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Ditopological texture spaces and intuitionistic sets

Lawrence M. Brown[a,*], Murat Diker[b]

[a] *Department of Mathematics, Faculty of Science, Hacettepe University, 06532 Beytepe, Ankara, Turkey*
[b] *Department of Science Education, Faculty of Education, Hacettepe University, 06532 Beytepe, Ankara, Turkey*

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Abstract

In this paper it is shown that the lattice of intuitionistic subsets of a set X in the sense of D. Çoker may be represented as a special type of texture space, called an intuitionistic texture on X, and various characterizations are given. It is established that intuitionistic topologies are mapped to ditopologies on the corresponding texture, and some notions of compactness and stability are considered. © 1998 Elsevier Science B.V. All rights reserved.

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1. Introduction

The notion of a texture space, under the name of fuzzy structure, was introduced by the first author at the 2nd BUFSA Conference on Fuzzy Systems and Artificial Intelligence held at Trabzon University in 1992. One motivation for the introduction of these spaces is to provide a point-set based setting for the study of fuzzy sets, and the reader may consult [6] for various characterizations, in terms of textures, of fuzzy lattices, L-fuzzy sets and generalized fuzzy sets in the sense of Nakajima [13].

The concept of a dichotomous topology, or ditopology for short, on a texture space is a natural generalization of a bitopology on the one hand, and of a fuzzy topology on the other. In particular, the study of ditopologies makes it possible to extend much of the theory of bitopological spaces to a fuzzy topological setting. Some results in this direction may be found in [3,5,8].

Our aim in the present paper is to show that the lattice of intuitionistic sets on a set X [7] may be regarded as a texture space, to characterize those texture spaces which arise in this way and to give some topological results. Although some basic notions relating to ditopological texture spaces were presented in [2,3], we will recall all necessary definitions and results for the benefit of the reader.

Definition 1.1. Let S be a set. Then $\mathscr{L} \subseteq \mathscr{P}(S)$ is called a *texturing* of S, and S is said to be *textured by* $\mathscr{L}$ if

(1) $(\mathscr{L}, \subseteq)$ is a complete lattice containing S and $\emptyset$ and for any index set I and $A_i \in \mathscr{L}$, $i \in I$, the meet $\bigwedge_{i\in I} A_i$ and the join $\bigvee_{i\in I} A_i$ in $\mathscr{L}$ are related with the intersection and union in $\mathscr{P}(S)$ by

* Corresponding author. E-mail: brown@eti.cc.hun.edu.tr.

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the equalities

$$\bigwedge_{i\in I} A_i = \bigcap_{i\in I} A_i$$

for all I, while

$$\bigvee_{i\in I} A_i = \bigcup_{i\in I} A_i$$

for all finite I.

(2) $\mathscr{L}$ is completely distributive.

(3) $\mathscr{L}$ separates the points of S. That is, given $s_1 \neq s_2$ in S we have $L \in \mathscr{L}$ with $s_1 \in L$, $s_2 \notin L$, or $L \in \mathscr{L}$ with $s_2 \in L$, $s_1 \notin L$.

If S is textured by $\mathscr{L}$ then $(S, \mathscr{L})$ is called a *texture space*, or simply a *texture*.

Remark. As we shall see, the texturings corresponding to lattices of intuitionistic sets will turn out to be closed under arbitrary unions as well as intersections, so the second equality in Definition 1.1(1) will also hold for arbitrary I. In the general case an infinite union may fail to belong to the texturing, in which case the join will be strictly larger than the union. A simple example would be the texturing $\mathscr{L} = \{(0,r] \mid r \in [0,1]\}$ of $S = (0,1]$.

The mapping $s \to P_s = \bigcap\{L \in \mathscr{L} \mid s \in L\}$ is a natural embedding of S in $\mathscr{L}$. Recall that an element $M \neq \emptyset$ of $\mathscr{L}$ is a *molecule* if $M \subseteq L_1 \cup L_2 \Rightarrow M \subseteq L_1$ or $M \subseteq L_2$ for all $L_1, L_2 \in \mathscr{L}$. Clearly, $\{P_s \mid s \in S\}$ is a set of molecules in $\mathscr{L}$ which is a *base for* $\mathscr{L}$ in the sense that

$$L = \bigvee_{s\in L} P_s = \bigcup_{s\in L} P_s$$

for all $L \in \mathscr{L}$ [10]. We call $\mathscr{L}$ *simple* if every molecule of $\mathscr{L}$ belong to the set $\{P_s \mid s \in S\}$.

A mapping $\gamma : \mathscr{L} \to \mathscr{L}$ is called a *complementation* if $\gamma^2(P) = P$ for all $P \in \mathscr{L}$ and $P \subseteq Q$ in $\mathscr{L}$ implies $\gamma(Q) \subseteq \gamma(P)$. A complementation is necessarily bijective. A texture with a complementation is said to be *complemented.*

For $\mathscr{L} = \mathscr{P}(S)$ and $\gamma(P) = S \backslash P$ we obtain a complemented simple texture which represents the crisp set structure of S. It is shown in [6] that fuzzy lattices correspond precisely to complemented simple textures, and characterizations of L-fuzzy sets in the sense of Goguen [11] and generalized fuzzy sets in the sense of Nakajima [13] are given within the class of complemented simple textures.

Intuitionistic sets were introduced by Çoker [7] as a special case of the notion of fuzzy intuitionistic set defined by Atanassov [1]. If X is a crisp set, an *intuitionistic subset of* X may be regarded as a pair (A,B) of subsets of X satisfying $A \cap B = \emptyset$. A partial order, denoted by $\subseteq$, on the set

$$\mathbb{I}(X) = \{(A,B) \mid A, B \in \mathscr{P}(X),\ A \cap B = \emptyset\}$$

of intuitionistic subsets of X is defined by

$$(A,B) \subseteq (C,D) \ \Leftrightarrow\ A \subseteq C \text{ and } D \subseteq B.$$

It is clear that $(\mathbb{I}(X), \subseteq)$ is a complete lattice, the meet (join) of (A_i,B_i), $i \in I$, being denoted by $\bigcap_{i\in I}(A_i,B_i)$ $(\bigcup_{i\in I}(A_i,B_i))$ and equal to $(\bigcap_{i\in I} A_i,\ \bigcup_{i\in I} B_i)$ $((\bigcup_{i\in I} A_i,\ \bigcap_{i\in I} B_i))$, respectively. The largest intuitionistic subset of X is $\underline{X} = (X, \emptyset)$, and the smallest $\underline{\emptyset} = (\emptyset, X)$. There is also a *natural complement* given by the operation $\prime : \mathbb{I}(X) \to \mathbb{I}(X)$, $(A,B) \mapsto (A,B)'$, defined by $(A,B)' = (B,A)$.

2. Representation theorems

Our aim in this section is to show that the lattice $\mathbb{I}(X)$ of intuitionistic subsets of a set X may be represented as a texture, and to obtain the characteristic properties of such a texture.

Theorem 2.1. *Let X be a set and $\mathbb{I}(X)$ the lattice of intuitionistic subsets of X. Define $I_X = (X \times \{0\}) \cup (X \times \{1\})$ and let $\varphi : \mathbb{I}(X) \to \mathscr{P}(I_X)$ be defined by*

$$\varphi((A,B)) = (A \times \{0\}) \cup ((X \backslash B) \times \{1\})$$

for all $(A,B) \in \mathbb{I}(X)$. Then $\mathscr{I}_X = \varphi(\mathbb{I}(X)) \subseteq \mathscr{P}(S)$ is a simple complemented texturing of I_X.

Proof. Take $\{L_i \mid i \in I\} \subseteq \mathscr{I}_X$. Then we have $\{(A_i, B_i) \mid i \in I\} \subseteq \mathbb{I}(X)$ with $L_i = \varphi((A_i, B_i)) = (A_i \times \{0\}) \cup$

$((X\backslash B_i) \times \{1\})$. Hence,

$$\bigcup_{i\in I} L_i = \bigcup_{i\in I}(A_i \times \{0\}) \cup ((X\backslash B_i) \times \{1\})$$
$$= \left(\left(\bigcup_{i\in I} A_i\right) \times \{0\}\right) \cup \left(\left(X \backslash \bigcap_{i\in I} B_i\right) \times \{1\}\right)$$
$$= \varphi\left(\left(\bigcup_{i\in I} A_i, \bigcap_{i\in I} B_i\right)\right) = \varphi\left(\bigcup_{i\in I}(A_i, B_i)\right).$$

Therefore, $\bigcup_{i\in I} L_i \in \mathscr{I}_X$. In exactly the same way $\bigcap_{i\in I} L_i \in \mathscr{I}_X$, which establishes that $(I_X, \mathscr{I}_X)$ is a complete lattice with

$$\bigvee_{i\in I} L_i = \bigcup_{i\in I} L_i \quad \text{and} \quad \bigwedge_{i\in I} L_i = \bigcap_{i\in I} L_i$$

for $L_i \in \mathscr{I}_X$, $i \in I$, and all index sets I. In view of these equalities, $(I_X, \mathscr{I}_X)$ is clearly completely distributive, and we also have

$$I_X = \varphi((X, \emptyset)) = \varphi(\underline{X}) \in \mathscr{I}_X$$

and

$$\emptyset = \varphi((\emptyset, X)) = \varphi(\underline{\emptyset}) \in \mathscr{I}_X.$$

Next $\mathscr{I}_X$ separates the points of I_X. Indeed, let $s_1, s_2 \in I_X$ and $s_1 \neq s_2$. There are three cases to consider:

Case 1: $s_1 = (x, 1)$, $x \in X$. For $L = \varphi((\emptyset, X\backslash\{x\}) = \{(x, 1)\} \in \mathscr{I}_X$ we have $s_1 \in L$ and $s_2 \notin L$.

Case 2: $s_2 = (x, 1)$, $x \in X$. Proof as for Case 1 with s_1 and s_2 interchanged.

Case 3: $s_1 = (x, 0)$, $s_2 = (y, 0)$, $x, y \in X$, $x \neq y$. Since $\{x\} \cap (X\backslash\{x\}) = \emptyset$, $L = \varphi((\{x\}, X\backslash\{x\})) = \{(x, 0), (x, 1)\} \in \mathscr{I}_X$ and $s_1 \in L$ and $s_2 \notin L$.

To show that $(I_X, \mathscr{I}_X)$ is simple, note that from the above we clearly have $P_{(x,1)} = \{(x, 1)\}$ and $P_{(x,0)} = \{(x, 0), (x, 1)\}$. Hence if $L = (A \times \{0\}) \cap ((X\backslash B) \times \{1\}) \in \mathscr{I}_X$ it must contain $(x, 1)$ for some $x \in X$. Take $y \in X$ with $y \neq x$ and define

$$L_1 = ((A\backslash\{x\}) \times \{0\}) \cup ((X\backslash(B \cup \{x\})) \times \{1\}),$$

$$L_2 = ((A\backslash\{y\}) \times \{0\}) \cup ((X\backslash(B \cup \{y\})) \times \{1\}).$$

Then $L \subseteq L_1 \cup L_2$ and $L \not\subseteq L_1$. If $y \in A$ or $y \notin B$ we also have $L \not\subseteq L_2$. Hence, if L is a molecule it cannot contain $(y, 0)$ or $(y, 1)$ for $x \neq y$, and so must have the form $L = \{(x, 1)\} = P_{(x,1)}$ or $L = \{(x, 0), (x, 1)\} = P_{(x,0)}$ as required.

Finally, to show that $(I_X, \mathscr{I}_X)$ is complemented, consider the function $\iota_X : \mathscr{I}_X \to \mathscr{I}_X$ defined by

$$\iota_X(L) = \varphi((B, A)),$$

where $L = \varphi((A, B))$, $(A, B) \in \mathbb{I}(X)$.

Clearly, ι_X satisfies $\iota_X^2(L) = L$ for each $L \in \mathscr{I}_X$. Take $L, L' \in \mathscr{I}_X$ with $L \subseteq L'$. We may write $L = (A \times \{0\}) \cup ((X\backslash B) \times \{1\})$, $L' = (A' \times \{0\}) \cup ((X\backslash B') \times \{1\})$. Since $L \subseteq L'$ then $A \subseteq A'$ and $X\backslash B \subseteq X\backslash B'$. Therefore $B' \subseteq B$ and $X\backslash A' \subseteq X\backslash A$, from which we deduce $\iota_X(L') \subseteq \iota_X(L)$. Thus ι_X is a complementation. □

The texture $(I_X, \mathscr{I}_X, \iota_X)$ will be called the *intuitionistic texture on* X.

We turn now to the question of characterizing those textures which are an intuitionistic texture on some X. Since we can only expect to be able to do this up to isomorphism, we must first make explicit the notion of isomorphism between textures.

Definition 2.2. Let $(S, \mathscr{L})$ and $(S', \mathscr{L}')$ be textures. A function $\psi : S \to S'$ is a *textural isomorphism* from $(S, \mathscr{L})$ to $(S', \mathscr{L}')$ if

(i) ψ is bijective.

(ii) $\forall L \in \mathscr{L}$ we have $\psi(L) \in \mathscr{L}'$.

(iii) $L \mapsto \psi(L)$ is a bijection $\psi : \mathscr{L} \to \mathscr{L}'$.

We say $(S, \mathscr{L})$ and $(S', \mathscr{L}')$ are *isomorphic* if there exists an isomorphism between them. We denote this by $(S, \mathscr{L}) \cong (S', \mathscr{L}')$.

If $(S, \mathscr{L}, \gamma)$ and $(S', \mathscr{L}', \gamma')$ are complemented textures and ψ satisfies the additional property

$$\forall L \in \mathscr{L} \text{ we have } \psi(\gamma(L)) = \gamma'(\psi(L))$$

then ψ is called a *complemented textural isomorphism*. When such an isomorphism exists we write $(S, \mathscr{L}, \gamma) \cong (S', \mathscr{L}', \gamma')$.

As expected, a textural isomorphism preserves arbitrary meets and joins:

Proposition 2.3. *Let* ψ *be a textural isomorphism from* $(S, \mathscr{L})$ *to* $(S', \mathscr{L}')$, *and* $\{L_i \mid i \in I\} \subseteq \mathscr{L}$. *Then*

(i) $\psi(\bigcap_{i\in I} L_i) = \bigcap_{i\in I} \psi(L_i)$, *and*

(ii) $\psi(\bigvee_{i\in I} L_i) = \bigvee_{i\in I} \psi(L_i)$.

Proof. (i) Trivial since $\psi : \mathscr{L} \to \mathscr{L}'$ preserves inclusion and $\psi : S \to S'$ is injective.

(ii) The inclusion $\bigvee_{i\in I} \psi(L_i) \subseteq \psi(\bigvee_{i\in I} L_i)$ is immediate since $\psi : \mathscr{L} \to \mathscr{L}'$ preserves inclusion. By Definition 2.2(iii) we have $L \in \mathscr{L}$ with $\psi(L) = \bigvee_{i\in I} \psi(L_i)$. Then $L_i \subseteq L$ for all $i \in I$. Indeed, if $L_i \not\subseteq L$ then $L_i \cap L \neq L_i$ and so $\psi(L_i \cap L) \neq \psi(L)$, since $\psi : \mathscr{L} \to \mathscr{L}'$ is injective. By (i), $\psi(L_i \cap L) = \psi(L_i) \cap \psi(L)$, which contradicts $\psi(L_i) \subseteq \psi(L)$. We now have $\bigvee_{i\in I} L_i \subseteq L$, whence $\psi(\bigvee_{i\in I} L_i) \subseteq \psi(L) = \bigvee_{i\in I} \psi(L_i)$. □

Definition 2.4. $(S, \mathscr{L}, \gamma)$ is said to be an *intuitionistic texture* if there exists a set X with $(S, \mathscr{L}, \gamma) \cong (I_X, \mathscr{I}_X, \iota_X)$.

The following theorem gives various characterizations of intuitionistic textures.

Theorem 2.5. *The following are equivalent for a complemented texture* $(S, \mathscr{L}, \gamma)$:
(i) $(S, \mathscr{L}, \gamma)$ *is an intuitionistic texture.*
(ii) *There exists* $T \in \mathscr{L}$ *satisfying*
  (a) $\gamma(T) = T$,
  (b) $P_t = \{t\}$ $\forall t \in T$, *and*
  (c) $\gamma(P_t) = S \backslash P_t$ $\forall t \in S \backslash T$.
(iii) *For every maximal molecule* M *in* $\mathscr{L}$ *there exists a unique molecule* N_M *with* $N_M \subset M$ *and* γ *satisfies*
  (a) $\gamma(N_M) = N_M \cup (S \backslash M)$, *and*
  (b) $\gamma(M) = S \backslash M$ *for all maximal molecules* M.
(iv) *There exists a set* $T \subseteq S$ *and a bijection* $f : T \to S \backslash T$ *such that*
  (a) $\mathscr{L} = \{A \cup B \mid A \subseteq T,\ B \subseteq f(A)\}$, *and*
  (b) $\gamma(A \cup B) = S \backslash (f(A) \cup f^{-1}(B))$, $A \subseteq T$ *and* $B \subseteq f(A)$.

Proof. (i) ⇒ (ii). Let ψ be a complemented textural isomorphism from $(S, \mathscr{L}, \gamma)$ to $(I_X, \mathscr{I}_X, \iota_X)$. Since $X \times \{1\} = \varphi(\emptyset, \emptyset)$ is an element of $\mathscr{I}(X)$ invariant under ι_X, it is immediate that $T = \psi^{-1}(X \times \{1\}) \in \mathscr{L}$ satisfies $\gamma(T) = T$, so establishing (a). For (b), take $t \in T$. Then $\psi(t) = (x, 1)$ for some $x \in X$ so $\{t\} = \psi^{-1}(\{(x,1)\}) = \psi^{-1}(\varphi(\emptyset, X \backslash \{x\})) \in \mathscr{L}$, which verifies $P_t = \{t\}$. The proof of (c) is left to the reader.

(ii) ⇒ (iii). Suppose that $(S, \mathscr{L}, \gamma)$ has the properties given in (ii). We begin by showing that for $s \in S \backslash T$ we have $P_s = \{s, t\}$ for some $t \in T$.

1. $P_s \cap (S \backslash T) = \{s\}$. For suppose $s_1 \in P_s \cap (S \backslash T)$ with $s_1 \neq s$. Then, since $\mathscr{L}$ separates points in S, $s \in P_s \cap (S \backslash P_{s_1}) \subset P_s$, while $P_s \cap (S \backslash P_{s_1}) = P_s \cap \gamma(P_{s_1}) \in \mathscr{L}$, which contradicts the definition of P_s.
2. $P_s \cap T \neq \emptyset$. Indeed $T \subseteq S \backslash P_s = \gamma(P_s) \in \mathscr{L} \Rightarrow P_s = \gamma^2(P_s) \subseteq \gamma(T) = T$, which is impossible as $s \in P_s$ and $s \notin T$.
3. For $t \in P_s \cap T$ we have $\gamma(P_t) = S \backslash \{s\}$. Indeed, $P_t = \{t\} \subseteq P_s \cap T \Rightarrow S \backslash \{s\} = (S \backslash P_s) \cup T = \gamma(P_s) \cup \gamma(T) = \gamma(P_s \cap T) \subseteq \gamma(P_t) \subset S$, from which the result follows at once.
4. $P_s \cap T = \{t\}$. Immediate from (3) and the injectivity of γ.

From (1) and (4) we have $P_s = \{s, t\}$. On the other hand, given $t \in T$ we have $s \in S \backslash T$ with $P_s = \{s, t\}$. To see this, note that $T \subset \gamma(P_t) \subset S$ so we may choose $s \in S \backslash \gamma(P_t)$. By the above, $P_s = \{s, t_1\}$ for some $t_1 \in T$. Using (3) we may verify $t_1 = t$ and hence that $P_s = \{s, t\}$.

Now let $N \subseteq T$ be a molecule in $\mathscr{L}$ and take $t \in N$. By the above, we may choose $s \in S \backslash T$ with $P_s = \{s, t\}$. Suppose we have $t_1 \in N$ with $t_1 \neq t$. Then $N \not\subseteq N \cap \gamma(P_s) \in \mathscr{L}$, $N \not\subseteq P_s \in \mathscr{L}$ but $N \subseteq (N \cap \gamma(P_s)) \cup P_s$, which is impossible. Hence $N = \{t\} = P_t$. In the same way if M is a molecule with $M \not\subseteq T$ then $M = P_s$ for some $s \in S \backslash T$. Hence, the maximal molecules have the form $M = P_s = \{s, t\}$, and there is a unique molecule $N_M = P_t = \{t\}$ satisfying $N_M \subset M$. Finally $\gamma(M) = \gamma(P_s) = S \backslash P_s = S \backslash M$ and $\gamma(N_M) = \gamma(P_t) = S \backslash \{s\} = \{t\} \cup (S \backslash \{s, t\}) = N_M \cup (S \backslash M)$ by (3).

(iii) ⇒ (iv). Let N be a non-maximal molecule. Since the set of molecules containing N is clearly inductive, by Zorn's Lemma there is a maximal molecule M with $N \subset M$. If $t \in N$ then $P_t \subseteq N \subset M \Rightarrow N = M_N = P_t$, while $P_t = \{t\}$ since $\mathscr{L}$ separates the points of S.

In much the same way we may verify that for a maximal molecule M we have $s, t \in S$ with $M = P_s = \{s, t\}$.

Define $T = \{t \mid P_t = \{t\}\}$. For $t \in T$ the molecule P_t is contained in a maximal molecule $M = P_s = \{s, t\}$. Moreover s is unique, for if $M_1 = \{s_1, t\}$ then $N_M = N_{M_1} = \{t\}$, whence $\gamma(N_M) = \gamma(N_{M_1}) \Rightarrow N_M \cup (S \backslash M) = N_{M_1} \cup (S \backslash M_1) \Rightarrow S \backslash \{s\} = S \backslash \{s_1\} \Rightarrow s = s_1$. Hence, setting $f(t) = s \Leftrightarrow t \in T$ and $\{s, t\}$ is a maximal molecule, defines a function $f : T \to S \backslash T$. Clearly, f is bijective.

For $L \in \mathscr{L}$ define $A = L \cap T$ and $B = L \cap (S \setminus T)$. Clearly $L = A \cup B$ and $A \subseteq T$. Also $s \in B$ implies $P_s = \{s, t\} \subseteq L$. Hence $t \in T \cap L$ and $s = f(t)$, so $B \subseteq f(A)$. Conversely, take $A \subseteq T$ and $B \subseteq f(A)$. Clearly, $A \cup B = \bigcup \{P_t \mid t \in A\} \cup \bigcup \{P_s \mid s \in B\} \in \mathscr{L}$, and we have verified that $\mathscr{L} = \{A \cup B \mid A \subseteq T,\ B \subseteq f(A)\}$.

Finally, for $L = A \cup B = \bigcup_{t \in A} P_t \cup \bigcup_{s \in B} P_s \in \mathscr{L}$ we have

$$\begin{aligned}\gamma(L) &= \bigcap_{t \in A} \gamma(P_t) \cap \bigcap_{s \in B} \gamma(P_s)\\ &= \bigcap_{t \in A} [\{t\} \cap (S \setminus \{f(t), t\})] \cap \bigcap_{s \in B} (S \setminus \{s, f^{-1}(s)\})\\ &= S \setminus (f(A) \cup f^{-1}(B)).\end{aligned}$$

(iv) $\Rightarrow$ (i). Let $X = S \setminus T$ and define $\psi : S \to I_X$ by

$$\psi(s) = \begin{cases} (f(t), 1), & t \in T,\\ (s, 0), & s \in X.\end{cases}$$

Clearly, ψ is bijective. Also, if $L = A \cup B \in \mathscr{L}$, $A \subseteq T$ and $B \subseteq f(A)$, then $\psi(L) = (B \times \{0\}) \cup (f(A) \times \{1\}) \in \varphi(\mathbb{I}(X))$, so it is easy to verify that $\psi : \mathscr{L} \to \mathscr{I}_X$ is also bijective. It remains to show that $\iota_X \circ \psi = \psi \circ \gamma$. But for $L = A \cup B$, $A \subseteq T$ and $B \subseteq f(A)$, we have

$$\begin{aligned}\iota_X(\psi(L)) &= \iota_X((B \times \{0\}) \cup (f(A) \times \{1\}))\\ &= ((X \setminus f(A)) \times \{0\}) \cup ((X \setminus B) \times \{1\})\\ &= \psi(f^{-1}(X \setminus B) \cup (X \setminus f(A)))\\ &= \psi(\gamma(A \cup B))\\ &= \psi(\gamma(L)).\end{aligned}$$

This verifies that ψ is a complemented textural isomorphism of $(S, \mathscr{L}, \gamma)$ with $(I_X, \mathscr{I}_X, \iota_X)$, so $(S, \mathscr{L}, \gamma)$ is an intuitionistic texture. □

The reader may refer to [6] for a corollary to Theorem 2.5 which puts intuitionistic sets in a fuzzy setting by showing that they are essentially L-fuzzy sets on X, where L is the fuzzy lattice $\{0, \frac{1}{2}, 1\}$.

We now present an example which shows that intuitionistic textures may arise naturally in everyday life.

Example 2.6. In a scheme for part-time home employment, each married woman in the region is eligible for employment, as is any man whose wife is working for the scheme. Under these circumstances, the set of all possible patterns of employment forms an intuitionistic texture on the set of potential employees. To see this let $E = W \cup M$, where W is the set of married women and M the set of married men in the region. If for $w \in W$ we denote by $h(w)$ the husband of w, the mapping $h : W \to M$ is a bijection. If A is the set of women and B the set of men employed on the scheme, we have $B \subseteq h(A)$ by hypothesis so the set $A \cup B$ of employees belongs to $\mathscr{E} = \{A \cup B \mid A \subseteq W, B \subseteq h(A)\}$, which is an intuitionistic texturing of the set E of potential employees (cf. Theorem 2.5(iv)). The complement of $A \cup B$ in this texturing is $E \setminus (h(A) \cup h^{-1}(B))$, which represents the pattern of employment in which a wife whose husband did not work has employment, as does a husband whose wife did not work.

If we wish to express the pattern of employment as an intuitionistic subset (U, V) of the set X of eligible families in the region, an examination of Theorem 2.1 shows that we must take U to be the set of families in which the husband is working for the scheme, and V the set of families in which the wife is not employed under the scheme.

3. Ditopological intuitionistic textures

In general, a texture is not closed under set complementation, and this has led to the consideration in [2] of the notion of a *dichotomous topology* (*ditopology*) on $(S, \mathscr{L})$. Specifically, (τ, κ) is called a *ditopology* on $(S, \mathscr{L})$ if

(1) $\tau \subseteq \mathscr{L}$ satisfies
  a. $S, \emptyset \in \tau$,
  b. $G_1, G_2 \in \tau \Rightarrow G_1 \cap G_2 \in \tau$,
  c. $G_\alpha \in \tau, \alpha \in A \Rightarrow \bigvee_\alpha G_\alpha \in \tau$ for arbitrary index set A, and

(2) $\kappa \subseteq \mathscr{L}$ satisfies
  a. $S, \emptyset \in \kappa$,
  b. $F_1, F_2 \in \kappa \Rightarrow F_1 \cup F_2 \in \kappa$,
  c. $F_\alpha \in \kappa, \alpha \in A \Rightarrow \bigwedge_\alpha F_\alpha \in \kappa$ for arbitrary index set A.

Hence a ditopology resembles a topology, except that the *open sets* (i.e. elements of τ) may be defined independently of the *closed sets* (i.e. elements of κ). In the case of a complemented texture $(S,\mathscr{L},\gamma)$ it might be that τ and κ are connected by the relation $\kappa=\{\gamma(G)\mid G\in\tau\}$, and in that case we speak of a *complemented ditopology*. By Proposition 2.3 it is clear that a (complemented) textural isomorphism maps a (complemented) ditopology to a (complemented) ditopology.

An *intuitionistic topology on X* is a subset $\mathscr{T}$ of $\mathbb{I}(X)$ containing $\underline{\emptyset}$ and $\underline{X}$, and closed under arbitrary unions and finite intersections (see [8] for the corresponding notion for intuitionistic fuzzy sets). The proof of the following theorem is straightforward, and is omitted.

Theorem 3.1. *Let X be a set and* $(I_X,\mathscr{I}_X,\iota_X)$ *the intuitonistic texture on X. Then*
(1) *If* $\mathscr{T}$ *is an intuitionistic topology on X,* $\tau=\{\varphi(T)\mid T\in\mathscr{T}\}$, $\kappa=\{\varphi(G')\mid G\in\mathscr{T}\}$ *defines a complemented ditopology on* $(I_X,\mathscr{I}_X,\iota_X)$.
(2) *If* (τ,κ) *is a complemented ditopology on* $(\mathscr{I}_X,\mathscr{I}_X,\iota_X)$ *then* $\mathscr{T}=\{\varphi^{-1}(G)\mid G\in\tau\}$ *is an intuitionistic topology on X.*

Let $(S,\mathscr{L},\gamma)$ be an intuitionistic texture and (τ,κ) a complemented ditopology on S. By Definition 2.4 we have a set X and a complemented textural isomorphism $\psi:S\to I_X$ of $(S,\mathscr{L},\gamma)$ with $(I_X,\mathscr{I}_X,\iota_X)$. If $\mathscr{T}$ is the intuitionistic topology on X corresponding to the complemented ditopology $(\psi(\tau),\psi(\kappa))$ on $(I_X,\mathscr{I}_X,\iota_X)$, as in Theorem 3.1, we will say that *$(X,\mathscr{T})$ is an intuitionistic topological space corresponding to $(S,\mathscr{L},\gamma,\tau,\kappa)$*, or that *$(S,\mathscr{L},\gamma,\tau,\kappa)$ is a complemented ditopological intuitionistic texture corresponding to $(X,\mathscr{T})$*.

We turn now to the question of compactness. First, we give some definitions and results applicable to general ditopological textures, before specializing to intuitionistic textures and their relation with intuitionistic topological spaces. On $(S,\mathscr{P}(S))$ a ditopology is formally equivalent to a bitopology, τ being one of the topologies and $\kappa'=\{S\backslash F\mid F\in\kappa\}$ the other. In the general case this formal link fails because a texturing need not be closed under set theoretic complement, but even so many bitopological concepts may be suitably redefined to apply to ditopological textures. With some changes in terminology the following notions reflect Kopperman's subdivision of bitopological joint compactness into compactness and stability properties [12].

Definition 3.2. Let (τ,κ) be a ditopology on the texture $(S,\mathscr{L})$. Then $(S,\mathscr{L},\tau,\kappa)$ is called
(i) *Compact* if whenever $S=\bigvee_{i\in I}G_i$, $G_i\in\tau$, $i\in I$, there is a finite subset J of I with $\bigcup_{j\in J}G_j=S$.
(ii) *Co-compact* if whenever $\bigcap_{i\in I}F_i=\emptyset$, $F_i\in\kappa$, $i\in I$, there is a finite subset J of I with $\bigcap_{j\in J}F_j=\emptyset$.
(iii) *Stable* if every $K\in\kappa$ with $K\neq S$ is compact, i.e. whenever $K\subseteq\bigvee_{i\in I}G_i$, $G_i\in\tau$, $i\in I$, there is a finite subset J of I with $K\subseteq\bigcup_{j\in J}G_j$.
(iv) *Co-stable* if every $G\in\tau$ with $G\neq\emptyset$ is co-compact, i.e. whenever $\bigcap_{i\in I}F_i\subseteq G$, $F_i\in\kappa$, $i\in I$, there is a finite subset J of I with $\bigcap_{j\in J}F_j\subseteq G$.

Proposition 3.3. *Let (τ,κ) be a complemented ditopology on $(S,\mathscr{L},\gamma)$. Then*
(1) *$(S,\mathscr{L},\gamma,\tau,\kappa)$ is compact if and only if it is co-compact.*
(2) *$(S,\mathscr{L},\gamma,\tau,\kappa)$ is stable if and only if it is co-stable.*

Proof. Immediate. □

In order to state the next theorem we require the following concepts.

Definition 3.4. Let (τ,κ) be a ditopology on $(S,\mathscr{L})$.
(1) A set $\mathscr{D}\subseteq\mathscr{L}\times\mathscr{L}$ is called a *difamily* on $(S,\mathscr{L})$. A difamily $\mathscr{D}$ satisfying $\mathscr{D}\subseteq\tau\times\kappa$ is *open and co-closed*, one satisfying $\mathscr{D}\subseteq\kappa\times\tau$ is *closed and co-open*.
(2) A difamily $\mathscr{D}$ has the *finite exclusion property* (fep) [2] if whenever $(F_i,G_i)\in\mathscr{D}$, $i=1,2,\dots,n$ we have $\bigcap_{i=1}^n F_i\not\subseteq\bigcup_{i=1}^n G_i$.
(3) A closed, co-open difamily $\mathscr{D}$ with $\bigcap\{F\mid F\in\operatorname{dom}\mathscr{D}\}\not\subseteq\bigvee\{G\mid G\in\operatorname{ran}\mathscr{D}\}$ is said to be *bound* in $(S,\mathscr{L},\tau,\kappa)$.
(4) A difamily $\mathscr{D}=\{(G_i,F_i)\mid i\in I\}$ is called a *dicover* [4] of $(S,\mathscr{L})$ if for all partitions I_1, I_2 of I (including the trivial partitions) we have

$$\bigcap_{i\in I_1}F_i\subseteq\bigvee_{i\in I_2}G_i.$$

We may now give

Theorem 3.5. *The following are equivalent for a ditopological texture* $(S, \mathscr{L}, \tau, \kappa)$.

(1) $(S, \mathscr{L}, \tau, \kappa)$ *is compact, co-compact, stable and co-stable.*
(2) *Every closed, co-open difamily with the finite exclusion property is bound.*
(3) *Every open, co-closed dicover has a finite sub-dicover.*

Proof. (1) ⇒ (2). Suppose that (1) holds, but that we have a closed, co-open difamily $\mathscr{B} = \{(F_i, G_i) \mid i \in I\}$ with the fep, which is not bound in $(S, \mathscr{L}, \tau, \kappa)$. Let $F = \bigcap_{i \in I} F_i$, so that $F \in \kappa$ and $F \subseteq \bigvee_{i \in I} G_i$. According as $F \neq S$ or $F = S$ we may use stability or compactness, respectively, to show the existence of a finite subset J_1 of I with $F \subseteq \bigcup_{j \in J_1} G_j$. Now let $G = \bigcup_{j \in J_1} G_j$, so that $G \in \tau$ and $\bigcap_{i \in I} F_i \subseteq G$. According as $G \neq \emptyset$ or $G = \emptyset$ we may use co-stability or co-compactness, respectively, to show that $\bigcap_{j \in J_2} F_j \subseteq G$ for some finite subset J_2 of I. Since now $\bigcap_{j \in J_1 \cup J_2} F_j \subseteq \bigcup_{j \in J_1 \cup J_2} G_j$ we have a contradiction to the fact that $\mathscr{B}$ has the fep.

(2) ⇒ (3). This was established in [4], but for the sake of completeness we outline the proof. Suppose that $\mathscr{C} = \{(G_i, F_i) \mid i \in I\}$ is an open, co-closed dicover with no finite sub-dicover. For $J \subseteq I$ write $J = J_1 \sqcup J_2$ if $\{J_1, J_2\}$ is a partition of J and define

$$\mathscr{P}_J = \{(J_1, J_2) \mid J = J_1 \sqcup J_2\}$$

and

$$\mathscr{P}_J^\star = \left\{ (J_1, J_2) \in \mathscr{P}_J \;\middle|\; \bigcap_{j \in J_1} F_j \not\subseteq \bigvee_{j \in J_2} G_j \right\}.$$

By hypothesis $\mathscr{P}_J^\star \neq \emptyset$ for all finite $J \subseteq I$. Now define

$$\mathscr{P}_J^{\star\star} = \{(J_1, J_2) \in \mathscr{P}_J^\star \mid \forall K \text{ finite}, J \subseteq K \subseteq I, \exists (K_1, K_2) \in \mathscr{P}_K^\star \text{ with } J \cap K_l = J_l,\ l = 1, 2\}.$$

It may be verified that $\mathscr{P}_J^{\star\star} \neq \emptyset$ for all finite $J \subseteq I$ [4, Lemma]. Now, consider the set $\mathscr{F}$ of functions f satisfying

(1) dom f is a set of finite subsets of I.
(2) $\forall J \in \operatorname{dom} f$, $f(J) = (f_1(J), f_2(J)) \in \mathscr{P}_J^{\star\star}$.
(3) $J_1, \ldots, J_n \in \operatorname{dom} f \Rightarrow J_1 \cup \cdots \cup J_n \in \operatorname{dom} f$.
(4) $J, K \in \operatorname{dom} f$, $J \subseteq K \Rightarrow f_l(J) = J \cap f_l(K)$, $l = 1, 2$.

We see that $\mathscr{F} \neq \emptyset$. We define a partial order on $\mathscr{F}$ by

$$f \leqslant g \Leftrightarrow \operatorname{dom} f \subseteq \operatorname{dom} g \text{ and } J \in \operatorname{dom} f \Rightarrow f(J) = g(J).$$

It is easy to verify that $(\mathscr{F}, \leqslant)$ is inductive. Hence, by Zorn's Lemma we may choose a maximal $g \in \mathscr{F}$. Using the maximality of g it may be proved that

$$\bigcup \operatorname{dom} g = I.$$

Now consider the family $\mathscr{B} = \{(\bigcap_{j \in g_1(J)} F_j, \bigcup_{j \in g_2(J)} G_j) \mid J \in \operatorname{dom} g\}$. It is easy to show that $\mathscr{B}$ has the fep, so by (2) we have

$$\bigcap_{J \in \operatorname{dom} g} \left(\bigcap_{j \in g_1(J)} F_j \right) \not\subseteq \bigvee_{J \in \operatorname{dom} g} \left(\bigcup_{j \in g_2(J)} G_j \right).$$

Let $I_1 = \bigcup \{g_1(J) \mid J \in \operatorname{dom} g\}$, $I_2 = I \setminus I_1$. Then (I_1, I_2) is a partition of I, and $I_2 \subseteq \bigcup \{g_2(J) \mid J \in \operatorname{dom} g\}$. This gives us

$$\bigcap_{J \in \operatorname{dom} g} \left(\bigcap_{j \in g_1(J)} F_j \right) = \bigcap_{i \in I_1} F_i \subseteq \bigvee_{i \in I_2} G_i \subseteq \bigvee_{J \in \operatorname{dom} g} \left(\bigcup_{j \in g_2(J)} G_j \right),$$

which is a contradiction.

(3) ⇒ (1). First take $G_i \in \tau$, $i \in I$, with $S = \bigvee_{i \in I} G_i$. For $i \in I$ let $F_i = \emptyset$. Then $\mathscr{C} = \{(G_i, F_i) \mid i \in I\}$ is an open, co-closed dicover, so has a finite sub-dicover $\{(G_j, F_j) \mid j \in J\}$. For the partition $J_1 = \emptyset$, $J_2 = J$ of J,

$$S = \bigcap_{j \in J_1} F_j \subseteq \bigcup_{j \in J_2} G_j,$$

whence $S = \bigcup_{j \in J} G_j$, and $(\mathscr{L}, \tau, \kappa)$ is compact. Co-compactness is proved in an analogous way.

To establish stability take $F \in \kappa$, $F \neq S$ and $G_i \in \tau$, $i \in I$, with $F \subseteq \bigvee_{i \in I} G_i$. Define $\mathscr{C} = \{(S, F)\} \cup \{(G_i, \emptyset) \mid i \in I\}$. It is clear that $\mathscr{C}$ is an open, co-closed dicover, and hence has a finite sub-dicover $\mathscr{C}_1$. If $\mathscr{C}_1 = \{(G_j, \emptyset) \mid j \in J\}$, J finite, then the fact that $\mathscr{C}_1$ is a dicover implies $\bigcup_{j \in J} G_j = S$, whence $F \subseteq \bigcup_{j \in J} G_j$. On the other hand, if $(S, F) \in \mathscr{C}_1$ then we again obtain

$F \subseteq \bigcup_{j\in J} G_j$ as required. Co-stability can be proved in a similar way. □

The condition in Theorem 3.5(2) was called compactness in [2–4]. It corresponds to the very important bitopological property of *joint compactness* (compactness of the joint topology). In the case of a complemented ditopology, Proposition 3.3 shows that Theorem 3.5(1) may be weakened to require one of compact, co-compact together with one of stable, co-stable.

Compactness in intuitionistic topological spaces is defined in the obvious way, and the proof of the following proposition is trivial and is omitted.

Proposition 3.6. *Let $(X, \mathscr{T})$ be an intuitionistic topological space corresponding to the complemented ditopological intuitionistic texture $(S, \mathscr{L}, \gamma, \tau, \kappa)$. Then*

(1) *The following are equivalent*:
   a. *$(S, \mathscr{L}, \gamma, \tau, \kappa)$ is compact.*
   b. *$(S, \mathscr{L}, \gamma, \tau, \kappa)$ is co-compact.*
   c. *$(X, \mathscr{T})$ is compact.*

(2) *The following are equivalent*:
   a. *$(S, \mathscr{L}, \gamma, \tau, \kappa)$ is stable.*
   b. *$(S, \mathscr{L}, \gamma, \tau, \kappa)$ is co-stable.*
   c. *Every proper closed subset of $(X, \mathscr{T})$ is compact.*

The following example shows that compactness does not imply stability, even for intuitionistic textures.

Example 3.7. Let $S = [0,1) \cup (1,2]$, $T = [0,1)$, and define the bijection $f : T \to S\setminus T$ by $f(t) = 2 - t$. Using Theorem 2.5(iv), we may verify that $\mathscr{L} = \{A \cup B \mid A \subseteq T,\ B \subseteq f(A)\}$ is an intuitionistic texturing of S, with complementation $\gamma(A \cup B) = S\setminus(f(A) \cup f^{-1}(B))$, $A \subseteq T$, $B \subseteq f(A)$.

For $0 \leqslant r \leqslant 1$ let $G(r) = [0,r) \cup (2 - r/2, 2]$, so that $G(r) \in \mathscr{L}$. If we let $\tau = \{G(r) \mid 0 \leqslant r \leqslant 1\} \cup \{S\}$, $\kappa = \{\gamma(G) \mid G \in \tau\}$ then (τ, κ) is a complemented ditopology on $(S, \mathscr{L}, \gamma)$. If $S = \bigvee_{i\in I} G_i = \bigcup_{i\in I} G_i$, $G_i \in \tau$, then for some $i \in I$ we have $G_i = S$ since, for example, $\frac{5}{4} \in S\setminus G(r)\ \forall r,\ 0 \leqslant r \leqslant 1$. Hence $(S, \mathscr{L}, \gamma, \tau, \kappa)$ is compact. On the other hand it is not stable. To see this note that $\gamma(G(r)) = [r/2, 1) \cup (1, 2 - r]$ so in particular $F = \gamma(G(1)) = [\frac{1}{2}, 1) \in \kappa$. Hence, $F \subseteq \bigvee_{n=2}^{\infty} G(1 - 1/n)$ since $\bigvee_{n=2}^{\infty} G(1 - 1/n) = G(1 - \bigwedge_{n=2}^{\infty} 1/n) = G(1) = [0,1) \cup (\frac{3}{2}, 2]$. However, F is not contained in any finite union of the sets $G(1 - 1/n)$.

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