mathbb{R}^n)}^2 \right] dt \\ & \leq c_4 \left( 1 + \frac{T}{r} + T \|h\|_{L^2(\mathbb{R}^n \setminus B(0,r))} \right), \quad \forall T \geq 0 \text{ and } \forall r \geq r_0. \end{aligned}$$

Passing to limit as  $r \rightarrow \infty$  in the last inequality, we get

$$\begin{aligned} & \limsup_{m \rightarrow \infty} \limsup_{l \rightarrow \infty} \int_0^T \left[ \|v_m(t) - v_l(t)\|_{H^2(\mathbb{R}^n)}^2 + \|v_{mt}(t) - v_{lt}(t)\|_{L^2(\mathbb{R}^n)}^2 \right] dt \\ & \leq c_5, \quad \forall T \geq 0. \end{aligned} \tag{3.25}$$

Multiplying (3.21) by  $2t(v_{mt} - v_{lt})$ , integrating over  $(0, T) \times \mathbb{R}^n$ , using integration by parts and considering (2.4), we find

$$\begin{aligned} & T \|\Delta(v_m(T) - v_l(T))\|_{L^2(\mathbb{R}^n)}^2 + T \|v_{mt}(T) - v_{lt}(T)\|_{L^2(\mathbb{R}^n)}^2 \\ & + T\lambda \|v_m(T) - v_l(T)\|_{L^2(\mathbb{R}^n)}^2 + 2\alpha_0 \int_0^T \int_{\mathbb{R}^n \setminus B(0,r)} t (v_{mt}(t) - v_{lt}(t))^2 dx dt \\ & \leq \int_0^T \|v_{mt}(t) - v_{lt}(t)\|_{L^2(\mathbb{R}^n)}^2 dt + \int_0^T \|\Delta(v_m(t) - v_l(t))\|_{L^2(\mathbb{R}^n)}^2 dt \\ & + \lambda \int_0^T \|v_m(t) - v_l(t)\|_{L^2(\mathbb{R}^n)}^2 dt + 2 \int_0^T \int_{\mathbb{R}^n} t (f(\|v_l(t)\|_{L^p(\mathbb{R}^n)}) |v_l(t, x)|^{p-2} v_l(t, x) \\ & - f(\|v_m(t)\|_{L^p(\mathbb{R}^n)}) |v_m(t, x)|^{p-2} v_m(t, x)) (v_{mt}(t, x) - v_{lt}(t, x)) dx dt. \end{aligned} \tag{3.26}$$

Multiplying (3.21) by  $t\eta_r(v_m - v_l)$ , integrating over  $(0, T) \times \mathbb{R}^n$  and using integration by parts, we get

$$\begin{aligned} & T \int_{\mathbb{R}^n} (v_{mt}(T, x) - v_{lt}(T, x)) \eta_r(x) (v_m(T, x) - v_l(T, x)) dx \\ & - \int_0^T \int_{\mathbb{R}^n} t \eta_r(x) (v_{mt}(t, x) - v_{lt}(t, x))^2 dx dt \\ & - \int_0^T \int_{\mathbb{R}^n} \eta_r(x) (v_{mt}(t, x) - v_{lt}(t, x)) (v_m(t, x) - v_l(t, x)) dx dt \\ & + \int_0^T \int_{\mathbb{R}^n} t (\Delta(v_m(t, x) - v_l(t, x)))^2 \eta_r(x) dx dt \\ & + 2 \sum_{i=1}^n \int_0^T \int_{\mathbb{R}^n} \Delta(v_m(t, x) - v_l(t, x)) t (\eta_r(x))_{x_i} (v_m(t, x) - v_l(t, x))_{x_i} dx dt \end{aligned}$$



$$\begin{aligned}
& + \int_0^T \int_{\mathbb{R}^n} \Delta (v_m(t, x) - v_l(t, x)) t \Delta (\eta_r(x)) (v_m(t, x) - v_l(t, x)) dx dt \\
& \quad + \frac{T}{2} \int_{\mathbb{R}^n} \alpha(x) (v_m(T, x) - v_l(T, x))^2 \eta_r(x) dx \\
& \quad - \frac{1}{2} \int_0^T \int_{\mathbb{R}^n} \alpha(x) \eta_r(x) (v_m(t, x) - v_l(t, x))^2 dx dt \\
& \quad + \lambda \int_0^T \int_{\mathbb{R}^n} t (v_m(t, x) - v_l(t, x))^2 \eta_r(x) dx dt \\
& \quad + \int_0^T \int_{\mathbb{R}^n} t (f(\|v_m(t)\|_{L^p(\mathbb{R}^n)}) |v_m(t, x)|^{p-2} v_m(t, x) \\
& \quad - f(\|v_l(t)\|_{L^p(\mathbb{R}^n)}) |v_l(t, x)|^{p-2} v_l(t, x)) \eta_r(x) (v_m(t, x) - v_l(t, x)) dx dt = 0.
\end{aligned}$$

Then, considering (2.3), we obtain

$$\begin{aligned}
& \int_0^T \int_{\mathbb{R}^n} t (\Delta (v_m(t, x) - v_l(t, x)))^2 \eta_r(x) dx dt \\
& \quad + \lambda \int_0^T \int_{\mathbb{R}^n} t (v_m(t, x) - v_l(t, x))^2 \eta_r(x) dx dt \\
& \leq -T \int_{\mathbb{R}^n} (v_{mt}(T, x) - v_{lt}(T, x)) \eta_r(x) (v_m(T, x) - v_l(T, x)) dx \\
& \quad + \int_0^T \int_{\mathbb{R}^n} t \eta_r(x) (v_{mt}(t, x) - v_{lt}(t, x))^2 dx dt \\
& \quad + \int_0^T \int_{\mathbb{R}^n} \eta_r(x) (v_{mt}(t, x) - v_{lt}(t, x)) (v_m(t, x) - v_l(t, x)) dx dt \\
& - 2 \sum_{i=1}^n \int_0^T \int_{\mathbb{R}^n} \Delta (v_m(t, x) - v_l(t, x)) t (\eta_r(x))_{x_i} (v_m(t, x) - v_l(t, x))_{x_i} dx dt \\
& \quad - \int_0^T \int_{\mathbb{R}^n} \Delta (v_m(t, x) - v_l(t, x)) t \Delta (\eta_r(x)) (v_m(t, x) - v_l(t, x)) dx dt \\
& \quad + \frac{1}{2} \int_0^T \int_{\mathbb{R}^n} \alpha(x) \eta_r(x) (v_m(t, x) - v_l(t, x))^2 dx dt \\
& \quad - \int_0^T \int_{\mathbb{R}^n} t (f(\|v_m(t)\|_{L^p(\mathbb{R}^n)}) |v_m(t, x)|^{p-2} v_m(t, x)
\end{aligned}$$

$$-f(\|v_l(t)\|_{L^p(\mathbb{R}^n)}) |v_l(t, x)|^{p-2} v_l(t, x) \\ \times \eta_r(x) (v_m(t, x) - v_l(t, x)) dx dt, \quad \forall T \geq 0 \text{ and } \forall r \geq r_0.$$

Taking into account (2.5) and (3.19) in the above inequality, we find

$$\begin{aligned} & \int_0^T \int_{\mathbb{R}^n} t (\Delta(v_m(t, x) - v_l(t, x)))^2 \eta_r(x) dx dt \\ & + \lambda \int_0^T \int_{\mathbb{R}^n} t (v_m(t, x) - v_l(t, x))^2 \eta_r(x) dx dt \\ & \leq T \left( \|v_{mt}(T, x) - v_{lt}(T, x)\|_{L^2(\mathbb{R}^n)}^2 + \|v_m(T, x) - v_l(T, x)\|_{L^2(\mathbb{R}^n)}^2 \right) \\ & + \int_0^T \int_{\mathbb{R}^n} t \eta_r(x) (v_{mt}(t, x) - v_{lt}(t, x))^2 dx dt + \int_0^T \int_{\mathbb{R}^n} \eta_r(x) (v_{mt}(t, x) - v_{lt}(t, x))^2 dx dt \\ & + \int_0^T \int_{\mathbb{R}^n} \eta_r(x) (v_m(t, x) - v_l(t, x))^2 dx dt \\ & + \frac{1}{2} \int_0^T \int_{\mathbb{R}^n} \alpha(x) \eta_r(x) (v_m(t, x) - v_l(t, x))^2 dx dt \\ & + c_6 \frac{T}{r} + \widetilde{K}_r^{m,l}(T), \quad \forall T \geq 0 \text{ and } \forall r \geq r_0, \end{aligned} \quad (3.27)$$

where

$$\begin{aligned} \widetilde{K}_r^{m,l}(T) := & \int_0^T t \left( f(\|v_l(t)\|_{L^p(\mathbb{R}^n)}) - f(\|v_m(t)\|_{L^p(\mathbb{R}^n)}) \right) \int_{\mathbb{R}^n} |v_l(t, x)|^{p-2} v_l(t, x) \eta_r(x) \\ & \times (v_m(t, x) - v_l(t, x)) dx dt, \end{aligned}$$

and considering (3.19) -(3.20)<sub>3</sub>, it is easy to see that

$$\sup_{m,l} \left\| \widetilde{K}_r^{m,l} \right\|_{C[0,T]} < \infty \quad \text{and} \quad \lim_{m \rightarrow \infty} \limsup_{l \rightarrow \infty} \left| \widetilde{K}_r^{m,l}(T) \right| = 0, \quad \forall T \geq 0.$$

Now, multiplying (3.27) by  $\delta > 0$  and adding to (3.26), we have

$$\begin{aligned} & T \|\Delta(v_m(T) - v_l(T))\|_{L^2(\mathbb{R}^n)}^2 + T \|v_{mt}(T) - v_{lt}(T)\|_{L^2(\mathbb{R}^n)}^2 \\ & + T\lambda \|v_m(T) - v_l(T)\|_{L^2(\mathbb{R}^n)}^2 + 2\alpha_0 \int_0^T \int_{\mathbb{R}^n \setminus B(0,r)} t (v_{mt}(t) - v_{lt}(t))^2 dx dt \\ & + \delta \int_0^T \int_{\mathbb{R}^n} t (\Delta(v_m(t, x) - v_l(t, x)))^2 \eta_r(x) dx dt \\ & + \delta\lambda \int_0^T \int_{\mathbb{R}^n} t (v_m(t, x) - v_l(t, x))^2 \eta_r(x) dx dt \leq \int_0^T \|v_{mt}(T) - v_{lt}(T)\|_{L^2(\mathbb{R}^n)}^2 dt \end{aligned}$$

$$\begin{aligned}
& + \int_0^T \|\Delta(v_m(t) - v_l(t))\|_{L^2(\mathbb{R}^n)}^2 dt + \lambda \int_0^T \|v_m(t) - v_l(t)\|_{L^2(\mathbb{R}^n)}^2 dt \\
& + 2 \int_0^T \int_{\mathbb{R}^n} t (f(\|v_l(t)\|_{L^p(\mathbb{R}^n)}) |v_l(t, x)|^{p-2} v_l(t, x) \\
& - f(\|v_m(t)\|_{L^p(\mathbb{R}^n)}) |v_m(t, x)|^{p-2} v_m(t, x)) (v_{mt}(t, x) - v_{lt}(t, x)) dx dt \\
& + \delta T \left( \|v_{mt}(T, x) - v_{lt}(T, x)\|_{L^2(\mathbb{R}^n)}^2 + \|v_m(T, x) - v_l(T, x)\|_{L^2(\mathbb{R}^n)}^2 \right) \\
& + \delta \int_0^T \int_{\mathbb{R}^n} t \eta_r(x) (v_{mt}(t, x) - v_{lt}(t, x))^2 dx dt \\
& + \delta \int_0^T \int_{\mathbb{R}^n} \eta_r(x) (v_{mt}(t, x) - v_{lt}(t, x))^2 dx dt \\
& + \delta \int_0^T \int_{\mathbb{R}^n} \eta_r(x) (v_m(t, x) - v_l(t, x))^2 dx dt \\
& + \frac{\delta}{2} \int_0^T \int_{\mathbb{R}^n} \alpha(x) \eta_r(x) (v_m(t, x) - v_l(t, x))^2 dx dt \\
& + c_6 \delta \frac{T}{r} + \delta \widetilde{K}_r^{m,l}(T), \quad \forall T \geq 0 \text{ and } \forall r \geq r_0. \tag{3.28}
\end{aligned}$$

Considering Lemma 3.2 in (3.28), for every  $\gamma > 0$ , we get

$$\begin{aligned}
& T \|\Delta(v_m(T) - v_l(T))\|_{L^2(\mathbb{R}^n)}^2 + T \|v_{mt}(T) - v_{lt}(T)\|_{L^2(\mathbb{R}^n)}^2 \\
& + T \lambda \|v_m(T) - v_l(T)\|_{L^2(\mathbb{R}^n)}^2 + 2\alpha_0 \int_0^T \int_{\mathbb{R}^n \setminus B(0,r)} t (v_{mt}(t) - v_{lt}(t))^2 dx dt \\
& + \delta \int_0^T \int_{\mathbb{R}^n} t (\Delta(v_m(t, x) - v_l(t, x)))^2 \eta_r(x) dx dt \\
& + \delta \lambda \int_0^T \int_{\mathbb{R}^n} t (v_m(t, x) - v_l(t, x))^2 \eta_r(x) dx dt \leq \int_0^T \|v_{mt}(t) - v_{lt}(t)\|_{L^2(\mathbb{R}^n)}^2 dt \\
& + \int_0^T \|\Delta(v_m(t) - v_l(t))\|_{L^2(\mathbb{R}^n)}^2 dt + \lambda \int_0^T \|v_m(t) - v_l(t)\|_{L^2(\mathbb{R}^n)}^2 dt \\
& + \gamma \int_0^T \tau E_{\mathbb{R}^n \setminus (B(0,2r))} (v_m(t) - v_l(t)) dt + c_\gamma \int_0^T E_{\mathbb{R}^n \setminus (B(0,2r))} (v_m(t) - v_l(t)) dt \\
& + c_\gamma \int_0^T t \left( \|\sqrt{a} v_{mt}(t)\|_{L^2(\mathbb{R}^n)}^2 + \|\sqrt{a} v_{lt}(t)\|_{L^2(\mathbb{R}^n)}^2 \right) E_{\mathbb{R}^n \setminus B(0,2r)} (v_m(t) - v_l(t)) dt
\end{aligned}$$

$$\begin{aligned}
& + |K_r^{m,l}(T)| + \delta T \left( \|v_{mt}(T, x) - v_{lt}(T, x)\|_{L^2(\mathbb{R}^n)}^2 + \|v_m(T, x) - v_l(T, x)\|_{L^2(\mathbb{R}^n)}^2 \right) \\
& \quad + \delta \int_0^T \int_{\mathbb{R}^n} t \eta_r(x) (v_{mt}(t, x) - v_{lt}(t, x))^2 dx dt \\
& + \delta \int_0^T \int_{\mathbb{R}^n} \eta_r(x) (v_{mt}(t, x) - v_{lt}(t, x))^2 dx dt + \delta \int_0^T \int_{\mathbb{R}^n} \eta_r(x) (v_m(t, x) - v_l(t, x))^2 dx dt \\
& \quad + \frac{\delta}{2} \int_0^T \int_{\mathbb{R}^n} \alpha(x) \eta_r(x) (v_m(t, x) - v_l(t, x))^2 dx dt \\
& \quad + c_6 \delta \frac{T}{r} + \delta \widetilde{K}_r^{m,l}(T), \quad \forall T \geq 0 \text{ and } \forall r \geq r_0.
\end{aligned}$$

Then, for sufficiently small  $\gamma$  and  $\delta$ , we obtain

$$\begin{aligned}
TE_{\mathbb{R}^n}(v_m(T) - v_l(T)) & \leq c_7 \int_0^T E_{\mathbb{R}^n}(v_m(t) - v_l(t)) dt \\
& + c_\gamma \int_0^T t \left( \|\sqrt{a}v_{mt}(t)\|_{L^2(\mathbb{R}^n)}^2 + \|\sqrt{a}v_{lt}(t)\|_{L^2(\mathbb{R}^n)}^2 \right) E_{\mathbb{R}^n}(v_m(t) - v_l(t)) dt \\
& + c_7 \left( \frac{T}{r} + |K_r^{m,l}(T)| + |\widetilde{K}_r^{m,l}(T)| \right), \quad \forall T \geq 0 \text{ and } \forall r \geq r_0.
\end{aligned}$$

Now, denoting  $y_{m,l}(t) := tE_{\mathbb{R}^n}(v_m(t) - v_l(t))$ , from the previous inequality, we have

$$\begin{aligned}
y_{m,l}(T) & \leq c_\gamma \int_0^T \left( \|\sqrt{a}v_{mt}(t)\|_{L^2(\mathbb{R}^n)}^2 + \|\sqrt{a}v_{lt}(t)\|_{L^2(\mathbb{R}^n)}^2 \right) y_{m,l}(t) dt \\
& \quad + c_7 \int_0^T E_{\mathbb{R}^n}(v_m(t) - v_l(t)) dt + \\
& \quad + c_7 \left( \frac{T}{r} + |K_r^{m,l}(T)| + |\widetilde{K}_r^{m,l}(T)| \right), \quad \forall T \geq 0 \text{ and } \forall r \geq r_0.
\end{aligned}$$

Applying Gronwall inequality and considering (2.6) and (3.19) in the above inequality, we get

$$\begin{aligned}
& TE_{\mathbb{R}^n}(v_m(T) - v_l(T)) \\
& \leq c_7 \int_0^T E_{\mathbb{R}^n}(v_m(t) - v_l(t)) dt + c_7 \left( \frac{T}{r} + |K_r^{m,l}(T)| + |\widetilde{K}_r^{m,l}(T)| \right) \\
& \quad + c_7 \int_0^T \left( \int_0^t E_{\mathbb{R}^n}(v_m(s) - v_l(s)) ds + \frac{t}{r} + |K_r^{m,l}(t)| + |\widetilde{K}_r^{m,l}(t)| \right) \\
& \quad \times \left( \|\sqrt{a}v_{mt}(t)\|_{L^2(\mathbb{R}^n)}^2 + \|\sqrt{a}v_{lt}(t)\|_{L^2(\mathbb{R}^n)}^2 \right) e^{c_\gamma \int_t^T \left( \|\sqrt{a}v_{m\tau}(\tau)\|_{L^2(\mathbb{R}^n)}^2 + \|\sqrt{a}v_{l\tau}(\tau)\|_{L^2(\mathbb{R}^n)}^2 \right) d\tau} ds
\end{aligned}$$

$$\begin{aligned}
&\leq c_7 \int_0^T E_{\mathbb{R}^n} (v_m(t) - v_l(t)) dt + c_7 \left( \frac{T}{r} + |K_r^{m,l}(T)| + |\widetilde{K}_r^{m,l}(T)| \right) \\
&+ c_8 \int_0^T E_{\mathbb{R}^n} (v_m(t) - v_l(t)) dt \int_0^T \left( \|\sqrt{a}v_{mt}(t)\|_{L^2(\mathbb{R}^n)}^2 + \|\sqrt{a}v_{lt}(t)\|_{L^2(\mathbb{R}^n)}^2 \right) \\
&\quad + c_8 \frac{T}{r} \int_0^T \left( \|\sqrt{a}v_{mt}(t)\|_{L^2(\mathbb{R}^n)}^2 + \|\sqrt{a}v_{lt}(t)\|_{L^2(\mathbb{R}^n)}^2 \right) dt \\
&+ c_8 \int_0^T \left( |K_r^{m,l}(t)| + |\widetilde{K}_r^{m,l}(t)| \right) \left( \|\sqrt{a}v_{mt}(t)\|_{L^2(\mathbb{R}^n)}^2 + \|\sqrt{a}v_{lt}(t)\|_{L^2(\mathbb{R}^n)}^2 \right) dt \\
&\leq c_7 \left( |K_r^{m,l}(T)| + |\widetilde{K}_r^{m,l}(T)| \right) + c_9 \int_0^T E_{\mathbb{R}^n} (v_m(t) - v_l(t)) dt + c_9 \frac{T}{r} \\
&\quad + c_9 \int_0^T \left( |K_r^{m,l}(t)| + |\widetilde{K}_r^{m,l}(t)| \right) dt, \quad \forall T \geq 0 \text{ and } \forall r \geq r_0.
\end{aligned}$$

By using Lebesgue dominated convergence theorem and considering (3.25) in the last inequality, we obtain

$$\limsup_{m \rightarrow \infty} \limsup_{l \rightarrow \infty} T E_{\mathbb{R}^n} (v_m(T) - v_l(T)) \leq c_{10} \left( 1 + \frac{T}{r} \right), \quad \forall T \geq 0 \text{ and } \forall r \geq r_0,$$

where the constant  $c_{10}$  as all previous constants  $c_i$  ( $i = \overline{1,9}$ ) is independent of  $T$  and  $r$ . By passing to limit as  $r \rightarrow \infty$  in the above inequality, we find

$$\limsup_{m \rightarrow \infty} \limsup_{l \rightarrow \infty} T E_{\mathbb{R}^n} (v_m(T) - v_l(T)) \leq c_{10}, \quad \forall T \geq 0,$$

which gives

$$\begin{aligned}
&\limsup_{m \rightarrow \infty} \limsup_{l \rightarrow \infty} \|S(T + t_{k_m} - T_0)\varphi_{k_m} - S(T + t_{k_l} - T_0)\varphi_{k_l}\|_{H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)} \\
&\leq \frac{c_{11}}{\sqrt{T}}, \quad \forall T > 0.
\end{aligned}$$

Choosing  $T = T_0$  in the previous inequality, we have

$$\limsup_{m \rightarrow \infty} \limsup_{l \rightarrow \infty} \|S(t_{k_m})\varphi_{k_m} - S(t_{k_l})\varphi_{k_l}\|_{H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)} \leq \frac{c_{11}}{\sqrt{T_0}}, \quad \forall T_0 > 0.$$

As a consequence, from the above sequential limit inequality, we get (3.18) which completes the proof.  $\square$

Now we are in a position to complete the proof of the Theorem 2.2. Since, by (2.3) and (2.6), problem (2.1)-(2.2) admits a strict Lyapunov function

$$\Phi(u(t)) = E_{\mathbb{R}^n}(u(t)) + \frac{1}{p} F \left( \|u(t)\|_{L^p(\mathbb{R}^n)}^p \right) - \int_{\mathbb{R}^n} h(x) u(t, x) dx,$$

applying [6, Corollary 7.5.7], we obtain the claim of Theorem 2.2.

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