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A Subclass of Analytic Functions Defined by Using Certain Operators of Fractional Calculus

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Abstract—Making use of certain operators of fractional calculus, we introduce a new class $\mathbb{F}_\delta(n,\lambda,\alpha)$ of functions which are analytic in the open unit disk $\mathcal{U}$ and obtain a necessary and sufficient condition for a function to be in the class $\mathbb{F}_\delta(n,\lambda,\alpha)$. We also determine the radii of close-to-convexity, starlikeness, and convexity. Finally, an application involving fractional calculus of functions in the class $\mathbb{F}_\delta(n,\lambda,\alpha)$ is considered.

Keywords—Analytic functions, Fractional calculus, Coefficient bounds, Distortion theorems, Close-to-convex functions, Starlike functions.

1. INTRODUCTION AND DEFINITIONS

Let $\mathbb{F}(n)$ denote the class of functions $f(z)$ of the form

$$f(z) = z - \sum_{k=n+1}^{\infty} a_k z^k, \qquad (a_k \geq 0;\ n \in \mathbb{N} := \{1,2,3,\dots\}), \tag{1.1}$$

which are analytic in the *open* unit disk

$$\mathcal{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}.$$

Let $\mathbb{F}_\delta(n,\lambda,\alpha)$ be the subclass of $\mathbb{F}(n)$ consisting of functions which also satisfy the inequality

$$\Re\left\{\Gamma(2-\delta)\, z^{\delta-1}\left[(1-\lambda)\, D_z^\delta f(z) + \lambda z\, D_z^{1+\delta} f(z)\right]\right\} > \alpha, \qquad (\delta+\alpha<1), \tag{1.2}$$

for some δ $(0 \leq \delta < 1)$, λ $(0 \leq \lambda \leq 1)$, and α $(0 \leq \alpha < 1)$, and for all $z \in \mathcal{U}$. Here, and *throughout this paper*, D_z^δ denotes an operator of fractional calculus, which is defined as follows (cf., e.g., [1,2]).

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DEFINITION 1. *The fractional integral of order μ is defined by*

$$D_z^{-\mu} f(z) = \frac{1}{\Gamma(\mu)} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{1-\mu}}\, d\zeta, \qquad (\mu > 0), \tag{1.3}$$

where $f(z)$ is an analytic function in a simply-connected region of the z-plane containing the origin, and the multiplicity of $(z-\zeta)^{\mu-1}$ is removed by requiring $\log(z-\zeta)$ to be real when $z - \zeta > 0$.

DEFINITION 2. *The fractional derivative of order μ is defined by*

$$D_z^{\mu} f(z) = \frac{1}{\Gamma(1-\mu)} \frac{d}{dz} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{\mu}}\, d\zeta, \qquad (0 \le \mu < 1), \tag{1.4}$$

where $f(z)$ is constrained, and the multiplicity of $(z-\zeta)^{-\mu}$ is removed, as in Definition 1.

DEFINITION 3. *Under the hypotheses of Definition 1, the fractional derivative of order $k+\mu$ is defined by*

$$D_z^{k+\mu} f(z) = \frac{d^k}{dz^k} D_z^{\mu} f(z), \qquad (0 \le \mu < 1;\ k \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}). \tag{1.5}$$

The object of the present paper is to investigate various interesting properties of functions belonging to the class $\mathbb{F}_\delta(n, \lambda, \alpha)$. We remark in passing that

$$\mathbb{F}_0(1, \lambda, \alpha) = \mathbb{F}_\lambda(\alpha), \qquad (0 \le \lambda \le 1;\ 0 \le \alpha < 1), \tag{1.6}$$

where the class $\mathbb{F}_\lambda(\alpha)$ was studied recently by Bhoosnurmath and Swamy [3].

2. A THEOREM ON COEFFICIENT BOUNDS

THEOREM 1. *A function $f(z) \in \mathbb{F}(n)$ is in the class $\mathbb{F}_\delta(n, \lambda, \alpha)$ if and only if*

$$\sum_{k=n+1}^{\infty} \frac{[1+\lambda(k-1-\delta)]\Gamma(k+1)}{\Gamma(k+1-\delta)} a_k \le 1 - \lambda\delta - \alpha, \qquad (\delta + \alpha < 1). \tag{2.1}$$

The result is sharp.

PROOF. Suppose that $f(z) \in \mathbb{F}_\delta(n, \lambda, \alpha)$. Then, we find from Definitions 1 and 3, and the inequality (1.2), that

$$\Re\left\{1 - \lambda\delta - \sum_{k=n+1}^{\infty} \frac{[1+\lambda(k-1-\delta)]\,\Gamma(k+1)}{\Gamma(k+1-\delta)} a_k\, z^{k-1}\right\} > \alpha, \qquad (z \in \mathcal{U}).$$

If we choose z to be real and let $z \to 1-$, we get

$$1 - \lambda\delta - \sum_{k=n+1}^{\infty} \frac{[1+\lambda(k-1-\delta)]\,\Gamma(k+1)}{\Gamma(k+1-\delta)} a_k \ge \alpha, \qquad (\delta + \alpha < 1;\ 0 \le \alpha < 1;\ 0 \le \delta < 1),$$

which is equivalent to the assertion (2.1) of Theorem 1.

Conversely, let us suppose that the inequality (2.1) holds true. Then, we have

$$\begin{aligned}&\left|\Gamma(2-\delta)\ z^{\delta-1}\ \left[(1-\lambda)\, D_z^{\delta} f(z) + z\lambda\, D_z^{1+\delta} f(z)\right] - 1 + \gamma\delta\right| \\ &\qquad = \left| - \sum_{k=n+1}^{\infty} \frac{[1+\lambda(k-1-\delta]\,\Gamma(k+1)}{\Gamma(k+1-\delta)} a_k\, z^{k-1}\right|\end{aligned}$$

$$\leq \sum_{k=n+1}^{\infty} \frac{[1+\lambda(k-1-\delta)]\,\Gamma(k+1)}{\Gamma(k+1-\delta)}\, a_k\, |z|^{k-1}$$
$$\leq 1-\lambda\delta-\alpha, \qquad (z \in \mathcal{U};\ \delta+\alpha<1;\ 0\leq\alpha<1;\ 0\leq\delta<1),$$

which implies that $f(z) \in \mathbb{F}_\delta(n,\lambda,\alpha)$.

Finally, we note that the assertion (2.1) of Theorem 1 is sharp, the extremal function being

$$f(z) = z - \frac{(1-\lambda\delta-\alpha)\,\Gamma(n+2-\delta)}{[1+\lambda(n-\delta)]\,\Gamma(n+2)}\, z^{n+1}, \qquad (n \in \mathbb{N}). \tag{2.2}$$

COROLLARY 1. *If* $f(z) \in \mathbb{F}_\delta(n,\lambda,\alpha)$, *then*

$$a_{n+1} \leq \frac{(1-\lambda\delta-\alpha)\,\Gamma(n+2-\delta)}{[1+\lambda(n-\delta)]\,\Gamma(n+2)}, \qquad (n \in \mathbb{N}). \tag{2.3}$$

COROLLARY 2. *A function* $f(z) \in \mathbb{F}(n)$ *is in the class* $\mathbb{F}_0(n,\lambda,\alpha)$ *if and only if*

$$\sum_{k=n+1}^{\infty} [1+\lambda(k-1)]\, a_k \leq 1-\alpha, \qquad (0\leq\lambda\leq 1;\ 0\leq\alpha<1). \tag{2.4}$$

COROLLARY 3. *(cf., [3, p. 90, Theorem 1]). A function* $f(z) \in \mathbb{F}(1)$ *is in the class* $\mathbb{F}_0(1,\lambda,\alpha)$ *if the only if*

$$\sum_{k=2}^{\infty} [1+\lambda(k-1)]\, a_k \leq 1-\alpha, \qquad (0\leq\lambda\leq 1;\ 0\leq\alpha<1). \tag{2.5}$$

COROLLARY 4. *If* $f(z) \in \mathbb{F}_0(n,1,\alpha)$, *then* $\Re\{f'(z)\} > \alpha$ *for all* $z \in \mathcal{U}$.

PROOF. Since $f(z) \in \mathbb{F}_0(n,1,\alpha)$, we have (cf., [4])

$$\sum_{k=n+1}^{\infty} k\, a_k \leq 1-\alpha, \qquad (0\leq\alpha<1). \tag{2.6}$$

The result now follows from Theorem 1.

COROLLARY 5. *If* $f(z) \in \mathbb{F}_0(n,0,\alpha)$, *then*

$$\Re\left\{\frac{f(z)}{z}\right\} > \frac{1-\alpha}{n+1}, \qquad (n \in \mathbb{N}).$$

PROOF. Since $f(z) \in \mathbb{F}_0(n,0,\alpha)$, we have

$$(n+1)\sum_{k=n+1}^{\infty} a_k \leq \sum_{k=n+1}^{\infty} k\, a_k \leq 1-\alpha, \qquad (0\leq\alpha<1;\ n\in\mathbb{N}), \tag{2.7}$$

by applying the known inequality (2.6). Therefore, we obtain

$$\sum_{k=n+1}^{\infty} a_k \leq \frac{1-\alpha}{n+1}, \qquad (n \in \mathbb{N}). \tag{2.8}$$

COROLLARY 6. *(cf., [3, p. 91, Corollary 1.2]). If* $f(z) \in \mathbb{F}_0(1,0,0)$, *then*

$$\Re\left\{\frac{f(z)}{z}\right\} > \frac{1}{2},$$

for all $z \in \mathcal{U}$.

THEOREM 2. *Let the function $f(z)$ defined by (1.1) and the function $g(z)$ defined by*

$$g(z) = z - \sum_{k=n+1}^{\infty} b_k z^k, \qquad (b_k \geq 0;\ n \in \mathbb{N}) \tag{2.9}$$

be in the same class $\mathbb{F}_\delta(n,\lambda,\alpha)$. Then, the function $h(z)$ defined by

$$h(z) = (1-\beta) f(z) + \beta g(z) = z - \sum_{k=n+1}^{\infty} c_k z^k,$$
$$(c_k := (1-\beta) a_k + \beta b_k \geq 0;\ 0 \leq \beta \leq 1;\ n \in \mathbb{N})$$

is also in the class $\mathbb{F}_\delta(n,\lambda,\alpha)$.

PROOF. By the hypotheses of Theorem 2, we find from (2.1) that

$$\begin{aligned}\sum_{k=n+1}^{\infty} \frac{[1+\lambda(k-1-\delta)]\,\Gamma(k+1)}{\Gamma(k+1-\delta)} c_k &= (1-\beta) \sum_{k=n+1}^{\infty} \frac{[1+\lambda(k-1-\delta)]\,\Gamma(k+1)}{\Gamma(k+1-\delta)} a_k \\ &\quad + \beta \sum_{k=n+1}^{\infty} \frac{[1+\lambda(k-1-\delta)]\,\Gamma(k+1)}{\Gamma(k+1-\delta)} b_k \\ &\leq (1-\beta)(1-\lambda\delta-\alpha) + \beta(1-\lambda\delta-\alpha) = 1-\delta\lambda-\alpha,\end{aligned}$$

which completes the proof of Theorem 2.

3. DISTORTION THEOREMS INVOLVING OPERATORS OF FRACTIONAL CALCULUS

THEOREM 3. *If $f(z) \in \mathbb{F}_\delta(n,\lambda,\alpha)$, then*

$$\left|D_z^{-\mu} f(z)\right| \leq \frac{|z|^{1+\mu}}{\Gamma(2+\mu)} \left(1 + \frac{(1-\lambda\delta-\alpha)\,\Gamma(2+\mu)\,\Gamma(n+2-\delta)}{[1+\lambda(n-\delta)]\,\Gamma(n+2+\mu)} |z|\right), \tag{3.1}$$

and

$$\left|D_z^{-\mu} f(z)\right| \geq \frac{|z|^{1+\mu}}{\Gamma(2+\mu)} \left(1 - \frac{(1-\lambda\delta-\alpha)\,\Gamma(2+\mu)\,\Gamma(n+2-\delta)}{[1+\lambda(n-\delta)]\,\Gamma(n+2+\mu)} |z|\right), \tag{3.2}$$

for $\mu > 0$ and $n \in \mathbb{N}$, and for all $z \in \mathcal{U}$.

PROOF. Suppose that $f(z) \in \mathbb{F}_\delta(n,\lambda,\alpha)$. Then, we find from (2.1) that

$$\frac{[1+\lambda(n-\delta)]\,\Gamma(n+2)}{\Gamma(n+2-\delta)} \sum_{k=n+1}^{\infty} a_k \leq \sum_{k=n+1}^{\infty} \frac{[1+\lambda(k-1-\delta)]\,\Gamma(k+1)}{\Gamma(k+1-\delta)} a_k, \tag{3.3}$$

which evidently yields

$$\sum_{k=n+1}^{\infty} a_k \leq \frac{(1-\lambda\delta-\alpha)\,\Gamma(n+2-\delta)}{[1+\lambda(n-\delta)]\,\Gamma(n+2)}, \qquad (n \in \mathbb{N}). \tag{3.4}$$

Making use of (3.4) and Definition 1, we have

$$\begin{aligned} D_z^{-\mu} f(z) &= \frac{z^{1+\mu}}{\Gamma(2+\mu)} \left(1 - \sum_{k=n+1}^{\infty} \frac{\Gamma(k+1)\,\Gamma(2+\mu)}{\Gamma(k+1+\mu)} a_k z^{k-1}\right) \\ &= \frac{z^{1+\mu}}{\Gamma(2+\mu)} \left(1 - \sum_{k=n+1}^{\infty} \Theta(k)\, a_k z^{k-1}\right), \end{aligned} \tag{3.5}$$

where, for convenience,

$$\Theta(k) = \frac{\Gamma(k+1)\,\Gamma(2+\mu)}{\Gamma(k+1+\mu)}, \qquad (\mu > 0;\ k \geq n+1;\ n \in \mathbb{N}).$$

Clearly, the function $\Theta(k)$ is decreasing in k, and we have

$$0 < \Theta(k) \leq \Theta(n+1) = \frac{\Gamma(n+2)\,\Gamma(2+\mu)}{\Gamma(n+2+\mu)}. \tag{3.6}$$

Thus, we find from (3.4)–(3.6) that

$$\begin{aligned}\left|D_z^{-\mu} f(z)\right| &\leq \frac{|z|^{1+\mu}}{\Gamma(2+\mu)} \left(1 + |z|\,\Theta(n+1) \sum_{k=n+1}^{\infty} a_k\right) \\ &\leq \frac{|z|^{1+\mu}}{\Gamma(2+\mu)} \left(1 + \frac{(1-\lambda\delta-\alpha)\,\Gamma(2+\mu)\,\Gamma(n+2-\delta)}{[1+\lambda(n-\delta)]\,\Gamma(n+2+\mu)}\,|z|\right),\end{aligned}$$

which is precisely the assertion (3.1), and that

$$\begin{aligned}\left|D_z^{-\mu} f(z)\right| &\geq \frac{|z|^{1+\mu}}{\Gamma(2+\mu)} \left(1 - |z|\,\Theta(n+1) \sum_{k=n+1}^{\infty} a_k\right) \\ &\geq \frac{|z|^{1+\mu}}{\Gamma(2+\mu)} \left(1 - \frac{(1-\lambda\delta-\alpha)\,\Gamma(2+\mu)\,\Gamma(n+2-\delta)}{[1+\lambda(n-\delta)]\,\Gamma(n+2+\mu)}\,|z|\right),\end{aligned}$$

which is the same as the assertion (3.2).

THEOREM 4. *If $f(z) \in \mathbb{F}_\delta(n,\lambda,\alpha)$, then*

$$\left|D_z^{\mu} f(z)\right| \leq \frac{|z|^{1-\mu}}{\Gamma(2-\mu)} \left(1 + \frac{(1-\lambda\delta-\alpha)\,\Gamma(2-\mu)\Gamma(n+2-\delta)}{[1+\lambda(n-\delta)]\Gamma(n+2-\mu)}\,|z|\right), \tag{3.7}$$

and

$$\left|D_z^{\mu} f(z)\right| \geq \frac{|z|^{1-\mu}}{\Gamma(2-\mu)} \left(1 - \frac{(1-\lambda\delta-\alpha)\,\Gamma(2-\mu)\,\Gamma(n+2-\delta)}{[1+\lambda(n-\delta)]\,\Gamma(n+2-\mu)}\,|z|\right), \tag{3.8}$$

for $0 \leq \mu < 1$ and $n \in \mathbb{N}$, and for all $z \in \mathcal{U}$.

PROOF. Suppose that $f(z) \in \mathbb{F}_\delta(n,\lambda,\alpha)$. Then, we find from (2.1) that

$$\frac{[1+\lambda(n-\delta)]\,\Gamma(n+1)}{\Gamma(n+2-\delta)} \sum_{k=n+1}^{\infty} k\,a_k \leq \sum_{k=n+1}^{\infty} \frac{[1+\lambda(k-1-\delta)]\,\Gamma(k+1)}{\Gamma(k+1-\delta)}\,a_k, \tag{3.9}$$

which evidently yields

$$\sum_{k=n+1}^{\infty} k\,a_k \leq \frac{(1-\lambda\delta-\alpha)\,\Gamma(n+2-\delta)}{[1+\lambda(n-\delta)]\,\Gamma(n+1)}, \qquad (0 \leq \lambda \leq 1;\ 0 \leq \delta < 1;\ n \in \mathbb{N}). \tag{3.10}$$

Now, making use of (3.10) and Definition 2, we have

$$\begin{aligned}D_z^{\mu} f(z) &= \frac{z^{1-\mu}}{\Gamma(2-\mu)} \left(1 - \sum_{k=n+1}^{\infty} \frac{\Gamma(k+1)\,\Gamma(2-\mu)}{\Gamma(k+1-\mu)}\,a_k\,z^{k-1}\right) \\ &= \frac{z^{1-\mu}}{\Gamma(2-\mu)} \left(1 - \sum_{k=n+1}^{\infty} \Phi(k)\,k\,a_k\,z^{k-1}\right),\end{aligned} \tag{3.11}$$

where, for convenience,

$$\Phi(k) = \frac{\Gamma(k)\,\Gamma(2-\mu)}{\Gamma(k+1-\mu)} \qquad (0 \le \mu < 1;\ k \ge n+1;\ n \in \mathbb{N}).$$

Since the function $\Phi(k)$ is decreasing in k, we also have

$$0 < \Phi(k) \le \Phi(n+1) = \frac{\Gamma(n+1)\,\Gamma(2-\mu)}{\Gamma(n+2-\mu)}. \tag{3.12}$$

Thus, we find from (3.10)–(3.12) that

$$\begin{aligned} \left|D_z^{\mu} f(z)\right| &\le \frac{|z|^{1-\mu}}{\Gamma(2-\mu)} \left(1 + |z|\,\Phi(n+1) \sum_{k=n+1}^{\infty} k\, a_k\right) \\ &\le \frac{|z|^{1-\mu}}{\Gamma(2-\mu)} \left(1 + \frac{(1-\lambda\delta-\alpha)\,\Gamma(2-\mu)\,\Gamma(n+2-\delta)}{[1+\lambda(n-\delta)]\,\Gamma(n+2-\mu)}\,|z|\right), \end{aligned}$$

which is precisely the assertion (3.7), and that

$$\begin{aligned} \left|D_z^{\mu} f(z)\right| &\ge \frac{|z|^{1-\mu}}{\Gamma(2-\mu)} \left(1 - |z|\,\Phi(n+1) \sum_{k=n+1}^{\infty} k\, a_k\right) \\ &\ge \frac{|z|^{1-\mu}}{\Gamma(2-\mu)} \left(1 - \frac{(1-\lambda\delta-\alpha)\,\Gamma(2-\mu)\,\Gamma(n+2-\delta)}{[1+\lambda(n-\delta)]\,\Gamma(n+1-\mu)}\,|z|\right), \end{aligned}$$

which is the same as the assertion (3.8).

THEOREM 5. *If $f(z) \in \mathbb{F}_\delta(n,\lambda,\alpha)$, then*

$$\left|D_z^{1+\delta} f(z)\right| \le \frac{|z|^{-\delta}}{\Gamma(1-\delta)} \left(1 + \frac{(1-\lambda\delta-\alpha)(n+1-\delta)\,\Gamma(1-\delta)}{1+\lambda(n-\delta)}\,|z|\right), \tag{3.13}$$

and

$$\left|D_z^{1+\delta} f(z)\right| \ge \frac{|z|^{-\delta}}{\Gamma(1-\delta)} \left(1 - \frac{(1-\lambda\delta-\alpha)(n+1-\delta)\,\Gamma(2-\delta)}{1+\lambda(n-\delta)}\,|z|\right), \tag{3.14}$$

for $0 \le \delta < 1$ and $n \in \mathbb{N}$, and for all $z \in \mathcal{U}$.

PROOF. Suppose that $f(z) \in \mathbb{F}_\delta(n,\lambda,\alpha)$. Then, we find from (2.1) that

$$\sum_{k=n+1}^{\infty} \frac{\Gamma(k+1)\,\Gamma(1-\delta)}{\Gamma(k-\delta)}\, a_k \le \frac{(1-\lambda\delta-\alpha)(n+1-\delta)\,\Gamma(1-\delta)}{1+\lambda(n-\delta)} \qquad (0 \le \lambda \le 1;\ 0 \le \delta < 1;\ n \in \mathbb{N}). \tag{3.15}$$

On the other hand, by applying Definition 3 (with $k = 1$ and $\mu = \delta$), we obtain

$$D_z^{1+\delta} f(z) = \frac{z^{-\delta}}{\Gamma(1-\delta)} \left(1 - \sum_{k=n+1}^{\infty} \frac{\Gamma(k+1)\,\Gamma(1-\delta)}{\Gamma(k-\delta)}\, a_k\, z^{k-1}\right). \tag{3.16}$$

Thus, by combining (3.15) and (3.16), we immediately get the assertions (3.13) and (3.14) of Theorem 5.

Setting $\delta = \mu = 0$ in Theorem 4, we have the following corollary.

COROLLARY 7. *If $f(z) \in \mathbb{F}_0(n,\lambda,\alpha)$, then*

$$|z| - \frac{1-\alpha}{1+\lambda n}\,|z|^2 \le |f(z)| \le |z| + \frac{1-\alpha}{1+\lambda n}\,|z|^2, \tag{3.17}$$

for all $z \in \mathcal{U}$ and $n \in \mathbb{N}$.

For $\delta = 0$, Theorem 5 yields the following corollary.

COROLLARY 8. *If $f(z) \in \mathbb{F}_0(n, \lambda, \alpha)$, then*

$$1 - \frac{(1-\alpha)(n+1)}{1+\lambda n}|z| \leq |f'(z)| \leq 1 + \frac{(1-\alpha)(n+1)}{1+\lambda n}|z|, \tag{3.18}$$

for all $z \in \mathcal{U}$ and $n \in \mathbb{N}$.

Next, setting $\delta = \mu = 0$ and $n = 1$ in Theorem 4 (or, simply, $n = 1$ in Corollary 7), we have the following corollary.

COROLLARY 9. *(cf., [3, p. 91, Theorem 2]). If $f(z) \in \mathbb{F}_0(1, \lambda, \alpha)$, then*

$$|z| - \frac{1-\alpha}{1+\lambda}|z|^2 \leq |f(z)| \leq |z| + \frac{1-\alpha}{1+\lambda}|z|^2, \tag{3.19}$$

for all $z \in \mathcal{U}$.

If we set $\delta = 0$ and $n = 1$ in Theorem 5 (or, alternatively, if we just let $n = 1$ in Corollary 8), we obtain the following corollary.

COROLLARY 10. *(cf., [3, p. 92, Theorem 3]). If $f(z) \in \mathbb{F}_0(1, \lambda, \alpha)$, then*

$$1 - \frac{2(1-\alpha)}{1+\lambda}|z| \leq |f'(z)| \leq 1 + \frac{2(1-\alpha)}{1+\lambda}|z|, \tag{3.20}$$

for all $z \in \mathcal{U}$.

Numerous further consequences of Theorems 3–5 (and of Corollaries 7–10) can indeed be deduced by specializing the various parameters involved.

4. RADII OF CLOSE-TO-CONVEXITY, STARLIKENESS, AND CONVEXITY

A function $f(z) \in \mathbb{F}(n)$ is said to be *close-to-convex of order β* if it satisfies the inequality (cf., [5,6])

$$\Re\{f'(z)\} > \beta, \tag{4.1}$$

for some β $(0 \leq \beta < 1)$ and for all $z \in \mathcal{U}$. On the other hand, a function $f(z) \in \mathbb{F}(n)$ is said to be *starlike of order β* if it satisfies the inequality (cf., [5,6])

$$\Re\left\{\frac{zf'(z)}{f(z)}\right\} > \beta, \tag{4.2}$$

for some β $(0 \leq \beta < 1)$ and for all $z \in \mathcal{U}$. Furthermore, a function $f(z) \in \mathbb{F}(n)$ is said to be *convex of order β* if and only if $zf'(z)$ is starlike of order β, that is, if it satisfies the inequality (cf., [5,6])

$$\Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \beta, \tag{4.3}$$

for some β $(0 \leq \beta < 1)$ and for all $z \in \mathcal{U}$.

THEOREM 6. *If $f(z) \in \mathbb{F}_\delta(n, \lambda, \alpha)$, then $f(z)$ is close-to-convex of order β in $|z| < r_1(\alpha, \lambda, \delta, \beta)$, where*

$$r_1(\alpha, \lambda, \delta, \beta) = \inf_k \left[\frac{(1-\beta)\,\Gamma(k)[1+\lambda(k-1-\delta)]}{(1-\lambda\delta-\alpha)\,\Gamma(k+1-\delta)}\right]^{1/(k-1)}, \qquad (k \geq n+1;\ n \in \mathbb{N}).$$

PROOF. It is sufficient to show that $|f'(z) - 1| < 1 - \beta$. Indeed, we have

$$|f'(z) - 1| \leq \sum_{k=n+1}^{\infty} k\, a_k\, |z|^{k-1} \leq 1 - \beta, \tag{4.4}$$

and

$$\sum_{k=n+1}^{\infty} \frac{\Gamma(k+1)[1 + \lambda(k - \delta - 1)]}{\Gamma(k + 1 - \delta)}\, a_k \leq 1 - \lambda\delta - \alpha. \tag{4.5}$$

Hence, (4.4) is true if

$$\frac{k|z|^{k-1}}{1-\beta} \leq \frac{\Gamma(k+1)[1 + \lambda(k - \delta - 1)]}{(1 - \lambda\delta - \alpha)\,\Gamma(k + 1 - \delta)}, \qquad (k \geq n+1;\ n \in \mathbb{N}). \tag{4.6}$$

Solving (4.6) for $|z|$, we obtain

$$|z| \leq \left[\frac{(1-\beta)\,\Gamma(k)[1 + \lambda(k - 1 - \delta)]}{(1 - \lambda\delta - \alpha)\,\Gamma(k + 1 - \delta)}\right]^{1/(k-1)}, \qquad (k \geq n+1;\ n \in \mathbb{N}),$$

which obviously proves Theorem 6.

THEOREM 7. *If $f(z) \in \mathbb{F}_\delta(n, \lambda, \alpha)$, then $f(z)$ is starlike of order β in*

$$|z| < r_2(\alpha, \lambda, \delta, \beta),$$

where

$$r_2(\alpha, \lambda, \delta, \beta) = \inf_k \left[\frac{(1-\beta)\,\Gamma(k+1)[1 + \lambda(k - 1 - \delta)]}{(k-\beta)(1 - \lambda\delta - \alpha)\,\Gamma(k + 1 - \delta)}\right]^{1/(k-1)}, \qquad (k \geq n+1;\ n \in \mathbb{N}).$$

PROOF. We must show that

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leq 1 - \beta, \qquad \text{for} \quad |z| < r_2(\alpha, \lambda, \delta, \beta).$$

In fact, we have

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leq \frac{\sum_{k=n+1}^{\infty} (k-1)\, a_k\, |z|^{k-1}}{1 - \sum_{k=n+1}^{\infty} a_k\, |z|^{k-1}} \leq 1 - \beta, \tag{4.7}$$

if

$$\frac{(k-\beta)|z|^{k-1}}{1-\beta} \leq \frac{\Gamma(k+1)[1 + \lambda(k - \delta - 1)]}{(1 - \lambda\delta - \alpha)\,\Gamma(k + 1 - \delta)}, \qquad (k \geq n+1;\ n \in \mathbb{N}), \tag{4.8}$$

which evidently proves Theorem 7.

COROLLARY 11. *If $f(z) \in \mathbb{F}_\delta(n, \lambda, \alpha)$, then $f(z)$ is convex of order β in*

$$|z| < r_3(\alpha, \lambda, \delta, \beta),$$

where

$$r_3(\alpha, \lambda, \delta, \beta) = \inf_k \left[\frac{(1-\beta)\Gamma(k)[1 + \lambda(k - 1 - \delta)]}{(k-\beta)(1 - \lambda\delta - \alpha)\Gamma(k + 1 - \delta)}\right]^{1/(k-1)}, \qquad (k \geq n+1;\ n \in \mathbb{N}).$$

COROLLARY 12. *If $f(z) \in \mathbb{F}_0(1, \lambda, \alpha)$, then $f(z)$ is close-to-convex of order β in $|z| < r_4(\alpha, \lambda, \beta)$, where*

$$r_4(\alpha, \lambda, \beta) = \inf_k \left[\frac{(1-\beta)[1 + \lambda(k-1)]}{k(1-\alpha)}\right]^{1/(k-1)}, \qquad (k \in \mathbb{N} \setminus \{1\}).$$

COROLLARY 13. *If $f(z) \in \mathbb{F}_0(1, \lambda, \alpha)$, then $f(z)$ is starlike of order β in $|z| < r_5(\alpha, \lambda, \beta)$, where*

$$r_5(\alpha, \lambda, \beta) = \inf_k \left[\frac{(1-\beta)[1 + \lambda(k-1)]}{(k-\beta)(1-\alpha)}\right]^{1/(k-1)}, \qquad (k \in \mathbb{N} \setminus \{1\}).$$

COROLLARY 14. *If $f(z) \in \mathbb{F}_0(1, \lambda, \alpha)$, then $f(z)$ is convex of order β in $|z| < r_6(\alpha, \lambda, \beta)$, where*

$$r_6(\alpha, \lambda, \beta) = \inf_k \left[\frac{(1-\beta)[1 + \lambda(k-1)]}{k(k-\beta)(1-\alpha)}\right]^{1/(k-1)}, \qquad (k \in \mathbb{N} \setminus \{1\}).$$

In their *special* cases when $\beta = 0$, Corollaries 12–14 were proved earlier by Bhoosnurmath and Swamy [3, pp. 93–94, Theorems 5 and 6].

REFERENCES

1. S. Owa, On the distortion theorems. I, *Kyungpook Math. J.* **18**, 53–59 (1978).
2. S. Owa and H.M. Srivastava, Univalent and starlike generalized hypergeometric functions, *Canad. J. Math.* **39**, 1057–1077 (1987).
3. S.S. Bhoosnurmath and S.R. Swamy, Certain classes of analytic functions with negative coefficients, *Indian J. Math.* **27**, 89–98 (1985).
4. H. Silverman, Univalent functions with negative coefficients, *Proc. Amer. Math. Soc.* **51**, 109–116 (1975).
5. P.L. Duren, *Univalent Functions,* Grundlehren der Mathematischen Wissenschaften, Vol. 259, Springer-Verlag, New York, (1983).
6. H.M. Srivastava and S. Owa, Editors, *Current Topics in Analytic Function Theory*, World Scientific, Singapore, (1992).