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A category approach to relation preserving functions in rough set theory ☆

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ABSTRACT

The category **Rel** whose objects are all pairs (U, r), where r is a relation on a universe U, and whose morphisms are relation-preserving mappings is a canonical example in category theory. One of the convenient categories for rough set systems on a single universe is **Rel** since the objects of **Rel** are approximation spaces. The morphisms of a ground category **dfTex** whose objects are textures can be characterized by definability. Therefore, we particularly investigate a textural counterpart of the category **Rel** denoted by **diRel** of textural approximation spaces and direlation preserving difunctions. In this respect, we prove that **diRel** is a topological category over **dfTex** and **Rel** is a full subcategory of **diRel**. In view of the textural arguments, we show that the preimage of a definable subset of an approximation space with respect to a relation preserving function is also definable in the category **Rere** of reflexive relations. Furthermore, we denote the category of all information system homomorphisms and all information systems by **IS** and we show that the category **ISO** of all information system homomorphisms and all object-irreducible information systems where the attribute functions are surjective is embeddable into **Rel**.

0. Introduction

It is well-known that the category **Rel** whose objects are all pairs (U, r), where r is a relation on a universe U, and whose morphisms are relation-preserving mappings is a canonical example in category theory [1]. In rough set theory, the pair (U, r), as an object of **Rel**, is called an approximation space. Recently, Juan Lu et al. studied a different version of this category denoted by M-**IndisSp** whose objects are M-indiscernibility spaces and the morphisms M-equivalence relation-preserving mappings where M is a fixed index set for the families of equivalence relations on a given universe [26]. If M is countable, then M-indiscernibility spaces are multiple-source approximation systems defined by Khan and Banerjee in [23]. Essentially, if the index set M is a singleton, then M-**IndisSp** turns into a category of approximation spaces and equivalence relations which is a full subcategory of **Rel**. In fact, an M-indiscernibility space is a dynamic relational system defined by Pagliani in [27]. Some applications on dynamic relational systems can be found in [22,37,38]. Recently, there have been developments in the subject of textural rough sets [11–15]. A ground category in texture space theory is **dfTex** whose objects are textures and morphisms are difunctions. Difunctions can be characterized using the prime concept of definability of rough set theory. That is, for any two textures $(U, \mathcal{U})$ and $(V, \mathcal{V})$, a direlation $(r, R) : (U, \mathcal{U}) \to (V, \mathcal{V})$ is a morphism in **dfTex** if and only if every subset $A \in \mathcal{U}$ is (r, R)-definable [8]. In the context of rough sets, this result can be stated as a fact that a relation $r : U \to V$ is a function if and only if every subset of V is r-definable. Recall that the presections with respect

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to direlations are natural generalizations of lower and upper approximations based on relations. Hence, from the textural point of view, one can observe that the lower and upper approximations of a set A can be given using the point-free formulations $\underline{app}_r(A) = U \setminus r^{-1}(V \setminus A)$ and $\overline{app}_r(A) = r^{-1}(A)$ for any subset $A \subseteq V$ where $r: U \to V$ is a relation [15]. These arguments help us to focus our attention on a new category denoted by **diRel** of textural approximation spaces and direlation preserving difunctions. One of the important full subcategories of **Rel** is the category of sets and reflexive relations denoted by **Rere** [1]. A textural approach shows that the preimage of a definable subset of an approximation space with respect to a relation preserving function is also definable in the category **Rere**. On the other hand, invariance of upper and lower approximations under information system homomorphisms are studied in [36]. Here, we note that information systems and information system homomorphisms form a category denoted by **IS**. An object function of an information system homomorphism between information systems is a relation preserving function with respect to equivalence relations determined by the attribute sets. Hence, definable sets of information systems can be also considered using the object functions of information system homomorphisms.

Note that the category **REL** of sets and relations is subject to rough set models on two universes and it is isomorphic to the category **R-APR** of power sets and approximation operators [16,30]. Hence, the categories **Rel** and **REL** have different directions. Some works on categorical results in rough set theory can be found in [2,3,16–19,25,26]. For the basic categorical results and terminology, we refer to [1].

This paper is organized as follows. In Section 1, we recall the motivation, and necessary concepts and results related to textures from [6–9,15]. Section 2 is devoted to direlation preserving difunctions. Here, we show that textural approximation spaces and direlations preserving difunctions form a category denoted by **diRel**. We prove that **Rel** is isomorphic to a full subcategory of **diRel**. Further, we show that **diRel** is a topological category and hence, it has products and sums. In Section 3 we discuss textural definability and bicontinuity. In particular, we prove that under pre-images of relation preserving difunctions, textural definability is preserved. Sections 4 and 5 are devoted to approximation spaces and information systems, respectively. First, we give some basic results related to definability and we discuss the category **Rel** and the category **IS** of information systems and information system homomorphisms.

1. Textures

Let U be a set. Then $\mathcal{U} \subseteq \mathcal{P}(U)$ is called a *texturing* of U, and $(U, \mathcal{U})$ is called a *texture space*, or simply a *texture* [6], if

(i) $(\mathcal{U}, \subseteq)$ is a complete lattice containing U and $\emptyset$, which has the property that arbitrary meets coincide with intersections, and finite joins coincide with unions,

(ii) $\mathcal{U}$ is completely distributive, that is, for all index set I, and for all $i \in I$, if J_i is an index set and if $A_i^j \in \mathcal{U}$, then we have

$$\bigcap_{i\in I} \bigvee_{j\in J_i} A_i^j = \bigvee_{\gamma\in\prod_i J_i} \bigcap A^i_{\gamma(i)}.$$

(iii) $\mathcal{U}$ separates the points of U, that is, given $u_1 \neq u_2$ in U there exists $A \in \mathcal{U}$ such that $u_1 \in A$, $u_2 \notin A$, or $u_2 \in A$, $u_1 \notin A$.

A complementation on $(U, \mathcal{U})$ is a mapping $c_U : \mathcal{U} \to \mathcal{U}$ satisfying the conditions

$$\forall A \in \mathcal{U}, \quad c_U^2(A) = A,$$
$$\forall A, B \in \mathcal{U}, \quad A \subseteq B \quad \Rightarrow \quad c_U(B) \subseteq c_U(A).$$

Then the triple $(U, \mathcal{U}, c_U)$ is said to be *a complemented texture space*.

In a texture $(U, \mathcal{U})$, p-sets and q-sets are defined by

$$P_u = \bigcap\{A \in \mathcal{U} \mid u \in A\} \quad \text{and} \quad Q_u = \bigvee\{A \in \mathcal{U} \mid u \notin A\},$$

respectively. The condition (ii), that is, the complete distributivity of $(U, \mathcal{U})$ is equivalent to the following statement [10].

(ii)′ For $A, B \in \mathcal{U}$, if $A \not\subseteq B$ then there exists $u \in U$ with $A \not\subseteq Q_u$ and $P_u \not\subseteq B$.

A nonempty set $A \in \mathcal{U}$ is *a molecule* if $\forall B, C \in \mathcal{U}$, $A \subseteq B \cup C \Rightarrow A \subseteq B$ or $A \subseteq C$. Clearly, p-sets are molecules of a texture space. A texture space $(U, \mathcal{U})$ is called *simple* if all molecules of the space are p-sets.

A trivial example of a texture is the pair $(U, \mathcal{P}(U))$ where $\mathcal{P}(U)$ is the power set of U. It is called *a discrete texture*. Clearly, $(U, \mathcal{P}(U))$ is simple and for $u \in U$ we have

$$P_u = \{u\} \quad \text{and} \quad Q_u = U \setminus \{u\}$$

and $c_U : \mathcal{P}(U) \to \mathcal{P}(U)$ is the ordinary complementation on $(U, \mathcal{P}(U))$ defined by $c_U(A) = U \setminus A$ for all $A \in \mathcal{P}(U)$. However, the basic motivation of textures is in fact, the natural correspondence between the fuzzy lattices and simple textures [7].

Let $(L, \leq, ')$ be a fuzzy lattice (Hutton algebra), that is, a complete, completely distributive lattice with an order reversing involution "$'$". Recall that $0 \neq m \in L$ is *join-irreducible*, if

$$\forall a, b \in L, \quad m \leq a \vee b \quad \Rightarrow \quad m \leq a \quad \text{or} \quad m \leq b.$$

Consider the sets

$M_L = \{m \mid m \text{ is join-irreducible in } L\}$,

$\mathcal{M}_L = \{\widehat{a} \mid a \in L\}$, and

$\widehat{a} = \{m \mid m \in M_L \text{ and } m \leq a\}$, for all $a \in L$.

Then the mapping $\widehat{\ } : L \to \mathcal{M}_L$ defined by $\forall a \in L,\ a \mapsto \widehat{a}$ is a lattice isomorphism and the triple $(M_L, \mathcal{M}_L, c_{M_L})$ is a complemented simple texture space which is called *a Hutton texture*. Here the complementation $c_{M_L} : \mathcal{M}_L \to \mathcal{M}_L$ is defined by

$$\forall a \in L, \quad c_{M_L}(\widehat{a}) = \widehat{a}'.$$

For $L = [0, 1]$, we may also consider the Hutton texture. The family $\mathcal{M} = \{(0, r] \mid r \in [0, 1]\}$ is a texture on $M = (0, 1]$ which is called the Hutton texture. Clearly, $\mathcal{M}$ is closed under arbitrary intersections. Then it is easy to see that it is a complete lattice with respect to set inclusion. Now let us take $(0, r], (0, s] \in \mathcal{M}$ where $(0, r] \not\subseteq (0, s]$. Then we have $s < r$. Choose a point $t \in [0, 1]$ where $s < t < r$. Since we have $P_t = Q_t = (0, t]$, we may write that $(0, r] \not\subseteq Q_t$ and $P_t \not\subseteq (0, s]$. Further, $\mathcal{M}$ is simple and the complementation $c_M : \mathcal{M} \to \mathcal{M}$ is defined by $\forall r \in (0, 1],\ c_M(0, r] = (0, 1 - r]$.

On the other hand, recall that a fuzzy subset α of U is a membership function $\alpha : U \to [0, 1]$. We denote the set of all fuzzy subsets of U by $\mathcal{F}(U)$. It is well known that $\mathcal{F}(U)$ is also a Hutton algebra with the pointwise ordering

$$\forall u \in U, \quad \alpha \leq \beta \quad \Longleftrightarrow \quad \alpha(u) \leq \beta(u)$$

and the order reversing involution $\alpha'(u) = 1 - \alpha(u)$. Here the join and the meet of fuzzy sets are considered as

$$(\alpha \wedge \beta)(u) = \alpha(u) \wedge \beta(u) \quad \text{and} \quad (\alpha \vee \beta)(u) = \alpha(u) \vee \beta(u)$$

for all $\alpha, \beta \in \mathcal{F}(U)$. Now consider the fuzzy points u_s of $\mathcal{F}(U)$ defined by

$$u_s(z) = \begin{cases} s, & \text{if } z = u \\ 0, & \text{if } z \neq u \end{cases}$$

for all $z \in U$. Let us take the sets:

$$\widehat{\alpha} = \{u_s \mid u_s \leq \alpha\},$$
$$\mathcal{M}_{\mathcal{F}(U)} = \{\widehat{\alpha} \mid \alpha \in \mathcal{F}(U)\}, \quad \text{and}$$
$$M_{\mathcal{F}(U)} = \{u_s \mid u_s \text{ is a fuzzy point in } \mathcal{F}(U)\}.$$

Then under the lattice isomorphism $\widehat{\ } : \mathcal{F}(U) \to \mathcal{M}_{\mathcal{F}(U)}$, the corresponding texture space will be $(M_{\mathcal{F}(U)}, \mathcal{M}_{\mathcal{F}(U)})$. Every fuzzy point u_s can be regarded as an ordered pair $(u, s) \in U \times (0, 1]$ and then we may write that $\widehat{\alpha} = \{(u, s) \mid s \leq \alpha(u)\}$. Therefore, it can be shown that the texture $(M_{\mathcal{F}(U)}, \mathcal{M}_{\mathcal{F}(U)})$ is isomorphic to the product texture

$$\big(U \times M,\ \mathcal{P}(U) \otimes \mathcal{M},\ c_{U \times M}\big)$$

of $(U, \mathcal{P}(U), c_U)$ and $(M, \mathcal{M}, c_M)$ while the complementation mapping is defined by

$$c_{U \times M}(\widehat{\alpha}) = \widehat{1 - \alpha}$$

for all $\alpha \in \mathcal{F}(U)$ [7]. Meanwhile, we immediately have that $\widehat{u_s} = \{u\} \times (0, s] = P_{(u,s)}$. This means that the texture $(U \times M,\ \mathcal{P}(U) \otimes \mathcal{M},\ c_{U \times M})$ is an alternative point-based setting for the fuzzy lattice $\mathcal{F}(U)$.

2. Direlations and rough set approximation operators

Direlations are morphisms between textures which are compatible with the structures of textures [8]. Let $(U, \mathcal{U})$, $(V, \mathcal{V})$ be texture spaces and let us consider the product texture $\mathcal{P}(U) \otimes \mathcal{V}$ of the texture spaces $(U, \mathcal{P}(U))$ and $(V, \mathcal{V})$ and denote the p-sets and the q-sets of $\mathcal{P}(U) \otimes \mathcal{V}$ by $\overline{P}_{(u,v)}$ and $\overline{Q}_{(u,v)}$, respectively (for product textures, see [7]). Note that here we take the discrete texture $\mathcal{P}(U)$ on U instead of the original texture $\mathcal{U}$. Further, we have

$$\overline{P}_{(u,v)} = \{u\} \times P_v \quad \text{and} \quad \overline{Q}_{(u,v)} = \big(\big(U \setminus \{u\}\big) \times V\big) \cup (U \times Q_v).$$

However, the p-sets and q-sets are considered with respect to texture $\mathcal{U}$ in the following conditions. This kind of choice on textures not only gives a natural way for the construction of morphisms of textures, but also provides a reasonable generalization for the approximation operators in rough set theory.

Now let us consider the following concepts.

(i) $r \in \mathcal{P}(U) \otimes \mathcal{V}$ is called a *relation* from $(U, \mathcal{U})$ to $(V, \mathcal{V})$ if it satisfies
R1 $r \not\subseteq \overline{Q}_{(u,v)}, P_{u'} \not\subseteq Q_u \Longrightarrow r \not\subseteq \overline{Q}_{(u',v)}$.
R2 $r \not\subseteq \overline{Q}_{(u,v)} \Longrightarrow \exists u' \in U$ such that $P_u \not\subseteq Q_{u'}$ and $r \not\subseteq \overline{Q}_{(u',v)}$.
(ii) $R \in \mathcal{P}(U) \otimes \mathcal{V}$ is called a *corelation* from $(U, \mathcal{U})$ to $(V, \mathcal{V})$ if it satisfies
CR1 $\overline{P}_{(u,v)} \not\subseteq R, P_u \not\subseteq Q_{u'} \Longrightarrow \overline{P}_{(u',v)} \not\subseteq R$.
CR2 $\overline{P}_{(u,v)} \not\subseteq R \Longrightarrow \exists u' \in U$ such that $P_{u'} \not\subseteq Q_u$ and $\overline{P}_{(u',v)} \not\subseteq R$.

A pair (r, R), where r is a relation and R a corelation from $(U, \mathcal{U})$ to $(V, \mathcal{V})$ is called a *direlation from* $(U, \mathcal{U})$ *to* $(V, \mathcal{V})$. Note that if (r, R) is a direlation from the texture $(U, \mathcal{P}(U))$ to $(V, \mathcal{P}(V))$, then r and R are ordinary relations from U to V, that is, $r, R \subseteq U \times V$ since $\mathcal{P}(U) \otimes \mathcal{P}(V) = \mathcal{P}(U \times V)$.

The identity direlation (i, I) on $(U, \mathcal{U})$ is defined by

$$i = \bigvee \{\overline{P}_{(u,u)} \mid u \in U\} \quad \text{and} \quad I = \bigcap \{\overline{Q}_{(u,u)} \mid u \in U^\flat\},$$

where $U^\flat = \{u \mid U \not\subseteq Q_u\}$. Recall that if (r, R) is a direlation on $(U, \mathcal{U})$, then r is *reflexive* if $i \subseteq r$ and R is *reflexive* if $R \subseteq I$. Then we say that (r, R) is *reflexive* if r and R are reflexive.

Now let (r, R) be a direlation from $(U, \mathcal{U})$ to $(V, \mathcal{V})$ where $(U, \mathcal{U})$ and $(V, \mathcal{V})$ are any two texture spaces. Then *the inverses* of r and R are defined by

$$r^{\leftarrow} = \bigcap \{\overline{Q}_{(v,u)} \mid r \not\subseteq \overline{Q}_{(u,v)}\} \quad \text{and} \quad R^{\leftarrow} = \bigvee \{\overline{P}_{(v,u)} \mid \overline{P}_{(u,v)} \not\subseteq R\},$$

respectively, where $r^{\leftarrow}$ is a corelation and $R^{\leftarrow}$ is a relation.

Further, the direlation $(r, R)^{\leftarrow} = (R^{\leftarrow}, r^{\leftarrow})$ from $(V, \mathcal{V})$ to $(U, \mathcal{U})$ is called the *inverse* of the direlation (r, R). Then (r, R) is called *symmetric* if $r = R^{\leftarrow}$ and $R = r^{\leftarrow}$.

The A-sections and the B-presections with respect to relation and corelation are given as

$$r^{\rightarrow} A = \bigcap \{Q_v \mid \forall u, r \not\subseteq \overline{Q}_{(u,v)} \Rightarrow A \subseteq Q_u\}$$
$$R^{\rightarrow} A = \bigvee \{P_v \mid \forall u, \overline{P}_{(u,v)} \not\subseteq R \Rightarrow P_u \subseteq A\}$$
$$r^{\leftarrow} B = \bigvee \{P_u \mid \forall v, r \not\subseteq \overline{Q}_{(u,v)} \Rightarrow P_v \subseteq B\}, \quad \text{and}$$
$$R^{\leftarrow} B = \bigcap \{Q_u \mid \forall v, \overline{P}_{(u,v)} \not\subseteq R \Rightarrow B \subseteq Q_v\}$$

for all $A \in \mathcal{U}$ and $B \in \mathcal{V}$, respectively.

Now let (U, r) be an approximation space, that is, let $r \subseteq U \times U$. Recall that a generalized rough set based on r is given by $(\underline{apr}_r A,\ \overline{apr}_r A)$ where

$$\underline{apr}_r A = \{x \mid \forall y \in U,\ (x, y) \in r \Longrightarrow y \in A\}, \quad \text{and}$$
$$\overline{apr}_r A = \{x \mid \exists y \in U,\ (x, y) \in r \text{ and } y \in A\}$$

for all $A \subseteq U$ (see e.g. [31–33]). On the other hand, the pair $(r,\ (U \times U) \setminus r)$ can be regarded as a complemented direlation on the discrete texture $(U, \mathcal{P}(U))$ where $R = U \times U \setminus r$. Conversely, if (r, R) is a complemented direlation on $(U, \mathcal{P}(U))$, then r and R are ordinary relations on U where $R = (U \times U) \setminus r$. Therefore, using the facts

(1) $r \not\subseteq \overline{Q}_{(u,v)} \Longleftrightarrow (u, v) \in r$, and
(2) $\overline{P}_{(u,v)} \not\subseteq R \Longleftrightarrow (u, v) \notin R$,

we immediately conclude that

$$(r^{\leftarrow} A,\ R^{\leftarrow} A) = (\underline{apr}_r A,\ \overline{apr}_r A)$$

for every set $A \in \mathcal{P}(U)$. Moreover, we see that if r is an ordinary relation on U, that is, $r \subseteq U \times U$, then

$$\forall X \subseteq U, \quad \underline{apr}_r X = U \setminus r^{-1}(U \setminus X) \quad \text{and} \quad \overline{apr}_r X = r^{-1}(X).$$

Therefore, a quadruple $(U, \mathcal{U}, r, R)$ where (r, R) is any direlation on a texture $(U, \mathcal{U})$ can be regarded as a natural generalization of an approximation space in rough set theory.

Now we may recall the composition of direlations.

Let $(U, \mathcal{U}), (V, \mathcal{V}), (W, \mathcal{W})$ be texture spaces. For any relation p from $(U, \mathcal{U})$ to $(V, \mathcal{V})$ and for any relation q from $(V, \mathcal{V})$ to $(W, \mathcal{W})$ their *composition* $q \circ p$ from $(U, \mathcal{U})$ to $(W, \mathcal{W})$ is defined by

$$q \circ p = \bigvee \{\overline{P}_{(u,w)} \mid \exists\, v \in V \text{ with } p \not\subseteq \overline{Q}_{(u,v)} \text{ and } q \not\subseteq \overline{Q}_{(v,w)}\}$$

and any corelation P from $(U,\mathcal{U})$ to $(V,\mathcal{V})$ and for any corelation Q from $(U,\mathcal{U})$ to $(V,\mathcal{V})$ their *composition* $Q \circ P$ from $(U,\mathcal{U})$ to $(W,\mathcal{W})$ defined by

$$Q \circ P = \bigcap \{\overline{Q}_{(u,w)} \mid \exists\, v \in V \text{ with } \overline{P}_{(u,v)} \not\subseteq P \text{ and } \overline{P}_{(v,w)} \not\subseteq Q\}.$$

Finally, the *composition* of the direlations (p, P), (q, Q) is the direlation

$$(q, Q) \circ (p, P) = (q \circ p, Q \circ P).$$

Further, r is *transitive* if $r \circ r \subseteq r$ and R is *transitive* if $R \subseteq R \circ R$. Then we say that (r, R) is *transitive* if r and R are transitive.

Now let c_U and c_V be the complementations on $(U,\mathcal{U})$ and $(V,\mathcal{V})$, respectively. The complement r' of the relation r is the corelation

$$r' = \bigcap \big\{\overline{Q}_{(u,v)} \mid \exists w, z \text{ with } r \not\subseteq \overline{Q}_{(w,z)}, c_U(Q_u) \not\subseteq Q_w \text{ and } P_z \not\subseteq c_V(P_v)\big\}.$$

The complement R' of the corelation R is the relation

$$R' = \bigvee \big\{\overline{P}_{(u,v)} \mid \exists w, z \text{ with } \overline{P}_{(w,z)} \not\subseteq R, P_w \not\subseteq c_U(P_u) \text{ and } c_V(Q_v) \not\subseteq Q_z\big\}.$$

The complement $(r, R)'$ of the direlation (r, R) is the direlation $(r, R)' = (R', r')$. A direlation (r, R) is called *complemented* if $r = R'$ and $R = r'$.

3. Direlation preserving difunctions

Difunctions are important tools for textures as morphisms of the category **dfTex** whose objects are textures [8]. *A difunction* (r, R) on a texture $(U,\mathcal{U})$ is a direlation satisfying the following two conditions:

DF1 For $u, v \in U$, $P_u \not\subseteq Q_v \Longrightarrow \exists\, w \in U$ with $r \not\subseteq \overline{Q}_{(u,w)}$ and $\overline{P}_{(v,w)} \not\subseteq R$.
DF2 For $u, v \in U$ and $w \in U$, $r \not\subseteq \overline{Q}_{(w,u)}$ and $\overline{P}_{(w,v)} \not\subseteq R \Longrightarrow P_v \not\subseteq Q_u$.

It is remarkable to note that a direlation on a texture $(U,\mathcal{U})$ is a difunction if and only if every set $A \in \mathcal{U}$ is (r, R)-definable [13].

Definition 3.1. (See [15].) Let (r, R) be a direlation on a texture $(U,\mathcal{U})$. Then the quadruple $(U,\mathcal{U}, r, R)$ is called *a textural approximation space*.

Definition 3.2. Let $(U,\mathcal{U}, r, R)$, $(V,\mathcal{V}, h, H)$ be textural approximation spaces. Then a difunction (f, F) from $(U,\mathcal{U}, r, R)$ to $(V,\mathcal{V}, h, H)$ is called

(i) *relation-preserving* if

$$\forall u, u' \in U, \quad r \not\subseteq \overline{Q}_{(u,u')} \quad \Longrightarrow \quad \exists v, v' \in V, \quad f \not\subseteq \overline{Q}_{(u,v)}, \quad \overline{P}_{(u',v')} \not\subseteq F \quad \text{and} \quad h \not\subseteq \overline{Q}_{(v,v')},$$

(ii) *corelation-preserving* if

$$\forall u, u' \in U, \quad \overline{P}_{(u,u')} \not\subseteq R \quad \Longrightarrow \quad \exists v, v' \in V, \quad f \not\subseteq \overline{Q}_{(u',v')}, \quad \overline{P}_{(u,v)} \not\subseteq F \quad \text{and} \quad \overline{P}_{(v,v')} \not\subseteq H.$$

If (f, F) is a relation and corelation-preserving difunction, then we say that (f, F) is *a direlation-preserving difunction*.

Now let us give the following result:

Theorem 3.3. *Let $(f, F) : (U,\mathcal{U}, r, R) \to (V,\mathcal{V}, h, H)$ and $(g, G) : (V,\mathcal{V}, h, H) \to (W,\mathcal{W}, k, K)$ be any two direlation preserving difunctions. Then*

(i) *$(g \circ f, G \circ F) : (U,\mathcal{U}, r, R) \to (W,\mathcal{W}, k, K)$ is also a direlation preserving difunction.*
(ii) *The identity direlation*

$$(i_U, I_U) : (U,\mathcal{U}, r, R) \to (U,\mathcal{U}, r, R)$$

is a direlation preserving difunction.

Proof. (i) Let $u, u' \in U$ and $r \not\subseteq \overline{Q}_{(u,u')}$. Then by the assumption there exist $v, v' \in V$ such that $f \not\subseteq \overline{Q}_{(u,v)}$, $\overline{P}_{(u',v')} \not\subseteq F$ and $h \not\subseteq \overline{Q}_{(v,v')}$. Since $h \not\subseteq \overline{Q}_{(v,v')}$, for some $w, w' \in W$, $g \not\subseteq \overline{Q}_{(v,w)}$, $\overline{P}_{(v',w')} \not\subseteq G$ and $k \not\subseteq \overline{Q}_{(w,w')}$. Then clearly, we have

$$g \circ f \not\subseteq \overline{Q}_{(u,w)}, \quad \overline{P}_{(u',w')} \not\subseteq G \circ F.$$

Since we also have $k \not\subseteq \overline{Q}_{(w,w')}$, $(g \circ f, G \circ F)$ a relation preserving difunction. Likewise, $(g \circ f, G \circ F)$ is a corelation preserving difunction.

(ii) Let $u, u' \in U$ and $r \not\subseteq \overline{Q}_{(u,u')}$. By R1, we have $u_1 \in U$ such that $P_u \not\subseteq Q_{u_1}$ and $r \not\subseteq \overline{Q}_{(u_1,u')}$. Now let us choose $u_2, u_3 \in U$ such that $r \not\subseteq \overline{Q}_{(u_2,u_3)}$ and $\overline{P}_{(u_2,u_3)} \not\subseteq \overline{Q}_{(u_1,u')}$. We have $u_2 = u_1$ and $P_{u_3} \not\subseteq Q_{u'}$. By definition of identity difunction $P_u \not\subseteq Q_{u_1}$ and $P_{u_3} \not\subseteq Q_{u'}$ implies that

$$i_U \not\subseteq \overline{Q}_{(u,u_1)} \quad \text{and} \quad \overline{P}_{(u',u_3)} \not\subseteq I_U.$$

Since we also have $r \not\subseteq \overline{Q}_{(u_1,u_3)}$, the identity direlation (i_U, I_U) is a relation preserving difunction. Similarly, it can be shown that (i_U, I_U) is also a corelation preserving difunction and this completes the proof of (ii). □

Theorem 3.4. *Textural approximation spaces and direlation preserving difunctions form a category denoted by* **diRel**.

Let r be an ordinary relation on U, that is, $r \subseteq U \times U$. It is known that reflexive approximation spaces and relation preserving functions determine a category denoted by **Rere** [1]. Here, we consider the textural version **diRere** of **Rere** as the category of reflexive textural approximation spaces and direlation-preserving difunctions.

If (U, r) is an approximation space, then $(U, \mathcal{P}(U), r, r')$ is a textural approximation space where r' is the set theoretical complement of r, that is, $(U \times U) \setminus r = r'$. Further, if $f : (U, r) \to (V, h)$ is a relation preserving function, then

$$\big(f, f'\big) : \big(U, \mathcal{P}(U), r, r'\big) \to \big(V, \mathcal{P}(V), h, h'\big)$$

is a direlation preserving difunction. Indeed, since (r, r') is a complemented direlation on $(U, \mathcal{P}(U))$, we have the following facts:

$$r \not\subseteq \overline{Q}_{(u,v)} \iff (u, v) \in r,$$
$$\overline{P}_{(u,v)} \not\subseteq r' \iff (u, v) \notin r'.$$

Now let $r \not\subseteq \overline{Q}_{(u,u')}$. Then $(u, u') \in r$ and since f is a relation preserving function, $(f(u), f(u')) \in h$. Let $f(u) = v$ and $f(u') = v'$. This means that $(u, v) \in f$ and $(u', v') \in f$, that is, $(u', v') \notin (U \times V) \setminus f = f'$. Therefore, $f \not\subseteq \overline{Q}_{(u,v)}$ and $\overline{P}_{(u',v')} \not\subseteq f'$. Further, we have $h \not\subseteq \overline{Q}_{(v,v')}$ and hence, (f, f') is a relation preserving difunction. Similarly, one can be shown that (f, f') is a corelation preserving difunction.

Now we may claim:

Theorem 3.5. *The mapping* $\mathfrak{T}$: **Rel** $\to$ **diRel** *defined by*

$$\mathfrak{T}\big((U, r)\big) = \big(U, \mathcal{P}(U), r, r'\big) \quad \text{and} \quad \mathfrak{T}(f) = \big(f, f'\big)$$

for all $(U, r) \in \mathrm{ob}(\mathbf{Rel})$ *and* $f \in \hom(\mathbf{Rel})$ *is a functor and full embedding.*

Proof. Let $(U, r), (V, h)$ and (W, k) be approximation spaces, $f : (U, r) \to (V, h)$ and $g : (V, h) \to (W, k)$ be morphisms in **Rel**. Let us consider the difunctions $(f, f') : (U, \mathcal{P}(U), r, r') \to (V, \mathcal{P}(V), h, h')$, and $(g, g') : (V, \mathcal{P}(V), h, h') \to (W, \mathcal{P}(W), k, k')$. By Lemma 3.1.(8) in [8], for the corelations f' and g', we have

$$f' \circ g' = \big(\big(f'\big)' \circ \big(g'\big)'\big)' = (f \circ g)'.$$

Therefore, by definition of $\mathfrak{T}$ and the composition of direlations, we also have

$$\mathfrak{T}(f \circ g) = \big(f \circ g, (f \circ g)'\big) = \big(f \circ g, f' \circ g'\big) = \big(f, f'\big) \circ \big(g, g'\big) = \mathfrak{T}(f) \circ \mathfrak{T}(g)$$

and

$$\mathfrak{T}(\Delta) = \mathfrak{T}\big(\big\{(u, u) \mid u \in U\big\}\big) = \big(\Delta, \Delta'\big) = \big(\Delta, (U \times U) \setminus \Delta\big)$$

where Δ and (Δ, Δ') are the identity function and identity difunction for U and the texture $(U, \mathcal{P}(U))$, respectively. Hence, $\mathfrak{T}$ is indeed a functor. Further, $\mathfrak{T}$ is injective on objects and it is faithful, that is, the home-set restriction $\mathfrak{T} : \hom_{\mathbf{Rel}}(U, V) \to \hom_{\mathbf{diRel}}(\mathfrak{T}(U), \mathfrak{T}(V))$ is injective. Indeed, for the morphisms $f, g : (U, r) \to (V, h)$ where $f \neq g$, we have $(f, f') \neq (g, g')$. It is also surjective. Since $\mathfrak{T}(U)$ and $\mathfrak{T}(V)$ are discrete textures, it is easy to see that if (f, F) is a morphism from $\mathfrak{T}(U)$ to $\mathfrak{T}(V)$, then $F = f'$. Therefore, $\mathfrak{T}(f) = (f, F)$ where $f : (U, r) \to (V, h)$ is a morphism in **Rel**. □

Clearly, if (U, r) is a reflexive approximation space, then $(U, \mathcal{P}(U), r, r')$ is a reflexive textural approximation space. In view of Theorem 3.5, it is easy to see that **Rere** is also a full subcategory of **diRere**.

Theorem 3.6. *Let $(U, \mathcal{U})$ be a texture space, $\{(V_j, \mathcal{V}_j, h_j, H_j) \mid j \in J\}$ be a family of textural approximation spaces, and $\{(f_j, F_j) \mid j \in J\}$ be a collection of difunctions from $(U, \mathcal{U})$ to $(V_j, \mathcal{V}_j)$. Then there exists a unique direlation (r, R) for which for all $j \in J$, (f_j, F_j) is a* **diRel***-morphism from $(U, \mathcal{U}, r, R)$ to $(V_j, \mathcal{V}_j, h_j, H_j)$ where*

$$r = \bigvee\{\overline{P}_{(u,u')} \mid \exists v, v' \in V_j,\, f_j \not\subseteq \overline{Q}_{(u,v)}, \overline{P}_{(u',v')} \not\subseteq F_j \text{ and } h_j \not\subseteq \overline{Q}_{(v,v')},\ \forall j \in J\}$$

and

$$R = \bigcap\{\overline{Q}_{(u,u')} \mid \exists v, v' \in V_j,\, f_j \not\subseteq \overline{Q}_{(u',v')}, \overline{P}_{(u,v)} \not\subseteq F_j \text{ and } \overline{P}_{(v,v')} \not\subseteq H_j,\ \forall j \in J\}.$$

Proof. Let us show that (r, R) is a direlation on $(U, \mathcal{U})$.

(R1) Let $r \not\subseteq \overline{Q}_{(u_1,u_2)}$ and $P_{u_3} \not\subseteq Q_{u_1}$. We show that $r \not\subseteq \overline{Q}_{(u_3,u_2)}$. By definition of r there exist $v, v' \in V$ such that for some $u' \in U$ we have

$$P_{u'} \not\subseteq Q_{u_2}, \quad f_j \not\subseteq \overline{Q}_{(u,v)}, \quad \overline{P}_{(u',v')} \not\subseteq F_j \quad \text{and} \quad h_j \not\subseteq \overline{Q}_{(v,v')}.$$

Since f_j is also a relation, $f_j \not\subseteq \overline{Q}_{(u_1,v)}$ and $P_{u_3} \not\subseteq Q_{u_1}$ implies that $f_j \not\subseteq \overline{Q}_{(u_3,v)}$. Let us choose a point $u_4 \in$ such that $P_{u'} \not\subseteq Q_{u_4}$ and $P_{u_4} \not\subseteq Q_{u_2}$. Now F_j is a corelation and so $\overline{P}_{(u',v')} \not\subseteq F_j$, $P_{u'} \not\subseteq Q_{u_4}$ gives that $\overline{P}_{(u',v')} \not\subseteq F_j$. Then by definition of r, we may write $\overline{P}_{(u_3,u_4)} \subseteq r$. However, $\overline{p}_{(u_3,u_4)} \not\subseteq \overline{Q}_{(u_3,u_4)}$ and hence we conclude that $r \not\subseteq \overline{Q}_{(u_3,u_2)}$.

(R2) Let $r \not\subseteq \overline{Q}_{(u_1,u_2)}$. Then for some $u, u' \in U$, we have

$$u = u_1, \qquad P_{u'} \not\subseteq Q_{u_2}, \qquad f_j \not\subseteq \overline{Q}_{(u_1,v)}, \qquad \overline{P}_{(u',v')} \not\subseteq F_j \quad \text{and} \quad h_j \not\subseteq \overline{Q}_{(v,v')}.$$

Since f_j is also a relation, for some $u_3 \in U$, we have $P_{u_1} \not\subseteq Q_{u_3}$ and $f_j \not\subseteq \overline{Q}_{(u_3,v)}$. Now let us choose a point $u_4 \in U$ such that $P_{u'} \not\subseteq Q_{u_4}$ and $P_{u_4} \not\subseteq Q_{u_2}$. Since F_j is a corelation, $P_{u'} \not\subseteq Q_{u_4}$ and $\overline{P}_{(u',v')} \not\subseteq F_j$ implies that $\overline{P}_{(u_4,v')} \not\subseteq F_j$. As a result, $f_j \not\subseteq \overline{Q}_{(u_3,v)}$ and $\overline{P}_{(u_4,v')} \not\subseteq F_j$ gives that $\overline{P}_{(u_3,u_4)} \subseteq r$. Clearly, we have $\overline{P}_{(u_3,u_4)} \not\subseteq \overline{Q}_{(u_3,u_2)}$, and so we find $r \not\subseteq \overline{Q}_{(u_3,u_2)}$. Since $P_{u_1} \not\subseteq Q_{u_3}$, the proof of R1 is complete.

The proof of CR1 and CR2 is omitted. Clearly, (f_j, F_j) is a **diRel**-morphism from $(U, \mathcal{U}, r, R)$ to $(V_j, \mathcal{V}_j, h_j, H_j)$. Indeed, if for all $u, u' \in U$, $r \not\subseteq \overline{Q}_{(u,u')}$, then by definition of r, there exist $v, v' \in V$ such that $f \not\subseteq \overline{Q}_{(u_1,v)}$, $\overline{P}_{(u'_1,v')} \not\subseteq F$ and $h \not\subseteq \overline{Q}_{(v,v')}$ where $\overline{P}_{(u_1,u'_1)} \not\subseteq \overline{Q}_{(u,u')}$. However, we have $u_1 = u$ and $P_{u'_1} \not\subseteq Q_{u'}$ and this gives that $f \not\subseteq \overline{Q}_{(u,v)}$, $\overline{P}_{(u',v')} \not\subseteq F$. Therefore, (f_j, F_j) is a relation preserving difunction. Likewise, (f_j, F_j) is a corelation preserving difunction.

Now we show that (r, R) is a unique direlation satisfying the conditions of theorem. Suppose that (e, E) is another direlation on $(U, \mathcal{U})$ satisfying the condition. Let us take $(W, \mathcal{W}, k, K) = (U, \mathcal{U}, e, E)$. Since $(i_U, I_U) : (U, \mathcal{U}, e, E) \to (U, \mathcal{U}, e, E)$ is a direlation preserving difunction, $(f_j \circ i_U, F_j \circ I_U) = (f_j, F_j)$ is a direlation preserving mapping from $(U, \mathcal{U}, e, E)$ to $(V_j, \mathcal{V}_j, h_j, H_j)$. Then we have $e \subseteq r$ and $R \subseteq E$. Indeed, let $e \not\subseteq \overline{Q}_{(u,u')}$. Choose $u'' \in U$ such that $e \not\subseteq \overline{Q}_{(u,u'')}$ and $P_{u''} \not\subseteq Q_{u'}$. Since (f_j, F_j) is relation preserving difunction, for some $v, v' \in U$, we may write

$$f \not\subseteq \overline{Q}_{(u,v)}, \qquad \overline{P}_{(u'',v')} \not\subseteq F \quad \text{and} \quad h \not\subseteq \overline{Q}_{(v,v')}$$

By definition of r, we have $\overline{P}_{(u,u'')} \subseteq r$. However, $\overline{P}_{(u,u'')} \not\subseteq \overline{Q}_{(u,u')}$ and this implies that $r \not\subseteq \overline{Q}_{(u,u')}$ and so we obtain $e \subseteq r$. Using a similar argument, it is easy to see that $R \subseteq E$. For the reverse inclusions, let $(W, \mathcal{W}, k, K) = (U, \mathcal{U}, r, R)$ and take the identity direlation $(i_U, I_U) : (U, \mathcal{U}, r, R) \to (U, \mathcal{U}, e, E)$. Since $(U, \mathcal{U}, e, E)$ satisfies the condition, $(f_j, F_j) : (U, \mathcal{U}, r, R) \to (V_j, \mathcal{V}_j, h_j, H_j)$ is a direlation preserving difunction. Then using a similar argument as above we may easily see that $r \subseteq e$ and $E \subseteq R$. □

Theorem 3.7. *Let $(W, \mathcal{W}, k, K)$ be a textural approximation space and $(g, G) : (W, \mathcal{W}) \longrightarrow (U, \mathcal{U})$ be any difunction. Then $(g, G) : (W, \mathcal{W}, k, K) \longrightarrow (U, \mathcal{U}, r, R)$ is a* **diRel***-morphism if and only if $(f_j \circ g, F_j \circ G) : (W, \mathcal{W}, k, K) \longrightarrow (V_j, \mathcal{V}_j, h_j, H_j)$ is a* **diRel***-morphism.*

Proof. Suppose that $(g, G) : (W, \mathcal{W}, k, K) \longrightarrow (U, \mathcal{U}, r, R)$ is a **diRel**-morphism. Let $k \not\subseteq \overline{Q}_{(w,w')}$. By the assumption, for some $u, u' \in U$, we have $g \not\subseteq \overline{Q}_{(w,u)}$, $\overline{P}_{(w',u')} \not\subseteq G$ and $r \not\subseteq \overline{Q}_{(u,u')}$. By definition of r, for some $v, v' \in V$ we have that $f_j \not\subseteq \overline{Q}_{(u,v)}$, $\overline{P}_{(u',v')} \not\subseteq F_j$ and $h_j \not\subseteq Q_{(v_1,v_2)}$. Finally, we obtain

$$f_j \circ g \not\subseteq \overline{Q}_{(w,v)}, \qquad \overline{P}_{(w',v')} \not\subseteq F_j \circ G \quad \text{and} \quad h_j \not\subseteq \overline{Q}_{(v,v')},$$

that is, $(f_j \circ g, F_j \circ G)$ is a relation preserving difunction. Dually, we may show that $(f_j \circ g, F_j \circ G)$ is a corelation preserving difunction and so $(f_j \circ g, F_j \circ G)$ is a **diRel**-morphism.

Now let $(f_j \circ g, F_j \circ G)$ be a **diRel**-morphism. Let $k \not\subseteq \overline{Q}_{(w,w')}$ where $w, w' \in W$. By the assumption for some $v, v' \in V_j$, we have

$$f_j \circ g \not\subseteq \overline{Q}_{(w,v)}, \qquad \overline{P}_{(w',v')} \not\subseteq F_j \circ G \quad \text{and} \quad h_j \not\subseteq \overline{Q}_{(v,v')}.$$

Then for some $u, u' \in U$, we may write $g \not\subseteq \overline{Q}_{(w,u)}$, $f_j \not\subseteq \overline{Q}_{(u,v)}$, $\overline{P}_{(w',u')} \not\subseteq G$ and $P_{(u',v')} \not\subseteq F_j$. Now $f_j \not\subseteq \overline{Q}_{(u,v)}$, $P_{(u',v')} \not\subseteq F_j$ and $h_j \not\subseteq \overline{Q}_{(v,v')}$ gives that $r \not\subseteq \overline{Q}_{(u,u')}$. Finally,

$$r \not\subseteq \overline{Q}_{(u,u')}, \qquad g \not\subseteq \overline{Q}_{(w,u)}, \quad \text{and} \quad \overline{P}_{(w',v')} \not\subseteq G$$

implies that (g, G) is a relation preserving difunction. Similarly, one can show that (g, G) is a corelation preserving difunction. $\square$

Let us consider the forgetful functor $\mathfrak{U} : \mathbf{diRel} \to \mathbf{dfTex}$.

Theorem 3.8. *The source $((f_j, F_j) : (U, \mathcal{U}, r, R) \to (V_j, \mathcal{V}_j, h_j, H_j))_{j\in J}$ in* **diRel** *is $\mathfrak{U}$-initial if and only if (r, R) is unique direlation on $(U, \mathcal{U})$ such that (f_j, F_j), $j \in J$ are direlation preserving difunctions.*

Proof. $\Longrightarrow$: Assume that (r, R) is not unique and let us consider a source $((f_j, F_j) : (U, \mathcal{U}, e, E) \to (V_j, \mathcal{V}_j, h_j, H_j))_{j\in J}$ in **diRel** with $(r, R) \neq (e, E)$. Let $(i_U, I_U) : (U, \mathcal{U}) \to (U, \mathcal{U})$ be the identity morphism in **dfTex**. Clearly, the diagram

$$\begin{array}{ccc} (U,\mathcal{U}) & \xrightarrow{(i_U, I_U)} & (U,\mathcal{U}) \\ \downarrow {\scriptstyle (f_j, F_j)} & \swarrow {\scriptstyle (f_j, F_j)} & \\ (V_j, \mathcal{V}_j) & & \end{array}$$

is commutative in **dfTex**, that is, $(f_j, F_j) = (f_j, F_j) \circ (i_U, I_U)$. Since the source $((f_j, F_j) : (U, \mathcal{U}, r, R) \to (V_j, \mathcal{V}_j, h_j, H_j))_{j\in J}$ in **diRel** is $\mathfrak{U}$-initial, the diagram

$$\begin{array}{ccc} (U,\mathcal{U},e,E) & \xrightarrow{(i_U, I_U)} & (U,\mathcal{U},r,R) \\ \downarrow {\scriptstyle (f_j, F_j)} & \swarrow {\scriptstyle (f_j, F_j)} & \\ (V_j, \mathcal{V}_j, h_j, H_j) & & \end{array}$$

is also commutative in **diRel**. Hence, by Theorem 7.5, $(i_U, I_U) : (U, \mathcal{U}, e, E) \to (U, \mathcal{U}, r, R)$ is also a morphism in **diRel**. Now it is easy to see that $e \subseteq r$ and $R \subseteq E$. To show this, let $e \not\subseteq \overline{Q}_{(u,u')}$. Since (i_U, I_U) is a relation preserving difunction, we have

$$i_U \not\subseteq \overline{Q}_{(u,u_1)}, \qquad \overline{P}_{(u',u_2)} \not\subseteq I_U \quad \text{and} \quad r \not\subseteq \overline{Q}_{(u_1,u_2)}$$

for some $u_1, u_2 \in U$. From the first two statements, we find $P_u \not\subseteq Q_{u_1}$ and $P_{u_2} \not\subseteq Q_{u'}$, respectively. By R1, we have $r \not\subseteq \overline{Q}_{(u,u_2)}$ and finally, we obtain $r \not\subseteq \overline{Q}_{(u,u')}$ and this gives us that $e \subseteq r$. The second and reverse inclusions are similar. Hence, we have the equalities $e = r$ and $E = R$.

$\Longleftarrow$: Suppose that (r, R) is unique direlation on $(U, \mathcal{U})$ such that (f_j, F_j), $j \in J$ are direlation preserving mappings. Now let us consider the following diagrams for the source $((g_j, G_j) : (W, \mathcal{W}, k, K) \to (V_j, \mathcal{V}_j, h_j, H_j))_{j\in J}$ and the morphism $(g_j, G_j) : (W, \mathcal{W}) \to (V_j, \mathcal{V}_j)$ in **diRel** and **dfTex**, respectively.

$$\begin{array}{ccc} (W,\mathcal{W},k,K) & \xrightarrow{(i_U, I_U)} & (U,\mathcal{U},r,R) \\ \downarrow {\scriptstyle (g_j, G_j)} & \swarrow {\scriptstyle (f_j, F_j)} & \\ (V_j, \mathcal{V}_j, h_j, H_j) & & \end{array} \qquad \begin{array}{ccc} (W,\mathcal{W}) & \xrightarrow{(i_U, I_U)} & (U,\mathcal{U}) \\ \downarrow {\scriptstyle (g_j, G_j)} & \swarrow {\scriptstyle (f_j, F_j)} & \\ (V_j, \mathcal{V}_j) & & \end{array}$$

Suppose that the right hand diagram is commutative in **dfTex**. Then we have

$$(f_j, F_j) \circ (i_U, I_U) = (g_j, G_j).$$

By the assumption (g_j, G_j) is a morphism in **diRel** and hence, by Theorem 3.7, (i_U, I_U) is also a morphism in **diRel**. Therefore, the left hand diagram is also commutative in **diRel**. As a result the source $((f_j, F_j) : (U, \mathcal{U}, r, R) \to (V_j, \mathcal{V}_j, h_j, H_j))_{j\in J}$ is $\mathfrak{U}$-initial. $\square$

Theorem 3.9. *The forgetful functor $\mathfrak{U}$: **diRel**→ **dfTex** is topological, that is, the category **diRel** is a topological category over **dfTex**.*

Proof. Let $((f_j, F_j) : (U, \mathcal{U}) \to (V_j, \mathcal{V}_j)_{j\in J})$ be a source in **dfTex** where $\mathfrak{U}(V_j, \mathcal{V}_j, h_j, H_j) = (V_j, \mathcal{V}_j)$ for $j \in J$. Consider the direlation (r, R) on $(U, \mathcal{U})$ given in Theorem 3.6. By Theorem 3.8, the source $((f_j, F_j) : (U, \mathcal{U}, r, R) \to (V_j, \mathcal{V}_j, h_j, H_j))_{j\in J}$ is a unique $\mathfrak{U}$-initial source and hence, **diRel** is a topological category over **dfTex**. □

Theorem 3.6 and Theorem 3.7 are also true for equivalence relations:

Lemma 3.10. *Let (r, R) and (h_j, H_j) be the direlations given in Theorem 3.6. If (h_j, H_j) is an equivalence direlation, then (r, R) is also an equivalence direlation.*

Proof. We give the proof leaving the dual parts to the interested reader. First let us show that (r, R) is reflexive. Suppose that $i_U \not\subseteq r$. Let us choose $u, u' \in U$ such that $i \not\subseteq \overline{Q}_{(u,u')}$ and $\overline{P}_{(u,u')} \not\subseteq r$. By definition of identity relation i_U, we have $P_u \not\subseteq Q_{u'}$. Then by DF1, there exists $v' \in V$ such that $f \not\subseteq \overline{Q}_{(u,v)}$ and $\overline{P}_{(u',v)} \not\subseteq F$. Take $v_1 \in V$ such that $f \not\subseteq \overline{Q}_{(u,v_1)}$ and $P_{v_1} \not\subseteq Q_v$. By definition of the identity relation i_V, we have $i_V \not\subseteq \overline{Q}_{(v_1,v)}$. By the assumption $i_V \subseteq h$ and so $h \not\subseteq \overline{Q}_{(v_1,v)}$. Finally, by definition of r we obtain $\overline{P}_{(u,u')} \subseteq r$ which is a contradiction.

Suppose that (h_j, H_j) symmetric, that is $H_j = h_j^{\leftarrow}$ and let $r \not\subseteq R^{\leftarrow}$. Let us choose $u, u' \in U$ such that $r \not\subseteq \overline{Q}_{(u,u')}$ and $\overline{P}_{(u,u')} \not\subseteq R^{\leftarrow}$. By definition of (r, R), we may write

$$f_j \not\subseteq \overline{Q}_{(u,v)}, \qquad \overline{P}_{(u',v')} \not\subseteq F_j \quad \text{and} \quad h_j \not\subseteq \overline{Q}_{(v,v')},$$

and for all $v_2, v_3 \in V$,

$$f_j \not\subseteq \overline{Q}_{(u,v_3)}, \qquad \overline{P}_{(u',v_2)} \not\subseteq F_j \quad \Longrightarrow \quad \overline{P}_{(v_2,v_3)} \subseteq H_j.$$

Then we have $\overline{P}_{(v',v)} \subseteq H_j = h_j^{\leftarrow}$ and by Lemma 2.4 in [8], $h \subseteq \overline{Q}_{(v,v')}$ is a contradiction.

Suppose that (h_j, H_j) is transitive. Then $h_j \circ h_j \subseteq h_j$. and $H_j \subseteq H_j \circ H_j$. Suppose that $r \circ r \not\subseteq r$. Let us choose $u, u' \in U$ such that $r \circ r \not\subseteq \overline{Q}_{(u,u')}$ and $\overline{P}_{(u,u')} \not\subseteq r$. By definition of composition, we have $r \not\subseteq \overline{Q}_{(u,u_2)}$ and $r \not\subseteq \overline{Q}_{(u_2,u')}$ for some $u_2 \in U$. By definition of r, for some $v, v_1, v_2, v_3 \in V$ we also have

$$f_j \not\subseteq \overline{Q}_{(u,v)}, \qquad \overline{P}_{(u_2,v_1)} \not\subseteq F_j\, h_j \not\subseteq \overline{Q}_{(v,v_1)} \quad \text{and} \quad f_j \not\subseteq \overline{Q}_{(u_2,v_2)}, \qquad \overline{P}_{(u',v_3)} \not\subseteq F_j, \qquad h_j \not\subseteq \overline{Q}_{(v_2,v_3)}.$$

Since $\overline{P}_{(u,u')} \not\subseteq r$, $f_j \not\subseteq \overline{Q}_{(u,v)}$ and $\overline{P}_{(u',v_3)} \not\subseteq F_j$ implies that $h_j \subseteq \overline{Q}_{(v_3)}$. By DF1, $\overline{P}_{(u_2,v_1)} \not\subseteq F_j$ and $f_j \not\subseteq \overline{Q}_{(u_2,v_2)}$ gives that $P_{v_1} \not\subseteq Q_{v_2}$. Therefore, $h_j \not\subseteq \overline{Q}_{(v,v_1)}$ implies that $h_j \not\subseteq \overline{Q}_{(v,v_2)}$. Now choose $v_4 \in V$ such that $h_j \not\subseteq \overline{Q}_{(v_2,v_4)}$ and $P_{v_4} \not\subseteq Q_{v_3}$. Then we find $\overline{P}_{(v,v_4)} \subseteq h_j \circ h_j$. Now we conclude that $h_j \circ h_j \not\subseteq \overline{Q}_{(v,v_3)}$, that is, $h_j \not\subseteq \overline{Q}_{(v,v_3)}$, which is a contradiction. □

Corollary 3.11. *The categories **diRere** and **diProst** are topological categories over **dfTex** with respect to forgetful functor $\mathfrak{U}$.*

By Example 11.4(1) and Proposition 13.15 in [1], we have the following.

Theorem 3.12. *The source $((f_j, F_j) : (U, \mathcal{U}, r, R) \to (V_j, \mathcal{V}_j, h_j, H_j))_{j\in J}$ is $\mathfrak{U}$-initial and $((f_j, F_j) : (U, \mathcal{U}) \to (V_j, \mathcal{V}_j))_{j\in J}$ is a product of $\mathfrak{U} \circ \mathfrak{D} : J \to$ **dfTex** if and only if $(f_j, F_j) : (U, \mathcal{U}, r, R) \to (V_j, \mathcal{V}_j, h_j, H_j)$ is a product of the functor $\mathfrak{D} : J \to$ **diRel** where J is regarded as a discrete category.*

The following is a result of Theorem 3.8 and Theorem 3.12.

Proposition 3.13. *The source $(f_j, F_j) : (U, \mathcal{U}, r, R) \to (V_j, \mathcal{V}_j, h_j, H_j)$ is a product of the family $(V_j, \mathcal{V}_j, h_j, H_j)_{j\in J}$ in **diRel** if and only if (r, R) is given as in Theorem 3.6 and $(f_j, F_j) : (U, \mathcal{U}) \to (V_j, \mathcal{V}_j)$ is a product of the family $(V_j, \mathcal{V}_j)_{j\in J}$ in **dfTex**.*

The following result gives the set-indexed product in **diRel**.

Corollary 3.14. *Let $(V, \mathcal{V})$ be the product of textures $(V_j, \mathcal{V}_j)_{j\in J}$. Then the source $((\pi_j, \Pi_j) : (V, \mathcal{V}, r, R) \to (V_j, \mathcal{V}_j, h_j, H_j))_{j\in J}$ is a product in **diRel** where $v = (v_j)_{j\in J}$, $v' = (v'_j)_{j\in J} \in \prod_{i\in J} V_j$,*

$$r = \bigvee\{\overline{P}_{(v,v')} \mid h_j \not\subseteq \overline{Q}_{(v_j,v'_j)}, \forall j \in J\} \quad \textit{and} \quad R = \bigcap\{\overline{Q}_{(v,v')} \mid P_{(v_j,v'_j)} \not\subseteq H_j, \forall j \in J\}.$$

Proof. By Theorem 3.4 in [8], $((\pi_j, \Pi_j) : (V, \mathcal{V}) \to (V_j, \mathcal{V}_j)_{j\in J})_{j\in J}$ is a product in **dfTex**. By Theorem 3.6, for (r, R), we have

$$r = \bigvee\{\overline{P}_{(v,v')} \mid \exists a_j, b_j \in V_j, \pi_j \not\subseteq \overline{Q}_{(v,a_j)}, \ \overline{P}_{(v',b_j)} \not\subseteq \Pi_j \text{ and } h_j \not\subseteq \overline{Q}_{(a_j,b_j)}, \forall j \in J\}$$

and

$$R=\bigcap\{\overline{Q}_{(v,v')} \mid \exists a_j,b_j\in V_j,\ \pi_j\not\subseteq \overline{Q}_{(v',b_j)},\ \overline{P}_{(v,a_j)}\not\subseteq \Pi_j \text{ and } \overline{P}_{(a_j,b_j)}\not\subseteq H_j, \forall j\in J\}.$$

Therefore, the equivalences

$$P_{v_j}\not\subseteq Q_{a_j} \iff \pi_j\not\subseteq \overline{Q}_{(v,a_j)} \quad \text{and} \quad P_{b_j}\not\subseteq Q_{v'_j} \iff \overline{P}_{(v',b_j)}\not\subseteq \Pi_j$$

implies that

$$r=\bigvee\{\overline{P}_{(v,v')} \mid \exists a_j,b_j\in V_j, P_{v_j}\not\subseteq Q_{a_j}, P_{b_j}\not\subseteq Q_{v'_j} \text{ and } h_j\not\subseteq \overline{Q}_{(a_j,b_j)}, \forall j\in J\}$$

and

$$R=\bigcap\{\overline{Q}_{(v,v')} \mid \exists a_j,b_j\in V_j,\ P_{a_j}\not\subseteq Q_{v_j}, P_{v'_j}\not\subseteq Q_{b_j} \text{ and } \overline{P}_{(a_j,b_j)}\not\subseteq H_j, \forall j\in J\}.$$

We have the following equivalences:

(i) $\exists a_j,b_j\in V_j,\ P_{v_j}\not\subseteq Q_{a_j}, P_{b_j}\not\subseteq Q_{v'_j}$ and $h_j\not\subseteq \overline{Q}_{(a_j,b_j)} \iff h_j\not\subseteq \overline{Q}_{(v_j,v'_j)}$

(ii) $\exists a_j,b_j\in V_j,\ P_{a_j}\not\subseteq Q_{v_j}, P_{v'_j}\not\subseteq Q_{b_j}$ and $\overline{P}_{(a_j,b_j)}\not\subseteq H_j \iff P_{(v_j,v'_j)}\not\subseteq H_j$.

We prove (i) leaving the dual proof of (ii). By R1, if $P_{v_j}\not\subseteq Q_{a_j}$ and $h_j\not\subseteq \overline{Q}_{(a_j,b_j)}$, then we have $h_j\not\subseteq \overline{Q}_{(v_j,b_j)}$. If $P_{b_j}\not\subseteq Q_{v'_j}$, then it is easy to see that $h_j\not\subseteq \overline{Q}_{(v_j,v'_j)}$. Conversely, suppose that $h_j\not\subseteq \overline{Q}_{(v_j,v'_j)}$. If we apply the condition R2, we have $P_{v_j}\not\subseteq Q_{a_j}$ and $h_j\not\subseteq \overline{Q}_{(a_j,v'_j)}$ for some $a_j\in V_j$. Further, we may find some $a'_j\in V_j$ such that $h_j\not\subseteq \overline{Q}_{(a_j,v'_j)}$ and $P_{b_j}\not\subseteq \overline{Q}_{v'_j}$. Hence, we obtain $r=\bigvee\{\overline{P}_{(v,v')} \mid h_j\not\subseteq \overline{Q}_{(v_j,v'_j)}, \forall j\in J\}$. Similarly, we have $R=\bigcap\{\overline{Q}_{(v,v')} \mid P_{(v_j,v'_j)}\not\subseteq H_j, \forall j\in J\}$. □

Corollary 3.15. *The categories* **diRel**, **diRere** *and* **diProst** *have products.*

Note that the product in **diRel** is a natural generalization of product in **Rel**. Indeed, if all textures are discrete in Corollary 3.14, then for $v=(v_j)_{j\in J}$ and $v'=(v'_j)_{j\in J}\in\prod_{i\in J}V_j$, we have

$$r=\bigvee\{\overline{P}_{(v,v')} \mid h_j\not\subseteq \overline{Q}_{(v_j,v'_j)}, \forall j\in J\}=\{(v,v') \mid h_j(v_j)=v'_j, \forall j\in J\}.$$

Since a topological functor is self-dual, the categories **diRel**, **diRere** and **diProst** have also co-products. Let us determine the co-products in **diRel**. Let $(V,\mathcal{V})$ be a texture space, $\{(V_j,\mathcal{V}_j,h_j,H_j) \mid j\in J\}$ be a family of textural approximation spaces, and $\{(f_j,F_j) \mid j\in J\}$ be a collection of difunctions from $(V_j,\mathcal{V}_j)$ to $(V,\mathcal{V})$. Then we may prove that there exists a unique direlation (h,H) on $(V,\mathcal{V})$ such that for all $j\in J$, (f_j,F_j) is a **diRel**-morphism from $(V_j,\mathcal{V}_j,h_j,H_j)$ to $(V,\mathcal{V},h,H)$ where

$$h=\bigvee\{\overline{P}_{(v,v')} \mid \exists j\in J, \exists a,b\in V_j,\ f_j\not\subseteq \overline{Q}_{(b,v')},\ \overline{P}_{(a,v)}\not\subseteq F_j \text{ and } h_j\not\subseteq \overline{Q}_{(a,b)}\}$$

and

$$H=\bigcap\{\overline{Q}_{(v,v')} \mid \exists j\in J, \exists a,b\in U_j,\ f_j\not\subseteq \overline{Q}_{(a,v)},\ \overline{P}_{(b,v')}\not\subseteq F_j \text{ and } \overline{P}_{(a,b)}\not\subseteq H_j\}.$$

Now we may determine the set-indexed co-products (sums) in **diRel** with respect to injection difunctions. Let $\{(V_j,\mathcal{V}_j) \mid j\in J\}$ be a family of textures with $V_j\cap V_k=\emptyset$ for $j\neq k$. Recall that a pair $(V,\mathcal{V})$ where

$$V=\bigcup_{j\in J}V_j \text{ and } \mathcal{V}=\{A\subseteq V \mid A\cap V_j\in\mathcal{U}_j, \forall j\in J\}$$

is called *a disjoint sum* of the textures $(V_j,\mathcal{V}_j)$, $j\in J$ and the jth-injection difunction $(e_j,E_j):V_j\to V$ [9] is defined by

$$e_j=\bigvee\{\overline{P}_{(z,\ (z,j))} \mid z\in V_j\}, \qquad E_j=\bigcap\{\overline{Q}_{(z,(z,j))} \mid z\in V_j\}.$$

It is easy to see that the sink

$$\big((e_j,E_j):(V_j,\mathcal{V}_j,h_j,H_j)\big)_{j\in J}\to(V,\mathcal{V},h,H)$$

is a coproduct in **diRel** where

$$h=\bigvee\{\overline{P}_{(v,v')} \mid h_j\not\subseteq \overline{Q}_{(v,v')}, \forall j\in J\} \quad \text{and} \quad H=\bigcap\{\overline{Q}_{(v,v')} \mid P_{(v,v')}\not\subseteq H_j, \forall j\in J\}.$$

For discrete textures, coproduct in **diRel** corresponds to coproduct in **Rel**.

$$h=\bigvee\{\overline{P}_{(v,v')} \mid h_j\not\subseteq \overline{Q}_{(v,v')}, \forall j\in J\}=\{(v,v') \mid h_j(v)=v', \forall j\in J\}.$$

4. Definability and bicontinuity

Theorem 4.1. *Let $(U, \mathcal{U}, r, R)$ and $(V, \mathcal{V}, h, H)$ be textural approximation spaces, and (f, F) be a difunction from $(U, \mathcal{U}, r, R)$ to $(V, \mathcal{V}, h, H)$. Then we have the following.*

(i) *(f, F) is a relation preserving difunction if and only if $f \circ r \subseteq h \circ f$.*
(ii) *(f, F) is a corelation preserving difunction if and only if $H \circ F \subseteq F \circ R$.*

Proof. (i) $\Longleftarrow$: Suppose that $f \circ r \subseteq h \circ f$ and $r \not\subseteq \overline{Q}_{(u,u')}$ for some $u, u' \in U$. Let us choose $u_1, u_2 \in U$ such that $r \not\subseteq \overline{Q}_{(u_1,u_2)}$ and $\overline{P}_{(u_1,u_2)} \not\subseteq \overline{Q}_{(u,u')}$. Clearly, we have $r \not\subseteq \overline{Q}_{(u,u_2)}$ and $P_{u_2} \not\subseteq Q_{u'}$. By DF1, we have

$$f \not\subseteq \overline{Q}_{(u_2,v)} \quad \text{and} \quad \overline{P}_{(u',v)} \not\subseteq F$$

for some $v \in V$. Choose a point $v_1 \in V$ such that $f \not\subseteq \overline{Q}_{(u_2,v_1)}$ and $P_{v_1} \not\subseteq Q_v$. Then $r \not\subseteq \overline{Q}_{(u,u_2)}$ and $f \not\subseteq \overline{Q}_{(u_2,v_1)}$ implies that $\overline{P}_{(u,v_1)} \subseteq f \circ r$ and this gives $f \circ r \not\subseteq \overline{Q}_{(u,v)}$. Now by the assumption, we have $h \circ f \not\subseteq \overline{Q}_{(u,v)}$. By definition of composition, for some $v' \in V$ we write $f \not\subseteq \overline{Q}_{(u,v')}$ and $h \not\subseteq \overline{Q}_{(v',v)}$. Then $v, v' \in V$ are the desired points for the theorem, that is, (f, F) is a relation preserving difunction.

$\Longrightarrow$: Let us show the inclusion $f \circ r \subseteq h \circ f$. Let $f \circ r \not\subseteq \overline{Q}_{(u_1,v_1)}$. Then there exist $u \in U$ and $v \in V$ such that $\overline{P}_{(u,v)} \not\subseteq \overline{Q}_{(u_1,v_1)}$ where

$$r \not\subseteq \overline{Q}_{(u,u_2)} \quad \text{and} \quad f \not\subseteq \overline{Q}_{(u_2,v)}$$

for some $u_2 \in U$. Then $r \not\subseteq \overline{Q}_{(u_1,u_2)}$ and $f \not\subseteq \overline{Q}_{(u_2,v_1)}$. By the assumption

$$f \not\subseteq \overline{Q}_{(u_1,v_3)}, \qquad \overline{P}_{(u_2,v_4)} \not\subseteq F \quad \text{and} \quad h \not\subseteq \overline{Q}_{(v_3,v_4)}$$

for some $v_3, v_4 \in V$. By DF2, $f \not\subseteq \overline{Q}_{(u_2,v_1)}$ and $\overline{P}_{(u_2,v_4)} \not\subseteq F$ implies that $P_{v_4} \not\subseteq Q_{v_1}$. Then $h \not\subseteq \overline{Q}_{(v_3,v_1)}$. Let us choose a point $v_2 \in V$ such that $h \not\subseteq \overline{Q}_{(v_3,v_2)}$ and $P_{v_2} \not\subseteq Q_{v_1}$. Clearly, we have $\overline{P}_{(u_1,v_2)} \subseteq h \circ f$. Note that $\overline{P}_{(u_1,v_2)} \not\subseteq \overline{Q}_{(u_1,v_1)}$ and we obtain $h \circ f \not\subseteq \overline{Q}_{(u_1,v_1)}$.

(ii) It is left to the reader. $\square$

Theorem 4.2. *Let $(U, \mathcal{U}, r, R)$ and $(V, \mathcal{V}, h, H)$ be textural approximation spaces, and (f, F) be a direlation preserving difunction from $(U, \mathcal{U}, r, R)$ to $(V, \mathcal{V}, h, H)$. If $B \in \mathcal{V}$ is (h, H)-definable, then $f^{\leftarrow}B$ or $F^{\leftarrow}B$ is (r, R)-definable.*

Proof. Let $B \in \mathcal{V}$ be (h, H)-definable, that is, $h^{\leftarrow}B = H^{\leftarrow}B$. Since (f, F) is a difunction, $f^{\leftarrow}h^{\leftarrow}B = F^{\leftarrow}H^{\leftarrow}B$ and so $(h \circ f)^{\leftarrow}B = (H \circ F)^{\leftarrow}B$. By Theorem 4.1(ii), we have $f \circ r \subseteq h \circ f$ and $H \circ F \subseteq F \circ R$. By Lemma 2.4(3) in [8], we conclude that $(h \circ f)^{\leftarrow} \subseteq (f \circ r)^{\leftarrow}$ and $(F \circ R)^{\leftarrow} \subseteq (H \circ F)^{\leftarrow}$. Therefore, by Lemma 2.7 in [8], we find

$$(F \circ R)^{\leftarrow}B \subseteq (H \circ F)^{\leftarrow}B = (h \circ f)^{\leftarrow}B \subseteq (f \circ r)^{\leftarrow}B,$$

that is, $R^{\leftarrow}F^{\leftarrow}B \subseteq r^{\leftarrow}f^{\leftarrow}B$. Since (r, R) is reflexive, we have already $r^{\leftarrow}f^{\leftarrow}B \subseteq R^{\leftarrow}F^{\leftarrow}B$ and hence,

$$r^{\leftarrow}f^{\leftarrow}B = R^{\leftarrow}F^{\leftarrow}B.$$

As a result, $f^{\leftarrow}B$ $(F^{\leftarrow}B)$ is (r, R)-definable. $\square$

Recall that [14] if (r, R) is a reflexive direlation on $(\mathcal{U}, U)$, then $(\tau(r), \kappa(R))$ is a ditopology on $(U, \mathcal{U})$ where

$$\tau(r) = \{A \in \mathcal{U} \mid r^{\leftarrow}A = A\} \quad \text{and} \quad \kappa(R) = \{A \in \mathcal{U} \mid R^{\leftarrow}A = A\}.$$

Theorem 4.3. *Let $(U, \mathcal{U}, r, R)$ and $(V, \mathcal{V}, h, H)$ be textural approximation spaces where (r, R) and (h, H) are reflexive direlations on $(U, \mathcal{U})$ and $(V, \mathcal{V})$, respectively. Further, let $(f, F) : (U, \mathcal{U}, r, R) \to (V, \mathcal{V}, h, H)$ be a direlation preserving difunction. Then $(f, F) : (U, \mathcal{U}, \tau(r), \kappa(r)) \to (V, \mathcal{V}, \tau(h), \kappa(h))$ is bicontinuous.*

Proof. Let $B \in \tau(h)$. Then $B = h^{\leftarrow}B$, and so $f^{\leftarrow}B = f^{\leftarrow}h^{\leftarrow}B$. Therefore,

$$f^{\leftarrow}B = (h \circ f)^{\leftarrow}B \subseteq (f \circ r)^{\leftarrow}B = r^{\leftarrow}f^{\leftarrow}B.$$

Since r is reflexive, by Theorem 4.4(i) in [15], we also have $r^{\leftarrow}f^{\leftarrow}B \subseteq B$. This gives that $r^{\leftarrow}f^{\leftarrow}B = B$, that is $f^{\leftarrow}B \in \tau(r)$. Hence (f, F) is continuous. The proof of cocontinuity of (f, F) is similar. $\square$

5. Definability and approximation spaces

Definability is one of the core concepts of rough set theory (see, [20,21,28,29,34,35]). Therefore, the connection between definable sets of approximation spaces may be highly important. To see this connection, let us consider first the following result given in [12] for $U = V$.

Theorem 5.1. *Let r be a relation from U to V. Then the following conditions are equivalent.*

(i) *r is a function from U to V.*
(ii) *$\forall Y \subseteq V$, Y is definable.*
(iii) *$\forall v \in V$, $\{v\}$ is definable.*

Proof. (i) $\Longrightarrow$ (ii) Let r be a function from U to V and suppose that $\underline{apr}_r Y \neq \overline{apr}_r Y$. Let $u \in U \setminus r^{-1}(V \setminus Y)$ and $u \notin r^{-1}(Y)$ for some $u \in U$. Then we have

$$u \notin r^{-1}(Y) \cup r^{-1}(V \setminus Y) = r^{-1}\big(V \cup (V \setminus Y)\big) = r^{-1}(U).$$

But this implies a contradiction, since $r^{-1}(V) = U$. Now let $u \notin U \setminus r^{-1}(V \setminus B)$ and $u \in r^{-1}(Y)$. Then $u \in r^{-1}(V \setminus Y)$ and this implies that $(u, v), (u, v') \in r$ for some $v, v' \in V$ where $v \neq v'$. Since r is a function, we also obtain a contradiction.

(ii) $\Longrightarrow$ (iii) Immediate.

(iii) $\Longrightarrow$ (i) Now suppose that for all $v \in V$, $\{v\}$ is definable. If r is not a function, then we have two cases:

(a) For some $u \in U$ and $v, v' \in V$ we have $(u, v), (u, v') \in r$ where $v \neq v'$, or
(b) For some $u \in U$, $u \notin r^{-1}(V)$.

Consider the case (a). By the assumption, we have $U \setminus r^{-1}(V \setminus \{v\}) = r^{-1}(\{v\})$. Since $v' \in V \setminus \{v\}$, $u \in r^{-1}(V \setminus \{v\})$, that is, $u \notin U \setminus r^{-1}(V \setminus \{v\})$. But $u \in r^{-1}(\{v\})$ implies a contradiction.

Now take the case (b). Since $\{v\}$ is definable, we have $U \setminus r^{-1}(V \setminus \{v\}) = r^{-1}(\{v\})$. Further, by (b), $u \notin r^{-1}(\{v\})$ and $u \notin r^{-1}(V \setminus \{v\})$ and so $u \in U \setminus r^{-1}(V \setminus \{v\})$ is a contradiction. $\square$

Theorem 5.2. *Let (U, r) and (V, h) be approximation spaces and $f : (U, r) \to (V, h)$ be a function. Then f is relation preserving if and only if $f \circ r \subseteq h \circ f$.*

Proof. $\Longrightarrow$: Suppose the contrary, that is, let $f \circ r \nsubseteq h \circ f$. Let us choose $u \in U$ and $v \in V$ such that $(u, v) \in f \circ r$ and $(u, v) \notin h \circ f$. Then we have

$$\forall v' \in V, \qquad \big(v', v\big) \in h \quad \Rightarrow \quad \big(u, v'\big) \notin f. \tag{1}$$

By definition of composition, we have $(u, u') \in r$ and $(u', v) \in f$ for some $u' \in U$. Since f is relation preserving mapping, we may write $(f(u), f(u')) \in h$. Let $f(u) = v'$ where $v' \in V$. Since $f(u') = v$, $(v', v) \in h$, and so by (1), $(u, v') \notin f$, that is, $f(u) \neq v'$ is a contradiction.

$\Longleftarrow$: Let $(u, u') \in r$. Since f is function from U to V, $f(u') = v$ for some $v \in V$, that is, $(u', v) \in f$. By definition of composition of relations, we have $(u, v) \in f \circ r$ and so $(u, v) \in h \circ f$. Therefore, $(u, v') \in f$ and $(v', v) \in h$ for some $v' \in V$. Hence, we obtain $(f(u), f(u')) \in h$. $\square$

By Theorem 5.1, if $f \subseteq U \times V$, then every subset of V is f-definable provided that f is a function. This result gives a connection between information systems for single universes, that is, inverse image of a definable set is also definable in the category **Rere** of reflexive relations:

Theorem 5.3. *Let (U, r) and (V, h) be approximation spaces where r and h are reflexive relations on U and V, respectively. Further, let $f : (U, r) \to (V, h)$ be a relation preserving mapping. If $Y \subseteq V$ is h-definable, then $f^{-1}(Y)$ is r-definable.*

Proof. Let $Y \subseteq V$ be an h-definable set, that is $\underline{apr}_h Y = \overline{apr}_h Y$. Since f is a function from U to V, by Theorem 5.1, every subset of V is f-definable, that is

$$\underline{apr}_f \underline{apr}_h Y = \overline{apr}_f \overline{apr}_h Y.$$

Then by Theorem 6.2 in [15], we have

$$\begin{aligned}
&U \setminus f^{-1}\big(V \setminus \big(V \setminus h^{-1}(V \setminus Y)\big)\big) = f^{-1}\big(h^{-1}(Y)\big) \\
&\quad \Longrightarrow \quad U \setminus f^{-1}\big(h^{-1}(V \setminus Y)\big) = f^{-1}\big(h^{-1}(Y)\big) \\
&\quad \Longrightarrow \quad U \setminus (h \circ f)^{-1}(V \setminus Y) = (h \circ f)^{-1}(Y).
\end{aligned}$$

Further, by Theorem 5.2, we also have the inclusion $(f\circ r)^{-1}\subseteq(h\circ f)^{-1}$ and this implies that

$$\begin{aligned}(f\circ r)^{-1}(Y)\subseteq(h\circ f)^{-1}(Y)&=U\setminus(h\circ f)^{-1}(V\setminus Y)\subseteq U\setminus(f\circ r)^{-1}(V\setminus Y)\\&=U\setminus r^{-1}\big(f^{-1}(V\setminus Y)\big)=U\setminus r^{-1}\big(U\setminus f^{-1}(Y)\big),\end{aligned}$$

that is, $r^{-1}(f^{-1}(Y))\subseteq U\setminus r^{-1}(U\setminus f^{-1}(Y))$ and so we obtain

$$\overline{apr}_r f^{-1}(Y)\subseteq\underline{apr}_r f^{-1}(Y).$$

Since r is reflexive, we conclude that

$$\overline{apr}_r f^{-1}(Y)=\underline{apr}_r f^{-1}(Y).\qquad\square$$

Theorem 5.4. *Let (U,r) and (V,h) be approximation spaces and let $f:(U,r)\to(V,h)$ be a relation preserving mapping. Then we have the following inclusions:*

(i) $\forall Y\subseteq V,\ f^{-1}(\underline{apr}_h(Y))\subseteq\underline{apr}_r(f^{-1}(Y))$.
(ii) $\forall X\subseteq U,\ f(\overline{apr}_r(X))\subseteq\overline{apr}_h(f(X))$.

Proof. (i) Let $Y\subseteq V$. By Theorem 5.2, we have

$$\begin{aligned}f^{-1}\big(\underline{apr}_h(Y)\big)&=f^{-1}\big(V\setminus h^{-1}(V\setminus Y)\big)=U\setminus f^{-1}\big(h^{-1}(V\setminus Y)\big)\\&=U\setminus(h\circ f)^{-1}(V\setminus Y)\subseteq U\setminus(f\circ r)^{-1}(V\setminus Y)\\&=U\setminus r^{-1}\big(f^{-1}(V\setminus Y)\big)=U\setminus r^{-1}\big(U\setminus f^{-1}(Y)\big)\\&=\underline{apr}_r\big(f^{-1}(Y)\big).\end{aligned}$$

(ii) If $X\subseteq U$, then by Theorem 5.2, we obtain

$$\begin{aligned}f\big(\overline{apr}_r(X)\big)&=f\big(r^{-1}(X)\big)=\big(f\circ r^{-1}\big)(X)\subseteq\big(h^{-1}\circ f\big)(X)\\&=h^{-1}\big(f(X)\big)=\overline{apr}_h\big(f(X)\big).\qquad\square\end{aligned}$$

Recall that if r is a reflexive relation on a universe U, then by Proposition 2 in [24], the family

$$\tau_r=\big\{X\subseteq U\mid\underline{apr}_r(X)=X\big\}$$

is a topology on U. Now we have the following:

Theorem 5.5. *Let (U,r) and (V,h) be approximation spaces where r and h are reflexive relations on U and V, respectively. Further, let $f:(U,r)\to(V,h)$ be a relation preserving mapping. Then $f:(U,\tau_r)\to(V,\tau_h)$ is continuous.*

Proof. Let $Y\in\tau_h$. Then by the preceding theorem, we have

$$f^{-1}(Y)=f^{-1}\big(\underline{apr}_h(Y)\big)\subseteq\underline{apr}_r\big(f^{-1}(Y)\big).$$

Since r is reflexive, this gives that $f^{-1}(Y)=\underline{apr}_r(f^{-1}(Y))$, that is, $f^{-1}(Y)\in\tau_r$. $\square$

6. Information systems

Recall that *an information system* is a quadruple $S=(U,AT,V,f)$ where U is a set of objects, AT is a set of attributes, $V=\bigcup_{a\in AT}V_a$ is a set of *values* of attributes and V_a is the domain of a where $f:U\times AT\to V$ is a description function such that $f(x,a)\in V_a$ for every $x\in U$ and $a\in AT$ [4,5]. Take $B\subseteq AT$ and $x,y\in U$. Then the equivalent relation

$$IND(B)=\big\{(x,y)\mid\forall a\in B,\ f(x,a)=f(y,a)\big\}$$

is called *B-indiscernibility relation*. Now let $S=(U,AT,V,f)$ and $S=(U',AT',V',f')$ be two information systems, let h_O be a mapping of U into U', let h_A be a mapping of AT into AT', and let h_D be a mapping of V into V'. The triple $h=(h_O,h_A,h_D)$ is called *a homomorphism* of S into S' [4,5], if for all $x\in U$ and $a\in AT$,

$$h_D\big(f(x,a)\big)=f'\big(h_O(x),h_A(a)\big).$$

Clearly, if $B\subseteq AT$, then $IND(B)=\{(x,y)\mid\forall a\in B,f(x,a)=f(y,a)\}$ is an equivalence relation on U. Now consider the approximation space $(U,IND(B))$ corresponding to the information system $S=(U,AT,V,f)$ for some $B\subseteq AT$.

Table 1
Information systems S and S'.

	a	b
x_1	1	2
x_2	2	1
x_3	2	1

	a'	b'
x'_1	1	2
x'_2	2	1
x'_3	1	2

Table 2
h_O and h_A.

x_1	x_2	x_3
x'_1	x'_3	x'_3

a	b
a'	b'

Proposition 6.1. *If $h = (h_O, h_A, h_D)$ is a homomorphism of S into S', then for all $B \subseteq AT$,*

$$h_O : \big(U, IND(B)\big) \to \big(U', IND\big(h_A(B)\big)\big)$$

is a relation preserving function.

Proof. Let $(x, y) \in IND(B)$. Then for all $a \in B$, we have $f(x, a) = f(y, a)$. Since h_D is a function, we also have $h_D(f(x, a)) = h_D(f(y, a))$. If we consider that h is a homomorphism of S into S', then for all $h_A(a) \in h_A(B)$, $f'(h_O(x), h_A(a)) = f'(h_O(y), h_A(a))$ and so $(h_O(x), h_O(y)) \in IND(h_A(B))$. $\square$

In general, the converse of Proposition 6.1 is not true:

Example 6.2. Let $S = (U, AT, V, f)$ and $S = (U', AT', V', f')$ be the information systems given as in Table 1. If the object and attribute function is given as in Table 2, then the triple $h = (h_O, h_A, h_D)$ is not an information system homomorphism since $h_D(f(x_3, a)) = 2$ and $f'(h_O(x_3), h_A(a)) = f'(x'_3, a') = 1$. However, $h_O : (U, AT) \to (U', AT')$ is a relation preserving function. To see this, consider the relations $IND(AT) = \{\{x_1\}, \{x_2, x_3\}\}$ and $IND(AT') = \{\{x'_1, x'_3\}, \{x'_2\}\}$ on U and U', respectively. Clearly, $(x_2, x_3) \in IND(AT)$ and $(h_O(x_2), h_O(x_3)) = (x'_3, x'_3) \in AT'$.

Corollary 6.3. *Let $h = (h_O, h_A, h_D)$ be a homomorphism of S into S', h_A be surjective and $Y \subseteq U'$. If the set Y is $IND(C)$-definable in S' for some $C \subseteq AT'$, then $h_O^{-1}(Y)$ is $IND(h_A^{-1}(C))$-definable in S.*

Proof. Immediate from Proposition 6.1 and Theorem 5.3. $\square$

Now for an information system $S = (U, AT, V, f)$, let us consider the lower and upper approximation operators,

$$\underline{apr}_{IND(B)}(X) = \underline{B}(X) \quad \text{and} \quad \overline{apr}_{IND(B)}(X) = \overline{B}(X),$$

respectively, where $B \subseteq AT$ and $X \subseteq U$.

The following result is already proved in [36].

Corollary 6.4. *Let $h = (h_O, h_A, h_D)$ be a homomorphism of S into S'. Then we have the following:*

(i) *If $B \subseteq AT$ and $X \subseteq U$, then $h_O(\overline{B}(X)) \subseteq \overline{h_A(B)}(h_O(X))$.*
(ii) *If $C \subseteq AT'$ and $Y \subseteq U'$, then $h_O^{-1}(\underline{C}(Y)) \subseteq \underline{h_A^{-1}(C)}(h_O^{-1}(Y))$.*

Proof. (i) By Proposition 6.1, $h_O : (U, IND(B)) \to (U', IND(h_A(B)))$ is a relation preserving function and then by Theorem 5.4(ii), we obtain the desired inclusion.

(ii) By Proposition 6.1,

$$h_O : \big(U, IND\big(h_A^{-1}(C)\big)\big) \to \big(U', IND\big(h_A\big(h_A^{-1}(C)\big)\big)\big)$$

is a relation preserving function. Since $h_A(h_A^{-1}(C)) \subseteq C$, the object function

$$h_O : \big(U, IND\big(h_A^{-1}(C)\big)\big) \to \big(U', IND(C)\big)$$

is also a relation preserving function. Then by Theorem 5.4(ii), we obtain the inclusion. $\square$

Theorem 6.5. *Information systems and homomorphisms form a category denoted by* **IS**.

Proof. Let $S = (U, AT, V, f)$, $S' = (U', AT', V', f')$, and $S'' = (U'', AT'', V'', f'')$ be information systems where $h : S \to S'$, $h' : S' \to S''$ be homomorphisms. Let us define the composition of h' and h as

$$h' \circ h = \left(h'_O \circ h_O,\ h'_A \circ h_A,\ h'_D \circ h_D\right).$$

Then

$$\begin{aligned}\left(h'_D \circ h_D\right)\left(f(x,a)\right) &= h'_D\left(h_D\left(\left(f(x,a)\right)\right)\right)\\ &= h'_D\left(f'\left(h_O(x),\ h_A(x)\right)\right)\\ &= f''\left(h'_O\left(h_O(x)\right),\ h'_A\left(h_A(x)\right)\right)\\ &= f''\left(\left(h'_O \circ h_O\right)(x),\ \left(h'_A \circ h_A\right)(x)\right)\end{aligned}$$

and so $h' \circ h : U \to U''$ is also an information system homomorphism. Further, the composition is associative and for any information system $S = (U, AT, V, f)$, $I_S = (I_O, I_A, I_D)$ is the identity homomorphism on S where

$$I_O : U \to U, \qquad I_A : AT \to AT \quad \text{and} \quad I_D : V \to V$$

are the identity functions, respectively. Indeed, for all $x \in U$ and $a \in AT$, we have

$$I_D\left(f(x,a)\right) = f(x,a) = f\left(I_O(x), I_A(x)\right). \qquad \square$$

Now let us consider the category of all information system homomorphisms and all object-irreducible information systems where the attribute functions are surjective. We denote this category by **ISO**.

Theorem 6.6. (i) *The mapping* $\mathfrak{B}$*:* **IS** $\to$ **Rel** *defined by*

$$\mathfrak{B}\left((U, AT, V, f)\right) = \left(U, IND(AT)\right) \quad \text{and} \quad \mathfrak{B}(h) = h_O$$

for all $(U, AT, V, f) \in \mathrm{ob}(\mathbf{IS})$ *and* $h \in \mathrm{hom}(\mathbf{IS})$ *is a functor.*

(ii) *If* $\mathfrak{B}_{\mathbf{ISO}}$ *is the restriction functor of* $\mathfrak{B}$*, then* $\mathfrak{B}_{\mathbf{ISO}}$*:* **ISO** $\to$ **Rel** *is an embedding.*

Proof. By Proposition 6.1,

$$h'_O \circ h_O : \left(U, IND(AT)\right) \to \left(U'', IND\left(\left(h'_A \circ h_A\right)(AT)\right)\right)$$

is a relation preserving function and hence,

$$\mathfrak{B}\left(\left(h' \circ h\right)\right) = h'_O \circ h_O = \mathfrak{B}\left(h'\right) \circ \mathfrak{B}(h).$$

Further, $\mathfrak{B}(I_S) = \mathfrak{B}(I_O, I_A, I_D) = I_O$.

(ii) By Theorem 4 in [36], h_O is uniquely determined by h_A and h_D. $\square$

7. Conclusion

In this work, we call attention to the category **Rel** of approximation spaces and relation preserving functions. To this end, we defined a textural version of **Rel** as the category **diRel** of textural approximation spaces and direlation preserving difunctions. We observed that **Rel** can be fully embedded into **diRel**. Moreover, we proved that **diRel** is a topological category and hence, it has product and sums. Furthermore, we discussed textural definability between textural approximation spaces under direlation preserving difunctions. We extended these arguments to approximation spaces. We showed that relation preserving functions preserve the definability under the inverse images. Information systems and information system homomorphisms form a category denoted by **IS**. If the information systems are object irreducible and the attribute functions are surjective, then we obtain a subcategory **ISO** of **IS**. We proved that **ISO** is embeddable into **Rel**.

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