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Research Article

An Unconventional Finite Difference Scheme for Modified Korteweg-de Vries Equation

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A numerical solution of the modified Korteweg-de Vries (MKdV) equation is presented by using a nonstandard finite difference (NSFD) scheme with theta method which includes the implicit Euler and a Crank-Nicolson type discretization. Local truncation error of the NSFD scheme and linear stability analysis are discussed. To test the accuracy and efficiency of the method, some numerical examples are given. The numerical results of NSFD scheme are compared with the exact solution and a standard finite difference scheme. The numerical results illustrate that the NSFD scheme is a robust numerical tool for the numerical integration of the MKdV equation.

1. Introduction

This paper is concerned with the nonstandard integration of modified Korteweg-de Vries (MKdV) equation

$$u_t + qu^2 u_x + ru_{xxx} = 0, \quad (x,t) \in [x_L, x_R] \times [0,T] \tag{1}$$

with initial condition

$$u(x,0) = u_0(x), \quad x \in [x_L, x_R] \tag{2}$$

and boundary conditions

$$\begin{aligned} u(x_L,t) &= f(t), \\ u(x_R,t) &= g(t), \\ & t \in [0,T], \end{aligned} \tag{3}$$

where $q, r \in \mathbb{R}$. The analytical solution of the MKdV equation (1) can be expressed as [1]

$$u(x,t) = \mp\sqrt{\frac{-6r}{q}} \tanh(x + 2rt). \tag{4}$$

It plays an important role in the study of nonlinear physics such as fluid physics and quantum field theory. It is a model equation for the weakly nonlinear long waves which occur in many different physical systems. It is an integrable equation and admits soliton solution obtained by means of the inverse scattering method and Hirota's direct method and by using Backlund transformations [2, 3]. It is well known that (1) has a solitary wave solution of the form

$$u(x,t) = \mp\sqrt{\frac{6r}{q}} k \operatorname{sech} k\left(x - rk^2 t\right). \tag{5}$$

Although the MKdV equation has been extensively studied by many authors in soliton theory, the solution (4) is never considered before in the literature. For the purpose of nonstandard integration, the kink soliton solution (4) will be used throughout the study. A nonstandard finite difference scheme can be constructed from the exact finite difference scheme [4]. An exact finite difference scheme can be constructed for any ordinary differential equation (ODE) or partial differential equation (PDE) from the analytical solution of the differential equation [5–7]. Among the various numerical techniques such as classical finite difference, finite volume, adaptive mesh, finite element, and spectral method for solving ODEs and PDEs, NSFD schemes have been proved to be one of the most efficient approaches in recent years. The authors in [8] proposed a nonstandard finite volume method for the numerical solution of a singularly perturbed Schrödinger equation. They have shown that the proposed nonstandard finite volume method is capable of reducing the computational cost associated with most classical schemes. They have highlighted that NSFD schemes have been efficient in tackling the deficiency of classical finite difference scheme for the approximation of solutions of several differential equation models. A nonstandard symplectic Runge-Kutta method is applied to Hamiltonian systems in [9]. In [9], it has been shown that nonstandard schemes are better than standard finite difference schemes in long time computations. Compared with some other methods, NSFD method is more stable [10].

Up to the author's knowledge, a NSFD scheme for the numerical solution of the MKdV equation (1) is never studied before. The aim of this paper is to designed a robust NSFD scheme for the numerical solution of the MKdV equation (1) that is better than the standard scheme in the numerical precision for large spatial step size which reduces the computational cost associated with most classical schemes. This paper is organized as follows. In the next section we begin with proposing the NSFD scheme for the MKdV equation (1). Stability and local truncation error of the NSFD scheme are examined in Section 3. In Section 4 some numerical experiments for the NSFD scheme are presented to show that our proposed method is efficient and accurate. Finally, we summarize our observation in Section 5.

2. Nonstandard Discretization

In this section, we will propose the NSFD model for the numerical solution of the MKdV equation (1). Firstly, we give three basic definitions and properties of the NSFD discretization proposed by Mickens [11, 12] to construct a NSFD scheme.

(1) The orders of the discrete derivatives must be exactly equal to the orders of the corresponding derivatives of the differential equations.

(2) Denominator functions for the discrete derivatives must, in general, be expressed in terms of more complicated functions of the step sizes than those conventionally used. For example, the discrete derivatives $u_t(x,t)$ and $u_x(x,t)$ are generalized as

$$\begin{aligned} u_t(x,t) &\cong \frac{u_j^{n+1}-u_j^n}{\Phi(\Delta t,\lambda)}, \quad \Phi(\Delta t,\lambda)=\Delta t+\mathcal{O}\left(\Delta t^2\right), \\ u_x(x,t) &\cong \frac{u_{j+1}^n-u_j^n}{\Gamma(\Delta x,\mu)}, \\ u_x(x,t) &\cong \frac{u_j^n-u_{j-1}^n}{\Gamma(\Delta x,\mu)}, \\ &\qquad \Gamma(\Delta x,\mu)=\Delta x+\mathcal{O}\left(\Delta x^2\right). \end{aligned} \tag{6}$$

(3) Nonlinear terms must, in general, be modeled nonlocally on the computational grid or lattice; for example,

$$\begin{aligned} \left(u_j^n\right)^2 &\approx u_{j+1}^n u_j^n, \\ \left(u_j^n\right)^2 &\approx \left(\frac{u_{j-1}^n+u_j^n+u_{j+1}^n}{3}\right)u_j^n. \\ \left(u_j^n\right)^3 &\approx \left(u_j^n\right)^2\left(\frac{u_{j+1}^n+u_{j-1}^n}{2}\right), \\ \left(u_j^n\right)^3 &\approx u_{j-1}^n u_j^n u_{j+1}^n. \end{aligned} \tag{7}$$

It is well known that a NSFD method is constructed from the exact finite difference schemes. But Mickens [11] discussed some difficulties of applying the nonstandard modeling rules in the actual construction of exact finite difference scheme for the MKdV equation. Some pitfalls in the procedures for constructing an exact finite difference schemes in terms of basic rules of the NSFD methods are investigated [11]. For the MKdV equation (1) two nonstandard finite difference models are proposed [11], namely, the explicit scheme

$$\begin{aligned} &\frac{U_j^{n+1}-U_j^n}{D_3(\Delta t)}+qU_j^{n+1}\left[\frac{\left(U_j^n\right)^2-\left(U_{j-1}^n\right)^2}{D_2(\Delta x)}\right] \\ &\quad+\frac{U_{j+2}^n-3U_{j+1}^n+3U_j^n-U_{j-1}^n}{D_1(\Delta x)^2 D_2(\Delta x)}=0 \end{aligned} \tag{8}$$

and the implicit scheme

$$\begin{aligned} &\frac{U_j^{n+1}-U_j^n}{D_3(\Delta t)}+qU_j^{n+1}\left[\frac{\left(U_j^n\right)^2-\left(U_{j-1}^n\right)^2}{D_2(\Delta x)}\right] \\ &\quad+\frac{U_{j+2}^{n+1}-3U_{j+1}^{n+1}+3U_j^{n+1}-U_{j-1}^{n+1}}{D_1(\Delta x)^2 D_2(\Delta x)}=0, \end{aligned} \tag{9}$$

where U_j^n is the approximation to the exact solution $u(x,t)$ at the mesh point (x_j,t_n) and $D_1(\Delta x)=\Delta x$, $D_2(\Delta x)=\Delta x$, and $D_3(\Delta t)=\Delta t$. The above construction processes do not give functional relation between the space and time step sizes which is not known yet (see Mickens [11], p: 228). The step sizes for exact schemes must satisfy some fixed conditions. In order to release these conditions for step size, we follow the way of Zhang et al. [13] and construct the following nonstandard-theta scheme [13]

$$\begin{aligned} &\frac{U_j^{n+1}-U_j^n}{\Phi} \\ &\quad+qU_j^nU_j^{n+1}\frac{\theta\left(U_j^{n+1}-U_{j-1}^{n+1}\right)+(1-\theta)\left(U_j^n-U_{j-1}^n\right)}{\Gamma} \\ &\quad+r\frac{\theta\left(U_{j+2}^{n+1}-2U_{j+1}^{n+1}+2U_{j-1}^{n+1}-U_{j-2}^{n+1}\right)}{\Psi} \\ &\quad+r\frac{(1-\theta)\left(U_{j+2}^n-2U_{j+1}^n+2U_{j-1}^n-U_{j-2}^n\right)}{\Psi}=0 \end{aligned} \tag{10}$$

for the numerical integration of the MKdV equation (1). Here Φ is the time step function and Γ and $\Psi = 2\Gamma^3$ are the space step functions; U_j^n is an approximation to the exact solution $u(x,t)$ at the mesh point (x_j, t_n). A standard finite difference scheme for the MKdV equation (1) can be proposed as

$$\begin{aligned}\frac{U_j^{n+1}-U_j^n}{\Delta t} &+ q\left(U_j^n\right)^2 \frac{\theta\left(U_j^{n+1}-U_{j-1}^{n+1}\right)+(1-\theta)\left(U_j^n-U_{j-1}^n\right)}{\Delta x} \\ &+ r\frac{\theta\left(U_{j+2}^{n+1}-2U_{j+1}^{n+1}+2U_{j-1}^{n+1}-U_{j-2}^{n+1}\right)}{2\Delta x^3} \\ &+ r\frac{(1-\theta)\left(U_{j+2}^{n}-2U_{j+1}^{n}+2U_{j-1}^{n}-U_{j-2}^{n}\right)}{2\Delta x^3} = 0.\end{aligned} \tag{11}$$

According to (10), we can write

$$\Phi = \frac{\left(U_j^n - U_j^{n+1}\right)\Psi\Gamma}{qU_j^nU_j^{n+1}\Psi A + r\Gamma B}, \tag{12}$$

where

$$\begin{aligned} A &= \theta\left(U_j^{n+1}-U_{j-1}^{n+1}\right)+(1-\theta)\left(U_j^n-U_{j-1}^n\right) \\ B &= \theta\left(U_{j+2}^{n+1}-2U_{j+1}^{n+1}+2U_{j-1}^{n+1}-U_{j-2}^{n+1}\right) \\ &\quad + (1-\theta)\left(U_{j+2}^{n}-2U_{j+1}^{n}+2U_{j-1}^{n}-U_{j-2}^{n}\right).\end{aligned} \tag{13}$$

When $\theta = 1$, we select $\Gamma(\Delta x) = \Gamma(h) = (e^{2h}-1)/2$ and hence $\Psi = 2\Gamma^3 = (e^{2h}-1)^3/4$. Then substituting Γ and Ψ into (12), we can rewrite Φ as

$$\begin{aligned}\Phi = {} & \frac{e^{4r\Delta t}-1}{4r} \times \left[4\left(1+se^{4r\Delta t}\right)\left(1+se^{4r\Delta t-2h}\right)\right. \\ & \left.\cdot\left(1+se^{4r\Delta t+4h}\right)\left(1+se^{4r\Delta t+2h}\right)\cdot\left(1+se^{4r\Delta t-4h}\right)\right] \\ & \times\left[6(s-1)\left(se^{4r\Delta t}-1\right)\left(1+se^{4r\Delta t+4h}\right)\right. \\ & \cdot\left(1+se^{4r\Delta t+2h}\right)\cdot\left(1+se^{4r\Delta t-4h}\right)2e^{4r\Delta t-2h}-4\left(e^{2h}\right. \\ & \left.+1\right)(s+1)\left(1+se^{4r\Delta t}\right)^2 \\ & \left.\cdot\left\{s^2e^{12r\Delta t-4h}-se^{8r\Delta t-6h}\left(e^{2h+1}\right)^2+e^{4r\Delta t-4h}\right\}\right]^{-1},\end{aligned} \tag{14}$$

where $s = s_j^n = e^{2(x_j+2rt_n)}$. If $\Gamma(\Delta x) = \Gamma(h) = h + \mathcal{O}(h^2)$ when $h \to 0$, $\Delta t \to 0$, after tedious calculation we obtain

$$\Phi(\Delta t) \longrightarrow \frac{e^{4r\Delta t}-1}{4r} \quad \text{with } \Phi(\Delta t) = \Delta t + \mathcal{O}\left(\Delta t^2\right). \tag{15}$$

Similarly, for $\theta = 0$ and $\theta = 1/2$, if Γ is selected to be

$$\Gamma(\Delta x) = \Gamma(h) = \frac{e^{2h}-1}{2} \tag{16}$$

we obtain the same "denominator function" Φ

$$\Phi \longrightarrow \frac{e^{4r\Delta t}-1}{4r}. \tag{17}$$

We note that $\Phi \to \Delta t$, $\Gamma \to \Delta x$, and $\Psi \to 2\Delta x^3$ as $(\Delta t, \Delta x) \to (0,0)$.

3. Stability and Local Truncation Error

In this section, stability and local truncation error of the nonstandard scheme (10) are examined. For stability analysis we use the von Neumann method. Since the method is applicable only for linear PDE, we consider the linearized MKdV equation

$$u_t + au_x + ru_{xxx} = 0, \quad (x,t) \in [x_L, x_R] \times [0,T], \tag{18}$$

where $a = \max\{qu^2(x,t)\}$ in the domain $(x,t) \in [x_L, x_R] \times [0,T]$. The application of the nonstandard-theta scheme (10) to the linear equation (18) yields

$$\begin{aligned}\frac{U_j^{n+1}-U_j^n}{\Phi} &+ a\frac{\theta\left(U_j^{n+1}-U_{j-1}^{n+1}\right)+(1-\theta)\left(U_j^n-U_{j-1}^n\right)}{\Gamma} \\ &+ r\frac{\theta\left(U_{j+2}^{n+1}-2U_{j+1}^{n+1}+2U_{j-1}^{n+1}-U_{j-2}^{n+1}\right)}{\Psi} \\ &+ r\frac{(1-\theta)\left(U_{j+2}^{n}-2U_{j+1}^{n}+2U_{j-1}^{n}-U_{j-2}^{n}\right)}{\Psi} = 0.\end{aligned} \tag{19}$$

We take the difference between the exact solution $u(x,t)$ at the mesh point (x_j, t_n) and the approximate solution U_j^n and define the error $\epsilon_j^n = u(x_j, t_n) - U_j^n$. Substituting $U_j^n = u(x_j, t_n) - \epsilon_j^n$ into the difference equation (19), we see that the error ϵ_j^n satisfies the same discrete equation

$$\begin{aligned}\frac{\epsilon_j^{n+1}-\epsilon_j^n}{\Phi} &+ a\frac{\theta\left(\epsilon_j^{n+1}-\epsilon_{j-1}^{n+1}\right)+(1-\theta)\left(\epsilon_j^n-\epsilon_{j-1}^n\right)}{\Gamma} \\ &+ r\frac{\theta\left(\epsilon_{j+2}^{n+1}-2\epsilon_{j+1}^{n+1}+2\epsilon_{j-1}^{n+1}-\epsilon_{j-2}^{n+1}\right)+}{\Psi} \\ &+ r\frac{(1-\theta)\left(\epsilon_{j+2}^{n}-2\epsilon_{j+1}^{n}+2\epsilon_{j-1}^{n}-\epsilon_{j-2}^{n}\right)}{\Psi} = 0.\end{aligned} \tag{20}$$

The von Neumann stability analysis uses the fact that every linear constant coefficient difference equation has a solution of the form

$$\epsilon_j^n = \left(e^{\alpha\Delta t}\right)^n e^{i\beta j\Delta x} = (\xi)^n e^{i\beta j\Delta x}, \quad \alpha, \beta \in \mathbb{R}, \ i^2 = -1. \tag{21}$$

The function ξ is determined from the difference equation by substituting the Fourier mode (21) into (20), and we obtain

$$\xi^{n+1}e^{i\beta j\Delta x}\left(1 + A_2 + iB_2\right) = \left(1 - A_1 - iB_1\right)\xi^n e^{i\beta j\Delta x}, \tag{22}$$

where

$$\begin{aligned} A_1 &= 2\frac{\Phi a}{\Gamma}(1-\theta)\sin^2\left(\frac{\beta\Delta x}{2}\right), \\ A_2 &= 2\frac{\Phi a}{\Gamma}\theta\sin^2\left(\frac{\beta\Delta x}{2}\right), \\ B_1 &= \frac{\Phi a}{\Gamma}(1-\theta)\sin(\beta\Delta x) \\ &\quad - 8\frac{\Phi r}{\Psi}(1-\theta)\sin(\beta\Delta x)\sin^2\left(\frac{\beta\Delta x}{2}\right), \\ B_2 &= \frac{\Phi a}{\Gamma}\theta\sin(\beta\Delta x) \\ &\quad - 8\frac{\Phi r}{\Psi}\theta\sin(\beta\Delta x)\sin^2\left(\frac{\beta\Delta x}{2}\right). \end{aligned} \tag{23}$$

Canceling the exponential term, we have $\xi^{n+1} = \rho\xi^n$, where

$$\rho = \frac{1 - A_1 - iB_1}{1 + A_2 + iB_2} \tag{24}$$

is the amplification factor of the method. If the method is to be stable, the stability requirement $|\rho| \le 1$ should be satisfied. Now we consider the following cases.

Crank-Nicolson Type Scheme. When $\theta = 1/2$, the amplification factor is simplified to

$$\rho = \frac{1 - A - iB}{1 + A + iB}, \tag{25}$$

where

$$\begin{aligned} A &= \frac{\Phi a}{\Gamma}\sin^2\left(\frac{\beta\Delta x}{2}\right) \\ B &= \frac{\Phi a}{2\Gamma}\sin(\beta\Delta x) - 4\frac{\Phi r}{\Psi}\sin(\beta\Delta x)\sin^2\left(\frac{\beta\Delta x}{2}\right). \end{aligned} \tag{26}$$

From (25) we get $|\rho|^2 = \rho\overline{\rho} \le 1$; hence the proposed method (10) is unconditionally stable for $\theta = 1/2$.

The Fully Implicit Scheme. When $\theta = 1$, the amplification factor is simplified to

$$\rho = \frac{1}{1 + A + iB}, \tag{27}$$

where

$$\begin{aligned} A &= 2\frac{\Phi a}{\Gamma}\sin^2\left(\frac{\beta\Delta x}{2}\right) \\ B &= \frac{\Phi a}{\Gamma}\sin(\beta\Delta x) - 8\frac{\Phi r}{\Psi}\sin(\beta\Delta x)\sin^2\left(\frac{\beta\Delta x}{2}\right). \end{aligned} \tag{28}$$

From (27) we get $|\rho|^2 = \rho\overline{\rho} \le 1$; hence the proposed method (10) is unconditionally stable for $\theta = 1$.

The Explicit Scheme. When $\theta = 0$, the amplification factor is simplified to

$$\rho = 1 - A - iB, \tag{29}$$

where

$$\begin{aligned} A &= 2\frac{\Phi a}{\Gamma}\sin^2\left(\frac{\beta\Delta x}{2}\right) \\ B &= \frac{\Phi a}{\Gamma}\sin(\beta\Delta x) - 8\frac{\Phi r}{\Psi}\sin(\beta\Delta x)\sin^2\left(\frac{\beta\Delta x}{2}\right). \end{aligned} \tag{30}$$

It is well known that the explicit methods are conditionally stable. Numerical experiments show that the amplification factor $|\rho| \le 1$ when $\Phi/\Gamma < 0.03$ (or $\Delta t/\Delta x < 0.1$) for the parameters $q = 0.3$ and $r = -0.5$.

Now, we will discuss the local truncation error of the nonstandard scheme (10). In theory, we can approximate the original problem as accurately as we wish by making the time step Δt and Δx small enough. It is said in this case that the approximation is consistent. The local truncation error and the stability play important roles in the convergence of the numerical method. In a convergent method, the order of the error is determined by the order of the local truncation error. The local truncation error T_j^n of the nonstandard scheme (10) is defined by

$$\begin{aligned} T_j^n &= \left(\partial_t u_j^n - u_t\left(x_j, t_n\right)\right) \\ &\quad + q\left(u_j^n u_n^{n+1}\partial_x u_j^n - u\left(x_j, t_n\right)^2 u_x\left(x_j, t_n\right)\right) \\ &\quad + r\left(\partial_{xxx} u_j^n - u_{xxx}\left(x_j, t_n\right)\right), \end{aligned} \tag{31}$$

where

$$\begin{aligned} \partial_t u_j^n &= \frac{u_j^{n+1} - u_j^n}{\Phi}, \\ \partial_x u_j^n &= \frac{\theta\left(u_j^{n+1} - u_{j-1}^{n+1}\right) + (1-\theta)\left(u_j^n - u_{j-1}^n\right)}{\Gamma}, \\ \partial_{xxx} u_j^n &= \frac{\theta\left(u_{j+2}^{n+1} - 2u_{j+1}^{n+1} + 2u_{j-1}^{n+1} - u_{j-2}^{n+1}\right)}{\Psi} \\ &\quad + \frac{(1-\theta)\left(u_{j+2}^n - 2u_{j+1}^n + 2u_{j-1}^n - u_{j-2}^n\right)}{\Psi} \end{aligned} \tag{32}$$

and u_j^n is the approximate solution for the MKDV equation (1) obtained from the nonstandard scheme (10) and $u(x_j, t_n)$ is the exact solution at the mesh point $(j\Delta x, n\Delta t)$. For $\theta = 1$, the principal part of the local truncation error is

$$\begin{aligned} T_j^n &= \left(\frac{\Delta t}{\Phi} - 1\right)u_t + \frac{\Delta t^2}{2\Phi}u_{tt} + \frac{\Delta t^3}{6\Phi}u_{ttt} \\ &\quad + \left(\frac{\Delta x}{\Gamma} - 1\right)qu^2 u_x - \frac{\Delta x^2}{2\Gamma}qu^2 u_{xx} \\ &\quad + \frac{\Delta x\Delta t}{\Gamma}qu^2 u_{xt} + \frac{\Delta x^3}{6\Gamma}qu^2 u_{xxx} \end{aligned}$$

$$
\begin{aligned}
&-\frac{\Delta x^2 \Delta t}{2\Gamma} q u^2 u_{xxt} + \frac{\Delta x \Delta t^2}{2\Gamma} q u^2 u_{xtt} \\
&+ \frac{\Delta x \Delta t}{\Gamma} q u u_{xt} - \frac{\Delta x^2 \Delta t}{2\Gamma} q u u_{xxt} \\
&+ \frac{\Delta x \Delta t^2}{\Gamma} q u u_t u_{xt} + \frac{\Delta x^3 \Delta t}{6\Gamma} q u u_t u_{xxx} \\
&- \frac{\Delta x^2 \Delta t^2}{2\Gamma} q u u_t u_{xxt} + \frac{\Delta x \Delta t^3}{2\Gamma} q u u_t u_{xtt} \\
&+ \frac{\Delta x \Delta t^2}{2\Gamma} q u u_{tt} u_x - \frac{\Delta x^2 \Delta t^2}{4\Gamma} q u u_{tt} u_{xx} \\
&+ \frac{\Delta x \Delta t^3}{2\Gamma} q u u_{tt} u_{xt} + \frac{\Delta x^3 \Delta t^2}{12\Gamma} q u u_{tt} u_{xxx} \\
&- \frac{\Delta x^2 \Delta t^3}{4\Gamma} q u u_{tt} u_{xxt} + r\left(\frac{2\Delta x^3}{\Psi} - 1\right) u_{xxx}.
\end{aligned} \tag{33}
$$

It is clear that $(\Phi, \Gamma, \Psi) \to (\Delta t, \Delta x, 2\Delta x^3)$ as $(\Delta t, \Delta x) \to (0, 0)$. Thus, we get the local truncation error $T_j^n = \mathcal{O}(\Delta t + \Delta x)$. Similarly, one can show that the local truncation error for $\theta = 0$ is first-order in time and space. The principal part of the local truncation error for $\theta = 1/2$ is

$$
\begin{aligned}
T_j^n = &\left(\frac{\Delta t}{\Phi} - 1\right) u_t + \frac{\Delta t^2}{2\Phi} u_{tt} + \frac{\Delta t^3}{6\Phi} u_{ttt} \\
&+ \left(\frac{\Delta x}{\Gamma} - 1\right) q u^2 u_x - \frac{\Delta x^2}{2\Gamma} q u^2 u_{xx} \\
&+ \frac{\Delta x \Delta t}{2\Gamma} q u^2 u_{xt} + \frac{\Delta x^3}{6\Gamma} q u^2 u_{xxx} \\
&- \frac{\Delta x^2 \Delta t}{4\Gamma} q u^2 u_{xxt} + \frac{\Delta x \Delta t^2}{4\Gamma} q u^2 u_{xtt} \\
&+ \frac{\Delta x \Delta t^2}{2\Gamma} q u u_x u_{tt} - \frac{\Delta x^2 \Delta t^2}{4\Gamma} q u u_{xx} u_{tt} \\
&- \frac{\Delta x^2 \Delta t^2}{4\Gamma} q u u_t u_{xxt} + \frac{\Delta x \Delta t^3}{4\Gamma} q u u_t u_{xtt} \\
&+ \frac{\Delta x \Delta t^4}{8\Gamma} q u u_{tt} u_{ttx} + \left(\frac{2\Delta x^3}{\Psi} - 1\right) r u_{xxx} \\
&+ \frac{\Delta x^3 \Delta t^2}{2\Psi} r u_{ttxxx} + \frac{\Delta x^5}{2\Psi} r u_{xxxxx}.
\end{aligned} \tag{34}
$$

It is clear that $(\Phi, \Gamma, \Psi) \to (\Delta t, \Delta x, 2\Delta x^3)$ as $(\Delta t, \Delta x) \to (0, 0)$. Thus, we get the local truncation error $T_j^n = \mathcal{O}(\Delta t^2 + \Delta x)$. We deduce that the nonstandard scheme (10) is consistent since the local truncation T_j^n tends to zero as Δt and Δx tend to zero. The centerpiece for the theory of convergence of linear difference approximations of time-dependent partial differential equations is the Lax Equivalence Theorem [14]. Since the proposed scheme (10) is consistent and stable, it is convergent according to the Lax theorem.

4. Numerical Results

In this section we present some numerical experiments to test the accuracy and efficiency of the proposed NSFD scheme (10) for the numerical solution of the MKdV equation (1) over the solution domains $x_L \le x \le x_R$ and $0 \le t \le T$. The solution domain is divided into equal intervals with length $\Delta x = (x_R - x_L)/M$ in the direction of the spatial variable x and $\Delta t = T/M$ in the direction of time t such that $x_j = x_L + j\Delta x$, $j = 0, 1, \ldots, J$, $t_n = n\Delta t$, and $n = 0, 1, \ldots, M$. The initial condition $u_0(x)$ and boundary conditions are taken from the exact solution (4)

$$
u_0(x) = \sqrt{\frac{-6r}{q}} \tanh(x), \quad x \in [x_L, x_R] \tag{35}
$$

$$
u(x_L, t) = \sqrt{\frac{-6r}{q}} \tanh(x_L + 2rt), \quad t \ge 0 \tag{36}
$$

$$
u(x_R, t) = \sqrt{\frac{-6r}{q}} \tanh(x_R + 2rt), \quad t \ge 0, \tag{37}
$$

respectively, where $r = -\beta/2$, $\beta, q \in \mathbb{R}$. We use the following error norms:

$$
\begin{aligned}
L_\infty &= \max_{0 \le j \le J} \left| u_e(j) - u_a(j) \right|, \\
L_2 &= \sqrt{\sum_{j=0}^{M} \left[u_e(j) - u_a(j) \right]^2},
\end{aligned} \tag{38}
$$

to assess the performance of the NSFD scheme. Here u_e is the exact solution obtained from (4) and u_a is the approximate solution obtained from the NSFD scheme (10).

Table 1 represents L_∞ and L_2 errors of the NSFD scheme (10) and the standard finite difference scheme (11) at different times for the MKdV equation (1) with $\theta = 1$, $\beta = 0.001$, and $q = 50$ in the spatial domain $-15 \le x \le 15$ with $\Delta x = 2.0$ and $\Delta t = 0.01$. From the table we see that the NSFD scheme (10) is more accurate than standard finite difference scheme (11) in all cases. We obtained similar results for $\theta = 0$ and $\theta = 1/2$. The absolute error is defined by

$$
\left| u(x_j, t_n) - U_j^n \right| \tag{39}
$$

for the standard scheme (11) and the nonstandard scheme (10) are provided in Table 2 at various x and t values. From the experiments it is readily seen that our method is more accurate than the standard method.

Table 3 also measures the accuracy and the versatility of the NSFD scheme (10) with $\theta = 1$ by using the absolute error at the mesh point (x_j, t_n). From the table we see that the nonstandard scheme is very accurate and efficient. In addition, we notice that increasing the value of the nonlinear term q does not affect the accuracy for large spatial step sizes. Similar results are obtained for $\theta = 0$ and $\theta = 1/2$.

It is well known that numerical instabilities occur in many discrete models unless certain numerical conditions on spatial and temporal step sizes are satisfied. For examples,

Table 1: L_∞ and L_2 errors for MKDV equation (1) with $\theta = 1$, $q = 50$, and $\beta = 0.001$, on $-15 \le x \le 15$ with $\Delta t = 0.01$ and $\Delta x = 2.0$. N: nonstandard; S: standard.

T	$\mathbf{L_\infty}$ **(N)**	$\mathbf{L_\infty}$ **(S)**	$\mathbf{L_2}$ **(N)**	$\mathbf{L_2}$ **(S)**
2.0	$6.261E-6$	$1.466E-5$	$8.007E-6$	$1.560E-5$
4.0	$1.251E-5$	$2.912E-5$	$1.601E-5$	$3.104E-5$
6.0	$1.872E-5$	$4.339E-5$	$2.402E-5$	$4.630E-5$
8.0	$2.492E-5$	$5.747E-5$	$3.202E-5$	$6.140E-5$
10.0	$3.111E-5$	$7.137E-5$	$4.002E-5$	$7.632E-5$

Table 2: Absolute errors for MKDV equation (1) with $\theta = 1$, $q = 0.3$, and $\beta = 0.1$ on $-15 \le x \le 15$ with $\Delta t = 0.01$ and $\Delta = x = 1.0$. N: nonstandard; S: standard.

x	t	N	S
-10	0.2	$1.203E-10$	$8.241E-10$
	0.5	$2.897E-10$	$2.251E-09$
	0.8	$4.466E-10$	$3.949E-09$
0	0.2	$1.983E-02$	$1.448E-02$
	0.5	$4.953E-02$	$3.636E-02$
	0.8	$7.915E-02$	$5.833E-02$
10	0.2	$3.298E-10$	$6.636E-10$
	0.5	$9.267E-10$	$1.969E-09$
	0.8	$1.668E-09$	$3.761E-09$

Table 3: Nonlinear effect: absolute errors for $\theta = 1$ and $\beta = 0.1$ on $-15 \le x \le 15$ with $\Delta x = 1.0$ and $T = 1$ with $\Delta t = 0.01$.

x	t	$q = 0.3$	$q = 10$	$q = 100$
-10	0.2	$1.203E-10$	$2.083E-11$	$6.588E-12$
	0.5	$2.897E-10$	$5.018E-11$	$1.587E-11$
	0.8	$4.466E-10$	$7.734E-11$	$2.446E-11$
0	0.2	$1.983E-02$	$3.434E-03$	$1.086E-03$
	0.5	$4.953E-02$	$8.579E-03$	$2.713E-03$
	0.8	$7.915E-02$	$1.371E-02$	$4.335E-03$
10	0.2	$3.298E-10$	$5.713E-11$	$1.807E-11$
	0.5	$9.267E-10$	$1.605E-10$	$5.075E-11$
	0.8	$1.668E-09$	$2.889E-10$	$9.136E-11$

forward and backward Euler and central difference for decay equations produce numerical instability for large step sizes [11]. Figure 1 compares the numerical solution of nonstandard scheme (10) and the exact wave solution (4). This picture represents the result of an integration with $q = 0.3$, $\beta = 0.1$, and $\theta = 1/2$, over the spatial domain $-15 \le x \le 15$ and temporal interval $0 \le t \le 1$ with spatial step size $\Delta x = 0.5$ and temporal step size $\Delta t = 0.001$. From the figure we see that nonstandard scheme well simulates the exact solution without showing any numerical instabilities. Figures 2 and 3 represent the L_∞ and L_2 errors and Figures 4 and 5 represent the absolute errors for various spatial step sizes of the NSFD scheme (10) and the standard finite difference scheme (11) for the MKdV equation (1) with the same set of parameters of Figure 1. From the figures it is obvious that the NSFD scheme is better than the standard scheme in the numerical precision for large spatial step size, but it is inferior for small spatial step size. We obtained similar results for $\theta = 0$ and $\theta = 1$.

A final issue to consider is the effect of the third-order dispersion coefficient $r = -\beta/2$ in (1). We see that for various values of β, as shown in Table 4, dispersion-dominated solution demonstrated that the error is increased at the place where the shock wave occurs. Both Tables 3 and 4 show that the errors are concentrated in the spatial region where there are steep solutions.

5. Conclusion

It is well known that explicit finite difference models for solution of differential equation require restriction on step size to prevent the numerical instabilities. For this reason, small step sizes are used to ensure the numerical stability. This causes

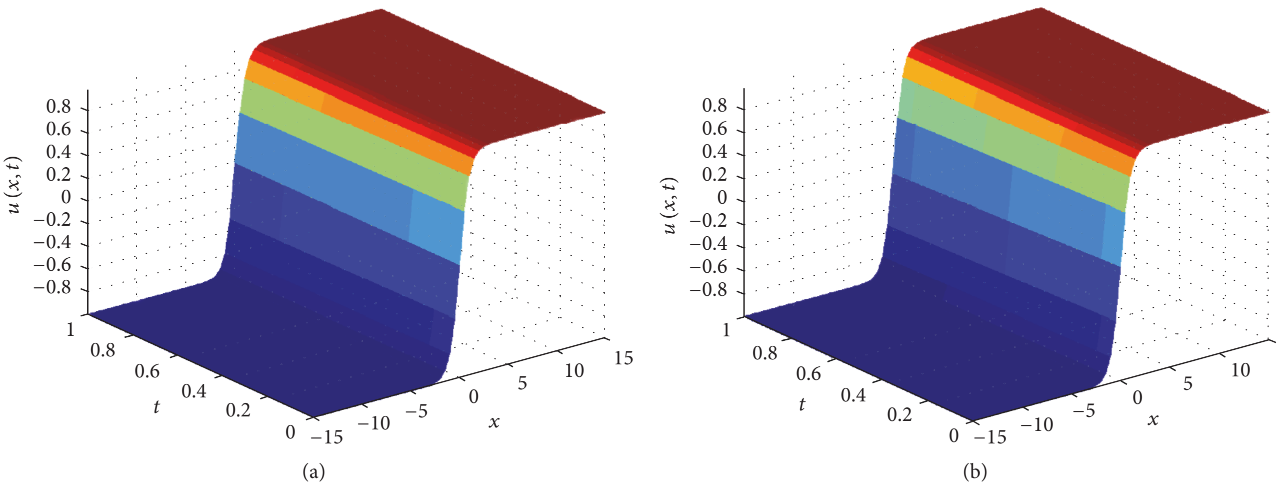


FIGURE 1: Surface for $\theta = 1/2$. Nonstandard (a). Exact (b).

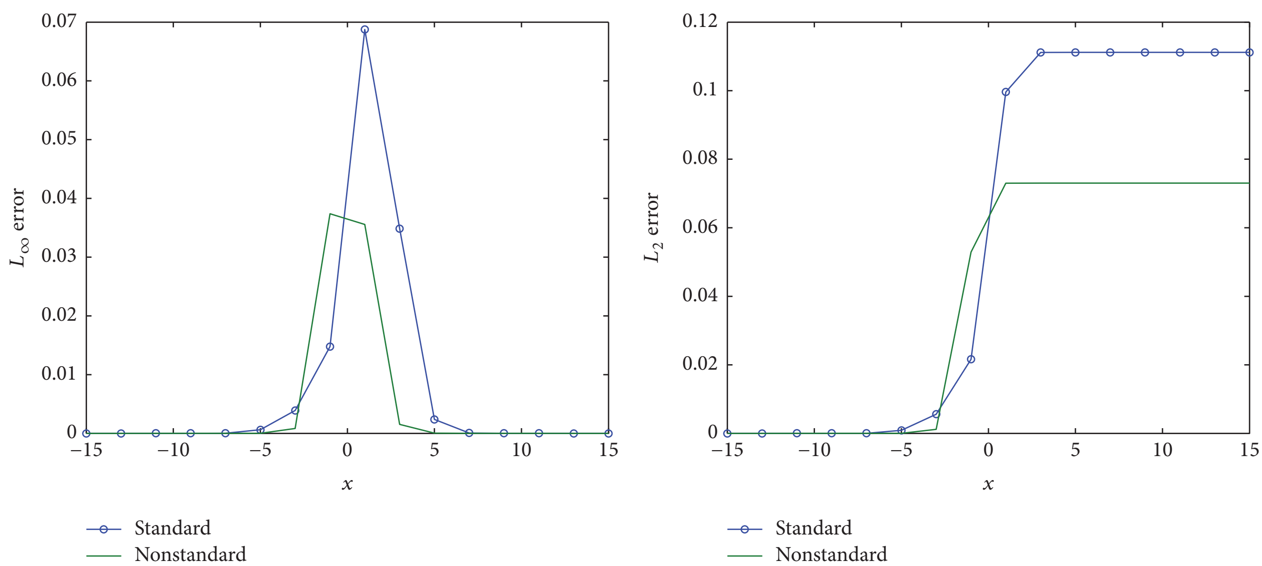


FIGURE 2: $\theta = 1/2$, L_∞ and L_2 errors for the NSFD scheme (10), and standard scheme (11): $\Delta x = 2$.

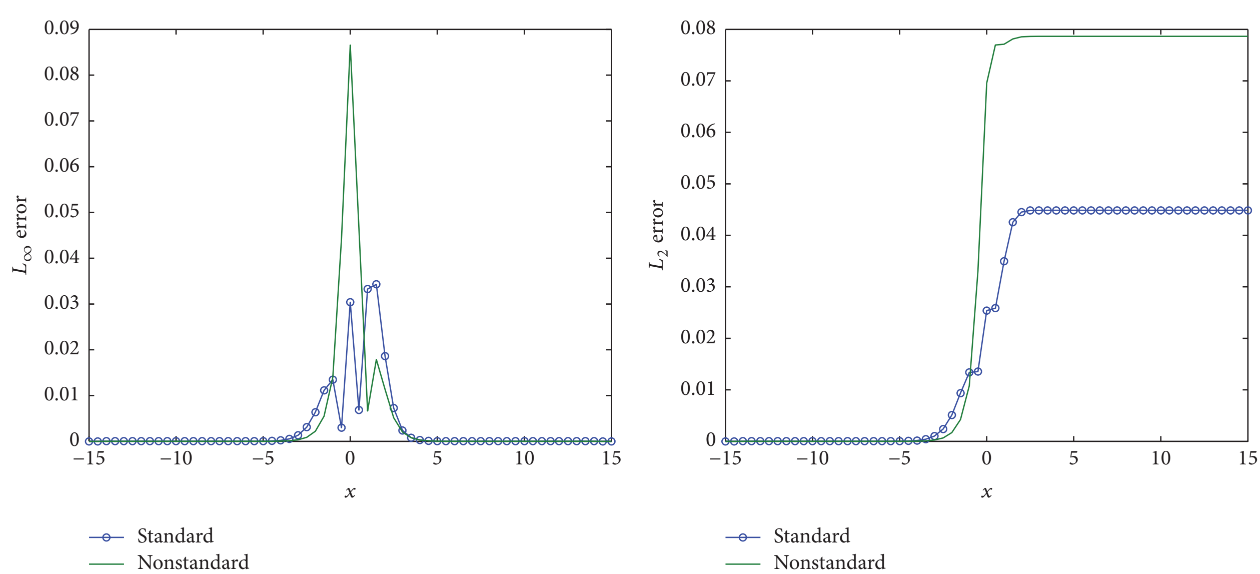
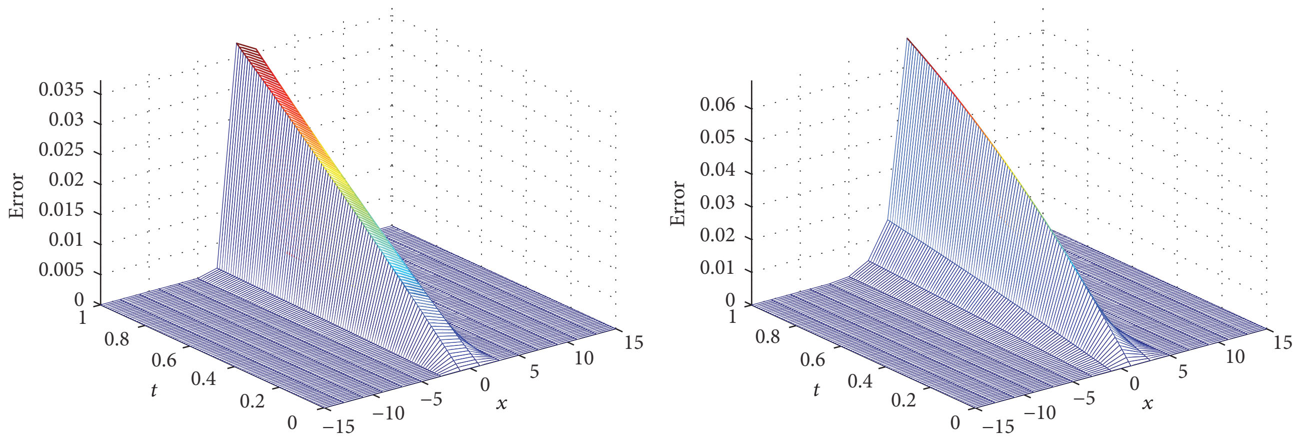
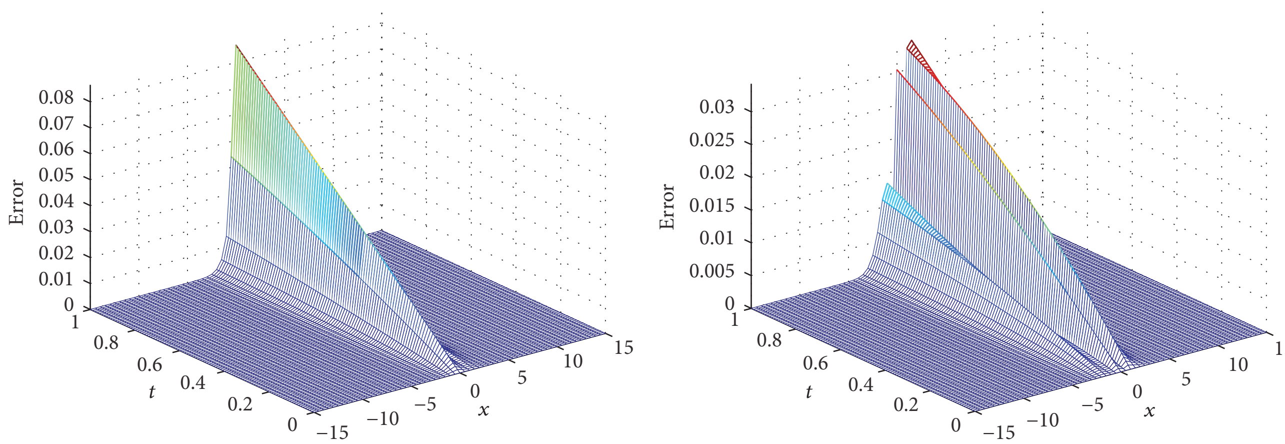


FIGURE 3: $\theta = 1/2$, L_∞ and L_2 errors for the NSFD scheme (10), and standard scheme (11): $\Delta x = 0.5$.



(a) (b)

Figure 4: $\theta = 1/2$, absolute errors for the NSFD scheme (10) (a) and standard scheme (11) (b). $\Delta t = 0.01$, $q = 0.3$, $\beta = 0.1$, and $\Delta x = 2$.



(a) (b)

Figure 5: $\theta = 1/2$, absolute errors for the NSFD scheme (10) (a) and standard scheme (11) (b). $\Delta t = 0.01$, $q = 0.3$, $\beta = 0.1$, and $\Delta x = 0.5$.

Table 4: Dispersion effect: absolute errors for $\theta = 1$ and $q = 0.3$ on $-15 \leq x \leq 15$ with $\Delta x = 1.0$ and $T = 1$ with $\Delta t = 0.01$.

x	t	$\beta = 0.001$	$\beta = 1.0$	$\beta = 10$
	0.2	$1.232E-13$	$3.060E-09$	$1.609E-08$
−10	0.5	$3.079E-13$	$5.364E-09$	$4.417E-09$
	0.8	$4.925E-13$	$6.117E-09$	$1.305E-09$
	0.2	$1.982E-05$	0.619	9.496
0	0.5	$4.957E-05$	1.448	9.609
	0.8	$7.931E-05$	2.080	9.310
	0.2	$3.047E-13$	$2.214E-08$	$8.238E-04$
10	0.5	$7.626E-13$	$2.011E-07$	$0.102E-04$
	0.8	$1.221E-12$	$1.230E-06$	$0.236E-05$

many grid points and hence round-off errors in numerical computations. Moreover the use of small step size is computationally expensive. In this study a nonstandard finite difference (NSFD) model is proposed for the numerical solution of the modified Korteweg-de Vries (MKdV) equation based on kink soliton solution. Local truncation error is discussed. Linear stability analysis is performed for the linearized equation. The numerical results obtained by the NSFD scheme is compared to the exact solution and a standard finite difference scheme. Proposed scheme indicates that NSFD scheme shows better performance for large step sizes. The effect of nonlinear term and the dispersive term is also studied numerically. The numerical results that have been obtained in this paper exhibit that the coefficient of the nonlinear term does not change the numerical results much. Tables and figures illustrate that the NSFD scheme can be a robust tool for numerical solution of various nonlinear problems such as 2D-KdV.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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