ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 29, Number 2, Summer 1999

DIRECT SUMS AND SUMMANDS OF WEAK CS-MODULES AND CONTINUOUS MODULES

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Introduction. In [**5**] it is left as a question whether direct sums and summands of weak CS-modules are weak CS or not. Some particular answers are given to the former question in [**5**, Lemma 1.10, Lemma 1.11, Theorem 1.12], and in the first part of this note we give a general result, Theorem 1.9, of which those assertions are corollaries, as well as the assertion that a finite direct sum of relatively injective weak CS-modules is weak CS, Corollary 1.10, the dual of which is proved for CS-modules by Harmanci and Smith in [**2**]. As for the latter one we give an affirmative answer for a module with C_3 property and a UC [**6**], in particular, nonsingular, module. Finally, in this section, we give a sufficient condition for a nonsingular module to be CESS. In the second part some properties of weak CS-modules in common with modules satisfying C_{11} [**8**] are investigated and a class of modules, direct summands of which are direct sums of uniform modules, Proposition 2.6, is introduced. In the third part a generalization of continuous modules is given, namely F-modules. Continuous modules are characterized in terms of F-modules satisfying the C_{11}-property. We eventually prove that a direct sum M of C_{11}, hence CS/continuous, modules is continuous if and only if M is an F-module.

In this paper R will denote a ring with identity and M a unitary right R-module. For any submodule K of M, the family of submodules N satisfying $K \cap N = 0$ has a maximal member by Zorn's lemma, which is called a *complement of* K *in* M. A submodule N of M is called a *complement in* M if N is a complement of a submodule of M. It is easy to see that a submodule is a complement in M if and only if it has no proper essential extensions in M. For $m \in M$, the *right annihilator of* m is the set of elements r of R such that $mr = 0$, and is denoted $ann(m)$. A module M is called *nonsingular* if no element of M except 0 has annihilator which is essential in R_R. A module M is said to be a CS-*module* if every complement in M is a direct summand of M,

Received by the editors on May 13, 1996, and in revised form on June 24, 1997.

equivalently, if every submodule of M is essential in a direct summand of M. M is said to satisfy the C_2 condition if any submodule of M which is isomorphic to a direct summand of M is itself a direct summand of M. M is said to satisfy the C_3 condition if the sum of any two direct summands of M with zero intersection is a direct summand of M. M is said to be a *UC-module* if, for any submodule K of M, there is a unique complement N of M such that K is essential in N. M is said to be a CESS-*module* if complements in M with essential socle are direct summands. A *weak CS-module* is a module, every semi-simple submodule of which is essential in a direct summand of M. A module M is said to satisfy the C_{11} property if every submodule of M has a complement which is a direct summand of M. M is said to satisfy the C_{12} condition if every submodule can be essentially embedded in a direct summand of M. For further details about modules satisfying C_{11}, and those satisfying C_{12}, see [**8**]. For weak CS-modules and CESS modules, see [**5**], and for UC modules, see [**6**]. M is said to have the property (A) if the ACC holds for annihilators of elements of M. In a module with (A) local summands are complements in M, see [**8**, Lemma 4.5].

1. Weak CS-modules. In this section we first attempt to offer some sufficient conditions for the direct summands of a weak CS-module to be weak CS. To this end we make the following definition. First recall the conditions C_2 and C_3 on a module M, and that C_2 implies C_3, see [**4**, Proposition 2.2 and Proposition 2.7].

Proposition 1.1. *Any direct summand of a module which is both weak* CS *and* UC *is weak* CS. *Note also that such a module is a* CESS *module.*

Proof. Let M be weak CS and UC and K be a direct summand of M and A, a semi-simple submodule of K. A is essential in a direct summand T of M, and A is essential in a complement Y of K. Y and T are complements in M, thus $Y = T$ and T is also a direct summand of K. So the proof is complete. □

By the above proposition any direct summand of a nonsingular module which is weak CS is weak CS.

Definition 1.2. (i) A module is called *weak quasi-continuous* if it is a weak CS-module satisfying C_3.

(ii) A module is called *weak continuous* if it is a weak CS-module satisfying C_2.

There exist examples of weak continuous modules which are not continuous.

Example 1.3. Let M be the $\mathbf{Z}$-module $\mathbf{Z}/p\mathbf{Z} \oplus \mathbf{Q}$. Then by [**8**, Example 4.2], M satisfies C_2 but not C_1. M is easily seen to be weak CS; thus, M is weak continuous but not continuous.

Theorem 1.4. *If a module M is weak quasi-continuous, then any direct summand is weak* CS.

Proof. Let $K \oplus K' = M$, and let A be a semi-simple submodule of K. By assumption there exists a direct summand L of M such that A is essential in L. Then $L \cap K' = 0$ obviously. Let $\pi : M \to K$ be the canonical projection. Therefore, we have $\pi(L) \oplus K' = L \oplus K'$. Also, since $L \oplus K'$ is a direct summand of M by the C_3 assumption, then so is $\pi(L)$, whence $\pi(L)$ is a direct summand of K, too. Since $A = \pi(A)$, A is in $\pi(L)$. Now since A is essential in L, $A \oplus K'$ is essential in $L \oplus K' = \pi(L) \oplus K'$ by [**3**, Corollary 5.1.7]. Thus $A = \pi(L) \cap (A \oplus K')$ is essential in $\pi(L) \cap (\pi(L) \oplus K') = \pi(L)$. Thus the result follows.

Corollary 1.5. (i) *Any direct summand of a weak quasi-continuous module is weak quasi-continuous.*

(ii) *Any direct summand of a weak continuous module is weak continuous.*

Proof. By [**4**, Proposition 2.2 and Proposition 2.7] and Theorem 1.4.

Definition 1.6. Let M be a module and A any submodule of M. If K is a direct summand of M such that A is essential in K, then we call K a *direct e-closure* of A.

Proposition 1.7. *In a weak quasi-continuous module, the direct e-closure of a semi-simple submodule is unique up to isomorphism.*

Proof. Let M be a weak quasi-continuous module and A a semi-simple submodule of M with direct e-closures K and L with $K \oplus K' = L \oplus L' = M$ for some submodules K' and L'. $L \cap K' = 0$ obviously. Let $\pi : M \to K$ be the canonical projection. Thus $L \oplus K' = \pi(L) \oplus K'$, and also since $L \oplus K'$ is a direct summand of M by the C_3 assumption, then so is $\pi(L)$ of M, hence of K. We remark that $\pi(L)$ is isomorphic to L and $A = \operatorname{Soc} L = \operatorname{Soc} K$. Thus $\operatorname{Soc}(L \oplus K') = A \oplus \operatorname{Soc} K' = \operatorname{Soc} \pi(L) \oplus \operatorname{Soc} K'$ and $\operatorname{Soc} \pi(L)$ is in A, thus $A = A \cap (\operatorname{Soc} \pi(L) \oplus \operatorname{Soc} K') = \operatorname{Soc} \pi(L) \oplus (A \cap \operatorname{Soc} K') = \operatorname{Soc} \pi(L)$. Thus A is in $\pi(L)$, whence $\pi(L)$ is essential in K. Also, since $\pi(L)$ is a direct summand of K, then $\pi(L) = K \cong L$. Therefore, the conclusion follows.

It is a question when a direct sum of weak CS modules is weak CS [**5**, Lemma 1.10, Lemma 1.11 and Theorem 1.12] provide some particular answers to this question. Here we give a theorem of which those assertions are corollaries.

Theorem 1.8. *Suppose $M = M_1 \oplus M_2$ is a direct sum of weak* CS *modules M_1 and M_2 where M_1 is M_2-injective. Then M is weak* CS.

Proof. Let A be a semi-simple submodule of M. Then there exists a submodule B of M such that $B \oplus (A \cap M_1) = A$. Since M_1 is weak CS, there exists a direct summand K of M_1 such that $A \cap M_1$ is essential in K and, by [**1**, Lemma 7.5], there exists a submodule M' of M such that $M' \oplus M_1 = M$ and $B \subseteq M'$. Then $M' \cong M_2$, thus M' is weak CS, so there exists a direct summand T of M' such that B is essential in T. Now we infer that $A = B \oplus (A \cap M_1)$ is essential in $T \oplus K$, which is a direct summand of $M' \oplus M_1$. Therefore, the conclusion follows.

Theorem 1.9. *If $M = M_1 \oplus \cdots \oplus M_n$, where M_i are weak* CS *and for each i, M_i is M_k-injective, $k > i$, then M is weak* CS.

Proof. By induction and Theorem 1.8.

Corollary 1.10. *A finite direct sum of relatively injective weak* CS *modules is weak* CS.

Proof. By Theorem 1.9.

Proposition 1.11. *Let M be a nonsingular module such that for any semi-simple submodule A of M there exists a complement K of A for which every homomorphism $f : A \oplus K \to M$ lifts to M. Then M is a* CESS*-module.*

Proof. Let L be a complement in M with essential socle A. By hypothesis there exists a complement K of A, hence of L, with the stated property. Now we claim first that every homomorphism $f : L \oplus K \to M$ lifts to M, then we will conclude by [**6**, Lemma 2] that L is a direct summand.

Let $f : L \oplus K \to M$ be a homomorphism. By hypothesis $f|_{A\oplus K}$ lifts to some homomorphism $g : M \to M$, i.e., $g|_{A\oplus K} = f|_{A\oplus K}$. We claim that $g|_L = f|_L$; then we can conclude at once that $g|_{L\oplus K} = f$. Suppose $m \in L - A$ and $f(m) \neq g(m)$; then $x = f(m) - g(m) \neq 0$. Consider the homomorphism $\Phi : R_R \to mR$ for which $\Phi(m) = mr$. Now since $mR \cap A$ is essential in mR, then so is $I = \Phi^{-1}(mR \cap A)$ in R_R. Now for any r in I, $xr = (f(m) - g(m))r = f(mr) - g(mr) = 0$ since mr is in A. Then I is in the right annihilator of x; thus $ann(x)$ is essential in R_R, which is a contradiction since $x \neq 0$ and M is nonsingular. Thus, the result follows.

2. Weak C_{11} and weak C_{12} modules. First recall the conditions C_{11} and C_{12}, see introduction.

A module M is said to be a *weak C_{11}-module* if every semi-simple submodule has a complement in M which is a direct summand of M, and will be denoted WC_{11}.

A module M is said to be a *weak C_{12}-module* if every semi-simple submodule can be imbedded in a direct summand of M by an essential monomorphism, and will be denoted WC_{12}.

We have been unable to find an example of a WC_{12} module which does not satisfy the C_{12} property.

Note that there exist weak C_{11} modules which fail to satisfy the C_{11} property. To demonstrate this fact, we give the following example. For further details, see [**7**, Example 11]:

Example 2.1. There exists a commutative valuation domain S such that every homomorphic image of S is a self-injective ring. There exists an ideal A of S such that the ring S/A has zero socle. Let $T = S/A$ and J be the unique maximal ideal of T. Let R be the subring of $T \oplus T$ defined by $R = \{(t, t') | t - t' \in J\}$. Now R_R fails to satisfy the C_{11} property by [**8**, Proposition 3.2 and Theorems 3.10]. However, it is weak CS. Thus, by Proposition 2.7 R_R is weak C_{11}.

Now we will first give a lemma and then characterize WC_{11} modules:

Lemma 2.2. *Let K be a complement in M and N be a submodule of M with $K \cap N = 0$. Then K is a complement of N in M if and only if $K \oplus N$ is essential in M.*

Proof. Necessity is obvious. Conversely, by Zorn's lemma, there exists a complement L of N containing K. Thus $N \oplus L$ is essential in M. By hypothesis $K \oplus N$ is essential in $L \oplus N$. Therefore, $K = L \cap (K \oplus N)$ is essential in $L \cap (L \oplus N) = L$. But, since K is a complement $K = L$, whence the result follows.

Proposition 2.3. *The following statements are equivalent for a module M:*

(i) *M is WC_{11}.*

(ii) *For any semi-simple submodule A of M there exists a direct summand K of M such that $A \cap K = 0$ and $A \oplus K$ is essential in M.*

(iii) *For any complement L in M with essential socle, there exists a complement of L which is a direct summand of M.*

(iv) *For any complement L in M with essential socle, there exists a direct summand K of M such that $K \cap L = 0$ and $K \oplus L$ is essential* *in* M.

Proof. (i) $\Leftrightarrow$ (ii) and (iii) $\Leftrightarrow$ (iv) by Lemma 2.2.

(ii) $\Rightarrow$ (iv). Let L be a complement in M with essential socle. By assumption, there exists a direct summand K of M such that $K \cap \operatorname{Soc} L = 0$ and $\operatorname{Soc} L \oplus K$ is essential in M. Thus, $K \cap L = 0$ obviously and, since Soc $L \oplus K$ is contained in $L \oplus K$, $L \oplus K$ is also essential in M.

(iii) $\Rightarrow$ (i). Let A be a semi-simple submodule of M. By Zorn's lemma there exists a complement L in M such that A is essential in L. Thus, L is a complement with essential socle. By hypothesis there is a complement K of L which is a direct summand of M. Since A is essential in L, K is also a complement of A.

In [**8**], it is left as a question whether any direct summand of a module satisfying C_{11} satisfies C_{11} or not. Now we are going to provide a sufficient condition that any direct summand of a WC_{11} module be WC_{11}. Before doing this, we prove

Lemma 2.4. *If* M *is a* WC_{11} *module satisfying* C_3 *and* $M = K \oplus K'$ *with* $\operatorname{Soc} K'$ *essential in* K', *then* K *is* WC_{11}.

Proof. Let A be a semi-simple submodule of K. By the WC_{11} assumption, there exists a direct summand L of M such that $A \oplus \operatorname{Soc} K' \oplus L$ is essential in M. Then we obviously have $L \cap K' = 0$. Now let $\pi : M \to K$ denote the canonical projection. Hence $\pi(L) \oplus K' = L \oplus K'$ is a direct summand of M by the C_3 assumption. Thus, $\pi(L)$ is a direct summand of M, hence of K. Now we claim that $A \cap \pi(L) = 0$ and $A \oplus \pi(L)$ is essential in K. Since $A \cap (L \oplus K')$ is semi-simple, $A \cap (L \oplus K') = A \cap \operatorname{Soc}\ (L \oplus K') = A \cap (\operatorname{Soc} L \oplus \operatorname{Soc} K')$ by [**3**, Corollary 9.1.5], which is in $A \cap (L \oplus \operatorname{Soc} K') = 0$. Thus, $A \cap (L \oplus K') = A \cap (\pi(L) \oplus K') = 0$. Therefore, $A \cap \pi(L) = 0$. Furthermore, $A + (L \oplus K') = A \oplus (L \oplus K')$. Now, since $A \oplus L \oplus \operatorname{Soc} K'$ is essential in $K \oplus K'$, then $A \oplus \pi(L) \oplus K' = A \oplus L \oplus K'$ is essential in $K \oplus K'$, whence $A \oplus \pi(L)$ is essential in K. Therefore, $\pi(L)$ is a complement of A in K by Lemma 2.2. Finally, we conclude that K is WC_{11}.

Corollary 2.5. *If M is a WC_{11} module satisfying C_3 and* $\operatorname{Soc} M$ *is essential in M, then any direct summand of M is WC_{11}.*

Proof. Any direct summand of M has essential socle.

Proposition 2.6. *Let M be a WC_{11} module with essential socle and satisfying C_3 and* (A)*. Then any direct summand of M is a direct sum of uniform submodules.*

Proof. Let $\Gamma = \{F \mid F$ is a family of direct summands of M which have simple socles and whose sum is direct$\}$. Γ is an inductive set. Let F be a maximal element in Γ and T be the direct sum of all submodules in F. T is a local summand by the C_3 assumption, thus a complement by (A) and [**8**, Lemma 4.5]. If $\operatorname{Soc} M$ is not in T, then we have a direct summand L of M such that $T \oplus L$ is essential in M. Note that L is WC_{11} and satisfies C_3 by Corollary 2.5, and $\operatorname{Soc} L$ is nonzero by hypothesis. Take some simple submodule N in L; then there exist submodules P and P' such that $P \oplus P' = L$ and $N \oplus P$ is essential in L. Hence $\operatorname{Soc} L = \operatorname{Soc} P \oplus \operatorname{Soc} P' = N \oplus \operatorname{Soc} P$. Thus, $\operatorname{Soc} P'$ is simple and essential in P'. But then $F \cup \{P'\}$ contradicts the maximality of F. Therefore, $\operatorname{Soc} M$ is in T, whence T is essential in M. Also, since T is a complement in M, $T = M$. Hence, the result follows for M. Now, since any direct summand of M satisfies the properties in the hypothesis by Corollary 2.5, the result follows immediately.

Proposition 2.7. *A weak* CS*-module is WC_{11}.*

Proof. If A is a semi-simple submodule of M, then there are submodules K and K' of M such that $M = K \oplus K'$ and A is essential in K, by hypothesis. Thus, since K' is a complement of K, then it is a complement of A.

Proposition 2.8. *A WC_{11}-module is WC_{12}.*

Proof. If A is a semi-simple submodule of M, then, by assumption, there exist submodules K and K' of M such that $K \oplus K' = M$ and

K is a complement of A. Then $(A \oplus K)/K$ is essential in $(K \oplus K')/K$ by [**3**, Lemma 5.2.5]. We know that there exist two isomorphisms $\Phi : A \to (A \oplus K)/K$ and $f : (K \oplus K')/K \to K'$. Hence their composition $f\Phi : A \to K'$ is the desired essential monomorphism. Thus the result follows.

Proposition 2.9. *If M is WC_{11}, then $M = K \oplus K'$ for some two submodules K and K' with $Soc\,K$ essential in K and $Soc\,K' = 0$.*

Proof. For $A = \operatorname{Soc} M$ there exist submodules K and K' such that $K \oplus K' = M$ and $A \oplus K'$ is essential in M. Now $A = \operatorname{Soc} M = \operatorname{Soc}(K \oplus K') = \operatorname{Soc} K \oplus \operatorname{Soc} K' = \operatorname{Soc} K$. Thus A is contained in K. Hence, $K \cap (A \oplus K') = A$ is essential in $K \cap (K \oplus K') = K$. Therefore, $\operatorname{Soc} K = \operatorname{Soc} M$ is essential in K, whence the result follows.

Theorem 2.10. *If $M = \oplus_{\alpha \in I} M_\alpha$ with $M_\alpha WC_{11}$ for each $\alpha \in I$, where I is any nonempty index set, then M is WC_{11}.*

Proof. Let A be a semi-simple submodule of M. Let $\alpha \in I$. By Proposition 2.3 there exists a direct summand K_α of M_α such that $(A \cap M_\alpha) \oplus K_\alpha = (A \oplus K_\alpha) \cap M_\alpha$ is essential in M_α. Let F be a nonempty subset of I containing α such that there exists a direct summand K of $\oplus_{\alpha \in F} M_\alpha$ with $(A \cap (\oplus_{\alpha \in F} M_\alpha)) \oplus K = (A \oplus K) \cap (\oplus_{\alpha \in F} M_\alpha)$ is essential in $\oplus_{\alpha \in F} M_\alpha$. Now let M_1 stand for $\oplus_{\alpha \in F} M_\alpha$ and suppose that $F \neq I$. Then choose some $\beta \in I$ which is not in F. By hypothesis, there exists a direct summand K' of M_β such that $((A \oplus \operatorname{Soc} K) \cap M_\beta) \oplus K'$ is essential in M_β. It is clear that $K \oplus K'$ is a direct summand of $M_1 \oplus M_\beta$. Now since A is semi-simple $A \cap (K \oplus K') = A \cap \operatorname{Soc}(K \oplus K') = A \cap (\operatorname{Soc} K \oplus \operatorname{Soc}\; K')$ which is a submodule of $A \cap (\operatorname{Soc} K \oplus K')$, which is the zero submodule since $(A \oplus \operatorname{Soc} K) \cap K' = 0$. It is left only to prove that $(A \oplus K \oplus K') \cap (M_1 \oplus M_\beta) = (A \cap (M_1 \oplus M_\beta)) \oplus K \oplus K'$ is essential in $M_1 \oplus M_\beta$. Now let Y stand for $A \cap M_1$. Then $Y \oplus K$ is essential in M_1. Also $K' \oplus ((A \oplus \operatorname{Soc} K) \cap M_\beta)$ is essential in M_β. Therefore $D = Y \oplus K \oplus K' \oplus ((A \oplus \operatorname{Soc} K) \cap M_\beta)$ is essential in $M_1 \oplus M_\beta$ by [**3**, Corollary 5.1.7]. Now, since $Y \oplus K \oplus K'$ and $((A \oplus \operatorname{Soc} K) \cap M_\beta)$ are both contained in $(A \oplus K \oplus K') \cap (M_1 \oplus M_\beta)$, then so is their sum D. And, since D is essential in $M_1 \oplus M_\beta$, then so is $(A \oplus K \oplus K') \cap (M_1 \oplus M_\beta)$. By repeating this argument, we can conclude that there exists a direct summand L of M such that $A \oplus L$ is essential in M. Hence, by Lemma 2.2, L is a complement of A. Therefore, the conclusion follows.

Corollary 2.11. *A direct sum of weak* CS*-modules is* WC_{11}.

Proposition 2.12. *If* M *is* WC_{12} *and* Soc M *is finitely generated, then* $M = K \oplus K'$ *for some submodules* K *and* K' *of* M *with* $Soc\, M$ *essential in* K *and* $Soc\, K' = 0$.

Proof. There exist submodules K and K' of M such that $M = K \oplus K'$, and an essential monomorphism $f : \mathrm{Soc}\, M \to K$, by WC_{12} assumption. Also $\mathrm{Soc}\, M = \oplus_{i=1}^{n} M_i$ for some simple submodules M_i and $n \in \mathbf{N}$. Thus, $f(\mathrm{Soc}\, M) = \oplus_{i=1}^{n} f(M_i)$ where each $f(M_i)$ is simple. Also, since $\oplus_{i=1}^{n} f(M_i)$ is essential in K, then $\mathrm{Soc}\, K = \oplus_{i=1}^{n} f(M_i)$. Thus, $\mathrm{Soc}\, M = \oplus_{i=1}^{n} M_i = \mathrm{Soc}\,(K \oplus K') = \mathrm{Soc}\, K \oplus \mathrm{Soc}\, K' = (\oplus_{i=1}^{n} f(M_i)) \oplus \mathrm{Soc}\, K'$. By the Remak-Krull-Schmidtt theorem, see [**3**, Theorem 7.3.1], $\mathrm{Soc}\, K' = 0$ and $\oplus_{i=1}^{n} M_i = \oplus_{i=1}^{n} f(M_i)$, whence the result follows.

3. Continuous modules.

Definition 3.1. A module is said to be an F *module*, respectively F_1 *module*, if any submodule which is isomorphic to a complement, respectively a complement with essential socle, in M is a complement in M.

There exist examples of F modules which are not continuous.

Example 3.1. Let K be a field and $V = K \times K$. Consider the ring R of 2×2 matrices of the form (a_{ij}) with $a_{11}, a_{22} \in K$, $a_{12} \in V$, $a_{21} = 0$ and $a_{11} = a_{22}$. Now the only right ideals of R are 0, R_R, I_1, I_2, I_3, $I(x, y)$ for any nonzero x and y in K, where I_1 is the set of (a_{ij}) with a_{11}, a_{22} and a_{21} all zero, and $a_{12} \in K \times 0$; I_2 is the set of (a_{ij}) with a_{11}, a_{22} and a_{21} all zero, and $a_{12} \in 0 \times K$; I_3 is the set of (a_{ij}) with a_{11}, a_{22} and a_{21} all zero and $a_{12} \in V$; $I(x, y)$ is the set of (a_{ij}) with a_{11}, a_{22} and a_{21} all zero and $a_{12} \in (x, y)K$. Now all the right ideals except I_3 are complements in R_R, and I_3 is not a complement since it is essential in R_R. I_3 is isomorphic to no other right ideal since it is the only right ideal of R which is two-dimensional over K. Thus, R_R is an F module but not CS, since R_R is the only nonzero direct summand of itself and $\dim V = 2$.

By the above example we see that the F condition does not imply continuity. But with regard to the C_3 condition we have the following.

Proposition 3.2. *An F module M satisfies the C_3 property.*

Proof. Let A and B be direct summands of M with zero intersection and $B \oplus B' = M$. By Zorn's lemma we can choose a complement L of B in M containing A. Thus, $L \oplus B = \pi(L) \oplus B$ is essential in M, where π is the canonical projection onto B'. Then $\pi(L)$ is essential in B'. Now since $\pi(L) \cong L$, $\pi(L)$ is a complement in M by assumption. Hence $\pi(L) = B'$, so $L \oplus B = B' \oplus B$. Thus, A being a direct summand of L, $A \oplus B$ is a direct summand of $L \oplus B$, whence the conclusion follows. □

Theorem 3.3. *M is continuous if and only if M is an F module satisfying C_{11}.*

Proof. Necessity is obvious. As for the converse, we first claim that M is CS. Let A be a submodule of M. Then, by the C_{11} assumption, there exist submodules K and K' such that $K \oplus K' = M$ and $A \oplus K$ is essential in M. By Zorn's lemma there exists a complement T such that A is essential in T. Thus $T \cap K = 0$. Now let $\pi : M \to K'$ be the canonical projection. Then $T \oplus K = \pi(\mathcal{T}) \oplus K$ is essential in M. Hence $\pi(T)$ is essential in K'. Also by the F assumption $\pi(T) = K'$, since it is isomorphic to T. Hence $T \oplus K = M$. Therefore, M is CS. It is easy to see that a module which is both CS and F is continuous. Therefore, the conclusion follows.

Corollary 3.4. *A direct sum $M = \oplus M_\alpha$ of C_{11}, hence CS/continuous, modules M_α is continuous if and only if M is an F module.*

Proof. By Theorem 3.3 and [**8**, Theorem 2.4]. □

The following proposition can be proved by the same technique as Theorem 3.3.

Proposition 3.5. *A module which is both WC_{11} and F_1 is a* CESS-*module.*

Proposition 3.6. *An F_1 module $M = \oplus M_\alpha$ is a* CESS *module if and only if each M_α is WC_{11}.*

Proof. Necessity part of the proof follows by the observation that any direct summand of a CESS module is CESS, and sufficiency follows by Theorem 2.9 and Proposition 3.5.

Acknowledgment. The author wishes to thank Adnan Tercan for providing examples and for motivating discussions.

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