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An optimal variational iteration method

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ABSTRACT

The variational iteration method is studied in the present work. The classical variational iteration method is improved and extended by introducing a new concept of a *convergence accelerating parameter*. A rigorous approach is later proposed for optimally determining the value of the convergence accelerating parameter. The consequences of the novel *optimal variational iteration method* are discussed through an application.

1. Introduction

The search for a better and easy to use tool for the solution of nonlinear equations illuminating the nonlinear phenomena of our life is intensifying.

Various methods, therefore, have been proposed for finding approximate solutions. One of the techniques that has become very popular recently is the variational iteration method, proposed in 1999 by He [1,2]. The method introduces a reliable and effective process for a wide variety of scientific and engineering applications, avoiding the restrictions and limitations of perturbation techniques. Numerous nonlinear problems have recently been treated by the method; see for instance, amongst many other references, [3–5].

We in the present work investigate the variational iteration technique with the aim of improving it further. For this purpose, a new idea of a convergence accelerating parameter in the classical variational iteration method is introduced. A novel approach is also proposed for optimally identifying the convergence accelerating parameter. An example is presented to illustrate the efficiency, effectiveness and accuracy of the new approach.

In the rest of the work, Section 2 gives the basis of the classical variational iteration method. A new approach is outlined in Section 3. Section 4 contains an illustrative example and this is followed by the conclusions in Section 5.

2. The variational iteration method

In this section, the basic ideas of the traditional variational iteration method are introduced. A description of the method [2] is given for handling the general nonlinear problem

$$L(u) + N(u) = g(t), \tag{2.1}$$

where u is the function to be solved, L is a linear operator, N is a nonlinear operator and $g(t)$ is a known analytic function. According to He's classical variational iteration scheme, we can construct a correction functional concerning (2.1) as follows:

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda(t)\{L(u_n(\tau)) + N(\tilde{u}_n(\tau)) - g(\tau)\}, \quad n \geq 0, \tag{2.2}$$

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where λ is a general Lagrange multiplier which can be identified optimally via variational theory. Here $\tilde{u}_n$ is considered as a restricted variation which means that $\delta\tilde{u}_n = 0$. Therefore, we first determine the Lagrange multiplier λ that will be identified optimally via integration by parts. The successive approximation $u_n(t)$, $n \geq 0$, of the solution $u(t)$ will be readily obtained upon using the Lagrange multiplier obtained and by using any selective function $u_0(t)$. The zeroth approximation $u_0(t)$ may be selected as any function that just satisfies, at least, the initial and boundary conditions. With λ determined, then several approximations $u_n(t)$, $n \geq 0$, follow immediately. Consequently, the exact solution may be obtained as

$$u(t) = \lim_{n\to\infty} u_n(t). \tag{2.3}$$

The variational iteration method outlined depends on the proper selection of the initial approximation $u_0(t)$. For approximation purposes, we approximate the solution of the nonlinear problem (2.1) by the nth-order term $u_n(t)$.

3. The optimal variational iteration method

It is plausibly possible to approach the variational iteration method (2.2) from a different mathematical perspective. To this end, let us rewrite (2.1) in the form

$$L(u) = a\{L(u) + N(u) - g(t)\} + L(u), \tag{3.4}$$

where a is a constant, which we name in what follows as *the convergence accelerating parameter*. After inverting the linear operator L, (3.4) can be converted into the following iterative formula:

$$u_{k+1}(t) = aL^{-1}\{L(u_k) + N(u_k) - g(t)\} + u_k(t). \tag{3.5}$$

The presence of the parameter a in (3.5) plays a very crucial role that may not be handled via the Lagrange multiplier of the classical variational iteration method in (2.2), for instance, in the case of failure of convergence of the classical variational iteration method. It is clear that when the parameter a is set to minus or plus unity, (3.5) turns out to be the classical variational iteration method (2.2); see the following example. The approach of (3.5), on the other hand, gives a much more flexible way of obtaining the successive iterations of the problem (3.4) through the variational iteration method in which an accelerating parameter a is plugged in. From a logical point of view, since the special choice of the convergence accelerating parameter a can be thought not to influence the solution, the region of validity of the parameter a might hence be identified by drawing *constant a curves* for particular values of the solution. This can be achieved for every physical problem by picking out a non-zero fixed value of the solution, plotting it against the parameter a and monitoring the interval of a for which only a slight change in the value is observed; see the following example for an implementation. A better and optimal value of the convergence accelerating parameter a can be determined at the order of approximation by the residual error

$$\mathrm{Res}(a) = \int_\alpha^\beta \{L(u_k(t)) + N(u_k(t)) - g(t)\}^2 \mathrm{d}t, \tag{3.6}$$

in which an assumption is imposed: that we wish to find the solution of (2.1) in the interval $[\alpha, \beta]$. One can easily minimize (3.6) by imposing the requirement

$$\frac{\mathrm{d\,Res}}{\mathrm{d}a} = 0.$$

Having identified the optimal convergence accelerating parameter in this fashion, we call the new iterative technique (3.5) together with (3.6) the *optimal variational iteration technique*.

It should be recalled here that if the exact square residual error $\mathrm{Res}(a)$ defined by (3.6) needs too much CPU time in practice, then to avoid the time-consuming computation, the constant a curves idea introduced above can be made use of.

4. An illustrative example

To illustrate the validity of the optimal variational iteration method outlined in Section 3, we consider the basic nonlinear initial-value problem

$$u' + u^2 = 1, \quad u(0) = 0, \quad t \in [0, 2]. \tag{4.7}$$

In compliance with the new idea, taking into account the linear operator in (4.7) $L = \frac{\mathrm{d}}{\mathrm{d}t}$ and further selecting the initial approximation as $u_0(t) = 0$, the variational iteration scheme (3.5) becomes

$$u_{k+1}(t) = u_k(t) + a\int_0^t \{u_k'(\tau) + u_k^2(\tau) - 1\}\mathrm{d}\tau. \tag{4.8}$$

We observe that $a = -1$ in (4.8) corresponds to the classical variational iteration method of He [2]. To work out the region of validity of the convergence acceleration parameter a, we plot the values of $u_6(0.1)$, $u_6(1)$ and $u_6(2)$ in Fig. 1, where the subscript is used to denote the order of approximation. It appears that a should at least lie within the interval $[-1, -1/2]$. For

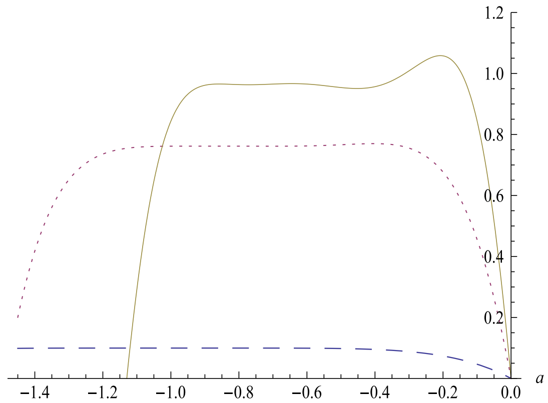

Fig. 1. The constant a curves for Eq. (4.7) from the sixth-order optimal variational iteration method: dashed curve: $u_6(t = 0.1)$, dotted curve: $u_6(t = 1)$ and straight curve: $u_6(t = 2)$.

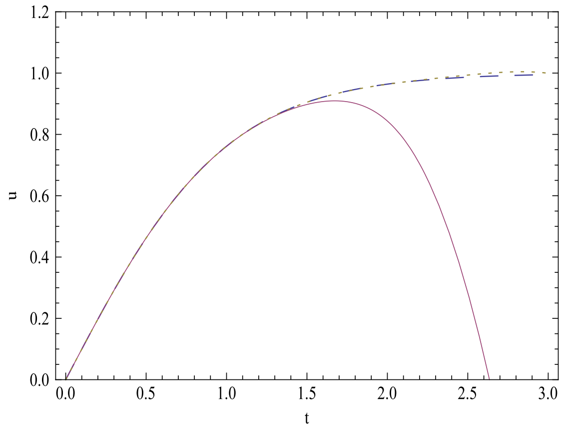

Fig. 2. Comparison of exact and approximate solutions for Eq. (4.7). The thick curve corresponds to the sixth-order classical variational iteration technique, the dashed curve to the exact solution and the dotted curve to the sixth-order optimal variational iteration method.

the best possible value within this region, the residual error at the sixth iteration was evaluated from (3.6) which gives rise to the optimal value a of $a = -0.8114$, resulting in a residual error 1.33×10^{-6}. On the other hand, the classical variational iteration method results in a residual error of 0.06687 at this order, which is much larger than the value found from the new iteration method introduced by us. Fig. 2 reveals that the classical variational iteration approximation actually diverges sharply from the exact solution as compared with the solution obtained via the optimal variational iteration method. We further provide an L^2 error defined as

$$\left[\int_0^2 (u_e(t) - u_6(t))^2 \mathrm{d}t\right]^{1/2}, \tag{4.9}$$

where $u_e(t) = \tanh t$ is the exact solution of (4.7). Error formula (4.9) produces 0.04 and 0.0002 respectively from the classical and optimal variational iteration methods. This constitutes plausible evidence for the validity and accuracy of our new variational iterative method (3.5).

In order to reinforce the efficiency of the new optimal variational iterative method, we now consider example (4.7) in the semi-infinite region $[0, \infty)$. With this purpose, we choose $L = \frac{\mathrm{d}}{\mathrm{d}t} + 1$ and $u_0(t) = 1 - \mathrm{e}^{-t}$. Then, the following variational iteration formula is obtained:

$$u_{k+1}(t) = u_k(t) + a \int_0^t \mathrm{e}^{(\tau - t)}\{u_k'(\tau) + u_k^2(\tau) - 1\}\mathrm{d}\tau, \tag{4.10}$$

which obviously corresponds to the classical variational iteration method provided that $a = -1$, giving a residual error of 0.0223 at the third-order approximation. However, the residual error in the semi-infinite domain ((3.6) with $b = \infty$) results in an optimal convergence accelerating value a of $a = -0.6716$, leading to the residual error 0.000173, which is favorably smaller. The third-order variational iteration solutions are compared in Fig. 3 with the exact solution. Fig. 3 clearly shows the effectiveness and accuracy of the optimal variational iteration technique introduced here. In conclusion, the new variational

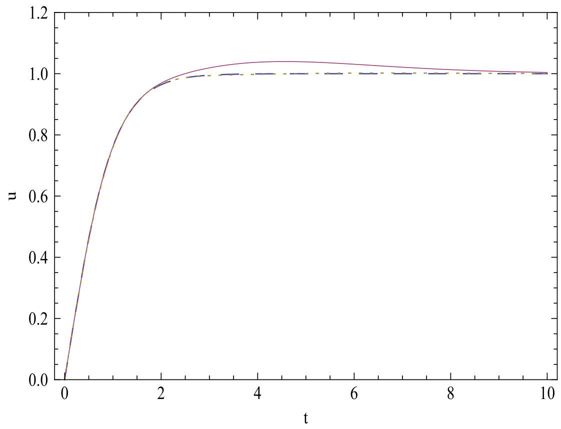

Fig. 3. Comparison of exact and approximate solutions for Eq. (4.7). The thick curve corresponds to the sixth-order classical variational iteration technique, the dashed curve to the exact solution and the dotted curve to the sixth-order optimal variational iteration method.

iteration approach seems to be more promising for finding better approximate solutions of nonlinear equations arising in science and engineering.

5. Concluding remarks

In this work, the variational iteration method has been analyzed with the aim of modifying and improving it to produce better approximates. A new *optimal variational iteration method* has been introduced which involves a *convergence accelerating parameter*. Additionally, a new concept of *constant parameter curves* has been brought to attention, that gives a strong clue as to the validity region of the convergence accelerating parameter. The optimal value of the convergence accelerating parameter can be shown to be effectively determined at the iteration considered from the residual error of the solution sought. The example presented justifies the efficiency and accuracy of the new approach remarkably, and can be easily adopted for the solution of nonlinear algebraic equations as well as the ordinary and partial differential equations of science and engineering.

References

[1] J.H. He, Variational iteration method—a kind of nonlinear analytical technique: some examples, Internat. J. Non-Linear Mech. 34 (1999) 699–708.
[2] J.H. He, Variational iteration method—some recent results and new interpretations, J. Comput. Appl. Math. 207 (2007) 3–17.
[3] A.M. Wazwaz, The variational iteration method for rational solutions for KdV, Burgers and cubic Boussinesq, J. Comput. Appl. Math. 1 (2007) 18–23.
[4] S. Momani, Z. Odibat, Comparison between homotopy perturbation method and the variational iteration method for linear fractional partial differential equations, Comput. Math. Appl. 54 (2007) 910–919.
[5] S. Das, Analytical solution of a fractional diffusion equation by variational iteration method, Comput. Math. Appl. 57 (2009) 361–366.