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# Versions of separability in bitopological spaces

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ABSTRACT

We study selective versions of separability in bitopological spaces. In particular, we investigate these properties in function spaces endowed with the topology of pointwise convergence and the compact-open topology.



## 1. Introduction

Many topological properties are defined or characterized in terms of the following two classical selection principles. Let $\mathcal{A}$ and $\mathcal{B}$ be sets consisting of families of subsets of an infinite set $X$. Then:

$\mathsf{S}_1(\mathcal{A},\mathcal{B})$ is the selection hypothesis: for each sequence $\langle A_n\colon n\in\mathbb{N}\rangle$ of elements of $\mathcal{A}$ there is a sequence $\langle b_n\colon n\in\mathbb{N}\rangle$ such that for each $n$, $b_n\in A_n$, and $\{b_n\colon n\in\mathbb{N}\}$ is an element of $\mathcal{B}$.

$\mathsf{S}_{fin}(\mathcal{A},\mathcal{B})$ denotes the selection hypothesis: for each sequence $\langle A_n\colon n\in\mathbb{N}\rangle$ of elements of $\mathcal{A}$ there is a sequence $\langle B_n\colon n\in\mathbb{N}\rangle$ of finite sets such that for each $n$, $B_n\subset A_n$, and $\bigcup_{n\in\mathbb{N}} B_n\in\mathcal{B}$.

The prototypes for these selection principles are the Rothberger covering property $\mathsf{S}_1(\mathcal{O},\mathcal{O})$ and the Menger property $\mathsf{S}_{fin}(\mathcal{O},\mathcal{O})$.

In recent years many papers about selection principles in topological spaces have appeared in the literature.

On the other hand, there are very few papers which deal with bitopological spaces and selection principles. They are mainly related to function spaces (see [7,14,17,19,22]). However, there is not a systematic study of selection principles theory in bitopological context. Here we study versions of separability in bitopological spaces.

Recently several papers on selective versions of separability have been published. Let $\mathcal{D}$ denote the family of dense subspaces of a topological space $X$. In [23] the selection principles $\mathsf{S}_{fin}(\mathcal{D},\mathcal{D})$ and $\mathsf{S}_1(\mathcal{D},\mathcal{D})$ have been introduced. These principles appeared in a natural way in a study of hyperspace topologies in [7,8]. Also, in [8] the notion of groupable (countable) dense sets was introduced; denote by $\mathcal{D}^{gp}$ the family of groupable dense subsets of a space. In [8] (see also [7]), the selection properties $\mathsf{S}_{fin}(\mathcal{D},\mathcal{D}^{gp})$ and $\mathsf{S}_1(\mathcal{D},\mathcal{D}^{gp})$ were introduced and studied in connection with hyperspace topologies. Later on, these four selection properties have been systematically studied by several authors [2–5,9,20]. In [4] the selection properties $\mathsf{S}_{fin}(\mathcal{D},\mathcal{D})$, $\mathsf{S}_1(\mathcal{D},\mathcal{D})$ and $\mathsf{S}_1(\mathcal{D},\mathcal{D}^{gp})$ are called M-separability, R-separability and GN-separability, respectively,

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while a bit modified property $\mathsf{S}_{fin}(\mathcal{D}, \mathcal{D}^{gp})$ is called H-separability (the last two after characterizations of the Hurewicz and Gerlits–Nagy properties given in [15]).

In this article we study these properties in the bitopological context, in particular in the space $\mathsf{C}(X)$ of all continuous real-valued functions on a Tychonoff space $X$, where $\mathsf{C}(X)$ is endowed with the topology $\tau_p$ of pointwise convergence and the compact-open topology $\tau_k$. Because both spaces $(\mathsf{C}(X), \tau_p)$ and $(\mathsf{C}(X), \tau_k)$ are homogeneous it is enough to study local properties of these spaces considering the constantly 0 function $\underline{0} \in \mathsf{C}(X)$. Recall that the sets $W(\underline{0}, F, \varepsilon) = \{f \in \mathsf{C}(X)\colon f(F) \subset (-\varepsilon, \varepsilon)\}$, $F$ finite (resp. compact) in $X$, $\varepsilon > 0$, form a local base of $\underline{0}$ in $(\mathsf{C}(X), \tau_p)$ (resp. $(\mathsf{C}(X), \tau_k)$).

This study naturally suggests to consider selection principles $\mathsf{S}_{fin}(\mathcal{K}, \Omega)$ and $\mathsf{S}_1(\mathcal{K}, \Omega)$ introduced in [11], as well as (versions of) the principles $\mathsf{S}_{fin}(\mathcal{K}, \Omega^{gp})$ and $\mathsf{S}_1(\mathcal{K}, \Omega^{gp})$ [7].

Recall that $\Omega$, $\mathcal{K}$, $\Omega^{gp}$ denote the families of $\omega$-covers, $k$-covers, groupable $\omega$-covers, respectively. An open cover $\mathcal{U}$ of a topological space $X$ is an *$\omega$-cover* (a *$k$-cover*) of $X$ if $X \notin \mathcal{U}$ and each finite (compact) subset of $X$ belongs to a member of $\mathcal{U}$. A countable $\omega$-cover $\mathcal{U}$ is groupable [15] if it can be partitioned into countable many pairwise disjoint finite sets $\mathcal{U}_n$ such that each finite subset of the space is contained in an element of $\mathcal{U}_n$ for all but finitely many $n$.

For undefined notions regarding selection principles in topological spaces we refer the reader to the survey papers [10, 13]. The books [1,16] are nice sources concerning function spaces.

At the end of this introduction we mention that there is a deep relationship between selection principles theory and game theory.

The symbol $\mathsf{G}_1(\mathcal{A}, \mathcal{B})$ denotes the infinitely long game for two players, ONE and TWO, who play a round for each positive integer. In the $n$-th round ONE chooses a set $A_n \in \mathcal{A}$, and TWO responds by choosing an element $b_n \in A_n$. TWO wins a play $(A_1, b_1; \ldots; A_n, b_n; \ldots)$ if $\{b_n\colon n \in \mathbb{N}\} \in \mathcal{B}$; otherwise, ONE wins.

$\mathsf{G}_{fin}(\mathcal{A}, \mathcal{B})$ denotes the game where in the $n$-th round ONE chooses a set $A_n \in \mathcal{A}$, while TWO responds by choosing a finite set $B_n \subset A_n$. A play $(A_1, B_1, \ldots, A_n, B_n, \ldots)$ is won by TWO if and only if $\bigcup_{n\in\mathbb{N}} B_n \in \mathcal{B}$.

It is evident that if ONE does not have a winning strategy in the game $\mathsf{G}_1(\mathcal{A}, \mathcal{B})$ (resp. $\mathsf{G}_{fin}(\mathcal{A}, \mathcal{B})$) then the selection hypothesis $\mathsf{S}_1(\mathcal{A}, \mathcal{B})$ (resp. $\mathsf{S}_{fin}(\mathcal{A}, \mathcal{B})$) is true. The converse implication need not be always true (though many properties described by selection principles can be characterized by the corresponding game).

## 2. Definitions

Throughout this paper $(X, \tau_1, \tau_2)$, sometimes written simply $X$, will be a bitopological space (shortly bispace), i.e. the set $X$ endowed with two topologies $\tau_1$ and $\tau_2$. For a subset $A$ of $X$, $\mathrm{Cl}_i(A)$ will denote the closure of $A$ in $(X, \tau_i)$, $i = 1, 2$.

We begin with some definitions.

Let $(X, \tau_1, \tau_2)$ be a bitopological space. Then:

1. ([6]) A subset $A$ of $X$ is *bidense* (here called *double dense* or shortly *d-dense*) in $X$ if $A$ is dense in both $(X, \tau_1)$ and $(X, \tau_2)$. $X$ is *d-separable* if there is a countable set $A$ which is $d$-dense in $X$.
2. A *pairwise $\pi$-base* of $X$ is a collection of pairs $(P_\alpha, Q_\alpha) \in \tau_1 \times \tau_2$, $\alpha \in \Lambda$, such that for every $(U, V) \in \tau_1 \times \tau_2$ there is $\alpha \in \Lambda$ such that $P_\alpha \subset V$ and $Q_\alpha \subset U$.
3. $X$ has countable *$(\tau_i, \tau_j)$-fan tightness* $(i \neq j;\ i, j = 1, 2)$ if for each $x \in X$ and each sequence $\langle A_n\colon n \in \mathbb{N}\rangle$ of subsets of $X$ such that $x \in \mathrm{Cl}_i(A_n)$ for each $n \in \mathbb{N}$, there are finite sets $F_n \subset A_n$, $n \in \mathbb{N}$, with $x \in \mathrm{Cl}_j(\bigcup_{n\in\mathbb{N}} F_n)$.
4. $X$ has countable *$(\tau_i, \tau_j)$-strong fan tightness* $(i \neq j;\ i, j = 1, 2)$, if for each $x \in X$ and each sequence $\langle A_n\colon n \in \mathbb{N}\rangle$ of subsets of $X$ such that $x \in \mathrm{Cl}_i(A_n)$ for each $n \in \mathbb{N}$, there are points $x_n \in A_n$, $n \in \mathbb{N}$, with $x \in \mathrm{Cl}_j(\{x_n\colon n \in \mathbb{N}\})$.

Denote by $\mathcal{D}_1$ and $\mathcal{D}_2$ the collections of all dense subsets of $(X, \tau_1)$ and $(X, \tau_2)$, respectively. We say that $X$ is:

- $\mathsf{M}_{(\tau_i,\tau_j)}$*-separable* $(i, j = 1, 2;\ i \neq j)$, if for each sequence $\langle D_n\colon n \in \mathbb{N}\rangle$ of elements of $\mathcal{D}_i$ there are finite sets $F_n \subset D_n$, $n \in \mathbb{N}$, such that $\bigcup_{n\in\mathbb{N}} F_n \in \mathcal{D}_j$, i.e. if $\mathsf{S}_{fin}(\mathcal{D}_i, \mathcal{D}_j)$ holds;
- $\mathsf{R}_{(\tau_i,\tau_j)}$*-separable* if $\mathsf{S}_1(\mathcal{D}_i, \mathcal{D}_j)$ holds;
- $\mathsf{H}_{(\tau_i,\tau_j)}$*-separable* if for each sequence $\langle D_n\colon n \in \mathbb{N}\rangle$ of elements of $\mathcal{D}_i$ there are finite sets $F_n \subset D_n$, $n \in \mathbb{N}$, such that each $\tau_j$-open subset of $X$ intersects $F_n$ for all but finitely many $n$;
- $\mathsf{GN}_{(\tau_i,\tau_j)}$*-separable* if $\mathsf{S}_1(\mathcal{D}_i, \mathcal{D}_j^{gp})$ holds.

Here $\mathcal{D}^{gp}$ is the collection of groupable dense subsets of a space; a countable dense subset $D$ of a space $Z$ is *groupable* if $D = \bigcup_{n\in\mathbb{N}} A_n$, each $A_n$ finite and each open set $U$ in $Z$ intersects all but finitely many $A_n$ [8].

Clearly, if $\tau_1 = \tau_2 = \tau$, then these definitions coincide with definitions of corresponding topological selective versions of separability of $(X, \tau)$.

Obviously,

$$\mathsf{GN}_{(\tau_i,\tau_j)}\text{-separability} \Rightarrow \mathsf{R}_{(\tau_i,\tau_j)}\text{-separability} \Rightarrow \mathsf{M}_{(\tau_i,\tau_j)}\text{-separability},$$

and

$$\mathsf{H}_{(\tau_i,\tau_j)}\text{-separability} \Rightarrow \mathsf{M}_{(\tau_i,\tau_j)}\text{-separability}.$$

## 3. Bitopological R-separability

In this section we give some results on bitopological **R**-separability.

**Fact 1.** $\mathsf{R}_{(\tau_i,\tau_j)}$-separability implies separability of $(X,\tau_j)$.

**Fact 2.** If $\tau_1 \leqslant \tau_2$ then:

2.1 $\mathsf{R}_{(\tau_1,\tau_2)}$-separability implies $(X,\tau_1)$ is R-separable (and, by Fact 1, $(X,\tau_2)$ is separable).
2.2 $(X,\tau_1)$ is R-separable $\Rightarrow$ $(X,\tau_1,\tau_2)$ is $\mathsf{R}_{(\tau_2,\tau_1)}$-separable $\Rightarrow$ $(X,\tau_1)$ is separable.

**Fact 3.** If $Y$ is either $d$-dense or $d$-open in $(X,\tau_1,\tau_2)$ and $X$ is $\mathsf{R}_{(\tau_i,\tau_j)}$-separable, then $Y$ is also $\mathsf{R}_{(\tau_i,\tau_j)}$-separable.

[It follows from the fact that each set which is $\tau_i$-dense (open) in $Y$ is $\tau_i$-dense (open) in $X$.]

**Theorem 1.** *If $X$ has a countable pairwise $\pi$-base, then $X$ is $\mathsf{R}_{(\tau_i,\tau_j)}$-separable, $i,j=1,2$.*

**Proof.** (For $i=1$, $j=2$) Let $\{(P_n,Q_n)\colon n\in\mathbb{N}\}$ be a countable pairwise $\pi$-base of $X$. Take any sequence $\langle D_n\colon n\in\mathbb{N}\rangle$ of dense subsets of $(X,\tau_1)$. For each $n\in\mathbb{N}$, the intersection $D_n\cap P_n$ is non-empty; take a point $x_n\in D_n\cap P_n$, $n\in\mathbb{N}$, and prove that the set $\{x_n\colon n\in\mathbb{N}\}$ is dense in $(X,\tau_2)$. Indeed, if $V\in\tau_2$, then there is $m\in\mathbb{N}$ such that $P_m\subset V$, hence $x_m\in V$. □

**Theorem 2.** *Let $(X,\tau_1,\tau_2)$ be a $d$-separable bitopological space with countable $(\tau_i,\tau_j)$-strong fan tightness. Then $X$ is $\mathsf{R}_{(\tau_i,\tau_j)}$-separable.*

**Proof.** We consider only the case $i=1$, $j=2$. Let $\langle D_n\colon n\in\mathbb{N}\rangle$ be a sequence of dense subsets of $(X,\tau_1)$, and let $A=\{a_n\colon n\in\mathbb{N}\}$ be a countable set dense in $(X,\tau_1)$ and in $(X,\tau_2)$. Fix a partition $\mathbb{N}=M_1\cup M_2\cup\cdots$ of $\mathbb{N}$ into infinite, pairwise disjoint sets. For each $n\in\mathbb{N}$, $a_n\in\bigcap_{k\in M_n}\mathrm{Cl}_1(D_k)$. Since $(\tau_1,\tau_2)$-strong fan tightness of $(X,\tau_1,\tau_2)$ is countable, there is a sequence $(x_k\colon k\in M_n)$ of points such that for each $k\in M_n$, $x_k\in D_k$ and $a_n\in\mathrm{Cl}_2(\{x_k\colon k\in M_n\})$. We claim that the set $\{x_n\colon n\in\mathbb{N}\}$ is dense in $(X,\tau_2)$. Let $U$ be a set open in $(X,\tau_2)$. Since $A$ is dense in $(X,\tau_2)$, there is some $a_m\in U$. But $a_m\in\mathrm{Cl}_2(\{x_k\colon k\in M_m\})$, hence $U\cap(\{x_k\colon k\in M_m\})\neq\emptyset$ and thus $U\cap\{x_n\colon n\in\mathbb{N}\}\neq\emptyset$. □

**Corollary 3.** *Let $(X,\tau_1,\tau_2)$ be a bitopological space such that $\tau_1\leqslant\tau_2$. If $(X,\tau_2)$ is separable and $X$ has countable $(\tau_i,\tau_j)$-strong fan tightness, then $(X,\tau_1,\tau_2)$ is $\mathsf{R}_{(\tau_i,\tau_j)}$-separable.*

As we mentioned, in what follows $\tau_p$ denotes the pointwise topology, and $\tau_k$ denotes the compact-open topology on the set $\mathsf{C}(X)$ of all continuous real-valued functions on a Tychonoff space $X$.

**Theorem 4.** *Let $X$ be a Tychonoff space such that $w(X)=\omega$. Then the following are equivalent*:

(1) $X\in\mathsf{S}_1(\mathcal{K},\Omega)$;
(2) $(\mathsf{C}(X),\tau_p,\tau_k)$ *is* $\mathsf{R}_{(\tau_k,\tau_p)}$*-separable.*

**Proof.** (1) $\Rightarrow$ (2): From $d(\mathsf{C}(X),\tau_k)=w(X)$ [16] it follows that $(\mathsf{C}(X),\tau_k)$ is separable. On the other hand, by Theorem 2.4 in [17], $(\tau_k,\tau_p)$-strong fan tightness of $(\mathsf{C}(X),\tau_p,\tau_k)$ is countable if (and only if) $X\in\mathsf{S}_1(\mathcal{K},\Omega)$. By Corollary 3 we obtain $(C(X),\tau_p,\tau_k)$ is $\mathsf{R}_{(\tau_k,\tau_p)}$-separable.

(2) $\Rightarrow$ (1): Let $\langle\mathcal{U}_n\colon n\in\mathbb{N}\rangle$ be a sequence of $k$-covers of $X$. For every $n\in\mathbb{N}$ let

$$A_n=\bigl\{f\in\mathsf{C}_k(X)\text{: there is } U\in\mathcal{U}_n,\ f(X\setminus U)=\{1\}\bigr\}.$$

We prove that each $A_n$ is dense in $\mathsf{C}_k(X)$. Let $f\in\mathsf{C}_k(X)$ and let $\bigcap_{i\leqslant m}W(K_i,V_i)$ be a basic neighbourhood of $f$. [Here $W(K_i,V_i)=\{f\in\mathsf{C}(X)\colon f(K_i)\subset V_i\}$, $K_i$ compact in $X$, $V_i$ open in $\mathbb{R}$.] The set $K=\bigcup_{i\leqslant m}K_i$ is compact, and since $\mathcal{U}_n$ is a $k$-cover of $X$, there is $U\in\mathcal{U}_n$ containing $K$. There is also $g\in\mathsf{C}_k(X)$ such that $g(X\setminus U)=\{1\}$ and $g\restriction K=f\restriction K$. Then $g\in\bigcap_{i\leqslant m}W(K_i,V_i)\cap A_n$.

Since $(\mathsf{C}(X),\tau_p,\tau_k)$ is $\mathsf{R}_{(\tau_k,\tau_p)}$-separable there are functions $f_n\in A_n$, $n\in\mathbb{N}$, such that the set $\{f_n\colon n\in\mathbb{N}\}$ is dense in $(\mathsf{C}(X),\tau_1)$. Let $U_n\in\mathcal{U}_n$ be a set for which $f_n(X\setminus U_n)=\{1\}$ holds. We claim that $\{U_n\colon n\in\mathbb{N}\}$ is an $\omega$-cover of $X$. Let $F$ be a finite subset of $X$. Suppose that for each $n\in\mathbb{N}$ there is a point $x_n\in F\setminus U_n$. This means that for each $n$, $f_n(x_n)=1$ which contradicts the fact that $\{f_n\colon n\in\mathbb{N}\}$ is dense in $(\mathsf{C}(X),\tau_p)$ (the open set $W(F,(-1,1))=\{g\in\mathsf{C}(X)\colon f(F)\subset(-1,1)\}$ does not contain any $f_m$, $m\in\mathbb{N}$). □

## 4. Bitopological M-separability

Because M-separability has many similarities with R-separability, we formulate here statements parallel to the corresponding statements from the previous section.

The proof of the following theorem is quite similar to the proof of Theorem 2 and thus will be omitted. Instead we prove Corollary 6 below parallel to Corollary 3.

**Theorem 5.** *Let $(X, \tau_1, \tau_2)$ be a d-separable bitopological space with countable $(\tau_i, \tau_j)$-fan tightness. Then $X$ is $\mathsf{M}_{(\tau_i,\tau_j)}$-separable.*

**Corollary 6.** *Let $(X, \tau_1, \tau_2)$ be a bitopological space with $\tau_1 \leqslant \tau_2$. If $(X, \tau_2)$ is separable and the $(\tau_i, \tau_j)$-fan tightness of $X$ is countable, then $(X, \tau_1, \tau_2)$ is $\mathsf{M}_{(\tau_i,\tau_j)}$-separable.*

**Proof.** $(i = 1,\ j = 2)$ Let $\langle D_n \colon n \in \mathbb{N}\rangle$ be a sequence of dense subsets of $(X, \tau_1)$. Fix a countable set $A = \{a_n \colon n \in \mathbb{N}\}$ dense in $(X, \tau_2)$, so also in $(X, \tau_1)$ because $\tau_1 \leqslant \tau_2$. Let $\mathbb{N} = M_1 \cup M_2 \cup \cdots$ be a partition of $\mathbb{N}$ into infinite, pairwise disjoint sets. For each $n$ we have $a_n \in \mathrm{Cl}_1(D_k)$ for each $k \in M_n$. Since $(\tau_1, \tau_2)$-fan tightness of $(X, \tau_1, \tau_2)$ is countable, there is a sequence $\langle F_k \colon k \in M_n\rangle$ of finite sets such that for each $k \in M_n$, $F_k \subset D_k$ and $a_n \in \mathrm{Cl}_2(\bigcup\{F_k \colon k \in M_n\})$. We claim that the sequence $\langle F_n \colon n \in \mathbb{N}\rangle$ witnesses for $\langle D_n \colon n \in \mathbb{N}\rangle$ that $(X, \tau_1, \tau_2)$ is $\mathsf{M}_{(\tau_1,\tau_2)}$-separable, i.e. that $\bigcup\{F_n \colon n \in \mathbb{N}\}$ is dense in $(X, \tau_2)$. Let $U$ be a set open in $(X, \tau_2)$. Then there is $m \in \mathbb{N}$ such that $a_m \in U$. But $a_m \in \mathrm{Cl}_2(\bigcup\{F_k \colon k \in M_m\})$ so that $U \cap (\bigcup\{F_k \colon k \in M_m\}) \neq \emptyset$, hence $\bigcup\{F_n \colon n \in \mathbb{N}\}$ is dense in $(X, \tau_2)$. □

**Theorem 7.** *For a Tychonoff space $X$ with countable base the following are equivalent:*

(1) $X \in \mathsf{S}_{fin}(\mathcal{K}, \Omega)$;
(2) $(\mathsf{C}(X), \tau_p, \tau_k)$ *is* $\mathsf{M}_{(\tau_k,\tau_p)}$*-separable.*

**Proof.** (1) ⇒ (2): The space $(\mathsf{C}(X), \tau_k)$ is separable since $d(\mathsf{C}(X), \tau_k) = w(X)$. On the other hand, by [17, Theorem 2.5], $(\tau_k, \tau_p)$-fan tightness of $(\mathsf{C}(X), \tau_p, \tau_k)$ is countable if and only if $X$ has selection property $\mathsf{S}_{fin}(\mathcal{K}, \Omega)$. Apply Corollary 6.

(2) ⇒ (1): Let $\langle \mathcal{U}_n \colon n \in \mathbb{N}\rangle$ be a sequence of $k$-covers of $X$. For every $n \in \mathbb{N}$ let

$$A_n = \big\{ f \in \mathsf{C}_k(X) \colon \text{there is } U \in \mathcal{U},\ f(X \setminus U) = \{1\} \big\}.$$

As in the proof of Theorem 4 one proves that each $A_n$ is dense in $\mathsf{C}_k(X)$.

Since $(\mathsf{C}(X), \tau_p, \tau_k)$ is $\mathsf{M}_{(\tau_k,\tau_p)}$-separable there are finite sets $B_n = \{f_{n,1}, \ldots, f_{n,m_n}\} \subset A_n$, $n \in \mathbb{N}$, such that the set $\bigcup_{n\in\mathbb{N}} B_n$ is dense in $(\mathsf{C}(X), \tau_p)$. For each $n$, let $\mathcal{V}_n = \{U_{n,1}, \ldots, U_{n,m_n} \colon f_{n,i}(X \setminus U_{n,i}) = \{1\},\ i \leqslant m_n\}$. Then $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$. We claim that $\bigcup_{n\in\mathbb{N}} \mathcal{V}_n$ is an $\omega$-cover of $X$. Let $F$ be a finite subset of $X$. The $\tau_p$-open set $W(\underline{0}, F, 1)$ meets $B_j$ for some $j \in \mathbb{N}$, i.e. there is a function $f_{j,m_j} \in B_j$ such that $f_{j,m_j}(x) \in (-1, 1)$ for each $x \in F$. This means $F \subset U_{j,m_j}$ as required. □

The unexpected result that separable Fréchet–Urysohn spaces are M-separable was shown in [3].

The following is a bitopological version of this result. Call a bispace $(X, \tau_1, \tau_2)$ $(\tau_i, \tau_j)$-*Fréchet–Urysohn* $(i \neq j;\ i, j = 1, 2)$, abbreviated as $(\tau_i, \tau_j)$-FU, if for each $A \subset X$ and each $x \in \mathrm{Cl}_i(A)$ there is a sequence $(a_n)_{n\in\mathbb{N}}$ in $A$ which $\tau_j$-converges to $x$.

**Theorem 8.** *Let $(X, \tau_1, \tau_2)$ be a d-separable bispace such that $\tau_1 \leqslant \tau_2$, $(X, \tau_1)$ is dense-in-itself, and $(X, \tau_2)$ is Hausdorff. If $X$ is $(\tau_1, \tau_2)$-FU, then $X$ is $\mathsf{M}_{(\tau_1,\tau_2)}$-separable.*

**Proof.** Let $A = \{a_n \colon n \in \mathbb{N}\}$ be a countable $d$-dense subset of $X$ and let $\langle D_n \colon n \in \mathbb{N}\rangle$ be a sequence of $\tau_1$-dense subsets of $X$. Take any $a \in A$. From $a \in \mathrm{Cl}_1(X \setminus \{a\})$ it follows that there is a sequence $(x_n)_{n\in\mathbb{N}}$ in $X \setminus \{a\}$ which $\tau_2$-converges to $a$. Further, for each $n \in \mathbb{N}$, $x_n \in \mathrm{Cl}_1(D_n)$ and thus there is a sequence $y_n = (y_{n,m})_{m\in\mathbb{N}}$ in $D_n$ $\tau_2$-converging to $x_n$. Clearly, $a \in \mathrm{Cl}_2(\bigcup_{n\in\mathbb{N}} y_n)$ and so $a \in \mathrm{Cl}_1(\bigcup_{n\in\mathbb{N}} y_n)$. There is a sequence $z = (z_m)_{m\in\mathbb{N}}$ in $\bigcup_{n\in\mathbb{N}} y_n$ $\tau_2$-converging to $a$. For each $n$ the intersection $y_n \cap z = F_n \subset D_n$ is finite because $y_n$ and $z$ $\tau_2$-converge to different points in the Hausdorff space $(X, \tau_2)$. Since $a \in \mathrm{Cl}_2(\{z_m \colon m \in \mathbb{N}\})$ and $z = \bigcup_{n\in\mathbb{N}} F_n$ we have $a \in \mathrm{Cl}_2(\bigcup_{n\in\mathbb{N}} F_n)$. This means that $X$ has countable $(\tau_1, \tau_2)$-fan tightness at each point $a \in A$.

Let $\mathbb{N} = M_1 \cup M_2 \cup \cdots$, where $M_i$, $i \in \mathbb{N}$ are infinite, pairwise disjoint. Fix $n \in \mathbb{N}$. Apply the fact that $X$ has countable $(\tau_1, \tau_2)$-fan tightness at $a_n$ to the sequence $\langle D_m \colon m \in M_n\rangle$ and find finite sets $F_m^n \subset D_m$, $m \in M_n$, such that $a_n \in \mathrm{Cl}_2(\bigcup_{m\in M_n} F_m^n)$. Then the sets $F_m^n$, $m \in M_n$, $n \in \mathbb{N}$ testify that $X$ is $\mathsf{M}_{(\tau_1,\tau_2)}$-separable because $A \subset \mathrm{Cl}_2(\bigcup_{n\in\mathbb{N}} \bigcup_{m\in M_n} F_m^n)$ and $A$ is $d$-dense in $X$. □

The following theorem, which is a bitopological version of [9, Prop. 4.1], gives an interesting implication. Recall that a bispace $(X, \tau_1, \tau_2)$ is $(\tau_i, \tau_j)$-*Pytkeev* $(i \neq j;\ i, j = 1, 2)$ [12] (see also [17]) if for each $A \subset X$ and each $x \in \mathrm{Cl}_i(A) \setminus A$ there are infinite sets $B_n \subset A$, $n \in \mathbb{N}$, such that each $\tau_j$-neighbourhood of $x$ contains some $B_n$.

**Theorem 9.** *Let $(X,\tau_1,\tau_2)$, $\tau_1 \leqslant \tau_2$, be a d-separable $(\tau_1,\tau_2)$-Pytkeev bispace. If X is $\mathsf{M}_{(\tau_1,\tau_2)}$-separable, then it is $\mathsf{R}_{(\tau_1,\tau_2)}$-separable.*

**Proof.** Fix a $d$-dense subset $A=\{a_n\colon n\in\mathbb{N}\}$ of $X$. Let $\langle D_n\colon n\in\mathbb{N}\rangle$ be a sequence of $\tau_1$-dense subsets of $X$. Let $\mathbb{N}=M_1\cup M_2\cup\cdots$ be a partition of $\mathbb{N}$ into infinite subsets. For each $n\in\mathbb{N}$ consider the sequence $\langle D_m\colon m\in M_n\rangle$. Since $X$ is $\mathsf{M}_{(\tau_1,\tau_2)}$-separable, there are finite sets $F_m\subset D_m$, $m\in M_n$, such that $Y_n:=\bigcup_{m\in M_n}F_m$ is $\tau_2$-dense in $X$. Let $K_1=\{n\in\mathbb{N}\colon a_n\in Y_n\}$ and $K_2=\mathbb{N}\setminus K_1$. For each $n\in K_2$ we have $a_n\in \mathrm{Cl}_1(Y_n)\setminus Y_n$ and since $X$ is $(\tau_1,\tau_2)$-Pytkeev there are infinite sets $B_{n,k}\subset Y_n$, $k\in\mathbb{N}$, such that each $\tau_2$-neighbourhood of $x$ contains some $B_{n,k}$. Clearly, each $B_{n,k}$ intersects infinitely many (finite) sets $F_m$, say $F_{m_1},\dots,F_{m_k}$. Pick a point $x_{n,m_k}\in B_{n,k}\cap F_{m_k}\subset D_{m_k}$. Let

$$z_n=\begin{cases} a_n, & \text{if } n\in K_1;\\ x_{n,m_k}, & \text{if } n\in K_2,\ k\in\mathbb{N}.\end{cases}$$

Then $\{z_n\colon n\in\mathbb{N}\}$ is $\tau_2$-dense in $X$, i.e. the sequence $\langle z_n\colon n\in\mathbb{N}\rangle$ witnesses for $\langle D_n\colon n\in\mathbb{N}\rangle$ that $X$ is $\mathsf{R}_{(\tau_1,\tau_2)}$-separable. □

## 5. Bitopological H- and GN-separability

A bispace $(X,\tau_1,\tau_2)$ has the *selectively $(\tau_i,\tau_j)$-Reznichenko property* ($i\neq j$; $i,j=1,2$), if for each sequence $\langle A_n\colon n\in\mathbb{N}\rangle$ of subsets of $X$ and each point $x\in\bigcap_{n\in\mathbb{N}}\mathrm{Cl}_i(A_n)$ there are finite, pairwise disjoint sets $B_n\subset A_n$, $n\in\mathbb{N}$, such that each $\tau_j$-neighbourhood of $x$ intersects $B_n$ for all but finitely many $n$.

This notion was introduced in [12]. If we do not require $B_n$'s to be disjoint, one obtains the notion of *weak selectively $(\tau_i,\tau_j)$-Reznichenko property* (compare with [18,21]).

**Theorem 10.** *For a Tychonoff space X with countable base the following are equivalent*:

(1) $(\mathsf{C}(X),\tau_p,\tau_k)$ *is* $\mathsf{H}_{(\tau_k,\tau_p)}$*-separable*;

(2) *For each sequence $\langle\mathcal{U}_n\colon n\in\mathbb{N}\rangle$ of k-covers of X there is a sequence $\langle\mathcal{V}_n\colon n\in\mathbb{N}\rangle$ of finite sets such that for each n, $\mathcal{V}_n\subset\mathcal{U}_n$ and each finite set $F\subset X$ is contained in an element of $\mathcal{V}_n$ for all but finitely many $n\in\mathbb{N}$.*

**Proof.** (1) ⇒ (2): Let $\langle\mathcal{U}_n\colon n\in\mathbb{N}\rangle$ be a sequence of $k$-covers of $X$. For every $n\in\mathbb{N}$ let

$$A_n=\big\{f\in\mathsf{C}_k(X)\colon \text{there is } U\in\mathcal{U}_n,\ f(X\setminus U)=\{1\}\big\}.$$

Each $A_n$ is dense in $(\mathsf{C}(X),\tau_k)$. By (1) there are finite sets $B_n\subset A_n$, $n\in\mathbb{N}$, such that each $\tau_p$-open set intersects $B_n$ for all but finitely many $n$. Let $\mathcal{V}_n$, $n\in\mathbb{N}$, be the family of sets $U_f\in\mathcal{U}_n$, $f\in B_n$, such that $f(X\setminus U_f)=\{1\}$. We prove that the sequence $\langle\mathcal{V}_n\colon n\in\mathbb{N}\rangle$ witnesses that $X$ satisfies (2).

Let $E$ be a finite subset of $X$. Then $W=W(E,(-1,1))$ is an open set in $(\mathsf{C}(X),\tau_p)$ and thus it meets $B_m$ for each $m$ bigger than some $m_0$; pick $f_m\in W\cap B_m$, $m>m_0$. Then for each $m>m_0$, $E\subset U_{f_m}\in\mathcal{V}_m$, as required in (2).

(2) ⇒ (1): We divide the proof into two parts.

**Claim 1.** $\mathsf{C}(X)$ *has the weak selectively $(\tau_k,\tau_p)$-Reznichenko property.*

Let $\langle A_n\colon n\in\mathbb{N}\rangle$ be a sequence of subsets of $\mathsf{C}(X)$ whose $\tau_k$-closures contain $\underline{0}$. For every compact set $K\subset X$ and every $m\in\mathbb{N}$ the $\tau_k$-neighborhood $W(\underline{0},K,1/m)$ of $\underline{0}$ intersects each $A_n$. It follows that for each $n\in\mathbb{N}$ there exists a function $f_{K,n,m}\in A_n$ satisfying $|f_{K,n,m}(x)|<1/m$ for each $x\in K$. For each $n$ set

$$\mathcal{U}_{n,m}=\big\{f^{\leftarrow}(-1/m,1/m)\colon m\in\mathbb{N},\ f\in A_n\big\}.$$

(We can view the indices $m,n$ in $\mathcal{U}_{n,m}$ as $\varphi(m,n)$ for some bijection $\varphi:\mathbb{N}^2\to\mathbb{N}$.) We claim that for each $n,m\in\mathbb{N}$, each compact subset of $X$ is contained in an element of $\mathcal{U}_{n,m}$. Indeed, if $C$ is a compact subset of $X$, then there is $f_{C,n,m}\in W(\underline{0},C,1/m)\cap A_n$. Hence $|f_{C,n,m}(x)|<1/m$ for each $x\in C$. This means $C\subset f^{\leftarrow}_{C,n,m}(-1/m,1/m)\in\mathcal{U}_{n,m}$.

Put $M=\{m\in\mathbb{N}\colon X\in\mathcal{U}_{n,m}\text{ for some } n\in\mathbb{N}\}$.

**Case 1.** $M$ is infinite.

There are $m_1<m_2<\cdots$ in $M$ and (the corresponding) $n_1,n_2,\dots$ in $\mathbb{N}$ such that $f^{\leftarrow}_{K_i,n_i,m_i}(-1/m_i,1/m_i)=X$ for all $i\in\mathbb{N}$ and some compact sets $K_i\subset X$. Let $W(\underline{0},F,\varepsilon)$ be a $\tau_p$-neighbourhood of $\underline{0}$. Pick $m_k$ such that $1/m_k<\varepsilon$. For every $m_i>m_k$ we have $f_{K_i,n_i,m_i}(x)\in(-1/m_i,1/m_i)$ for each $x\in X$ and so $f_{K_i,n_i,m_i}\in W(\underline{0},F,1/m_i)\subset W(\underline{0},F,\varepsilon)$. This means that the sequence $(f_{K_i,n_i,m_i}\colon i\in\mathbb{N})$ $\tau_p$-converges to $\underline{0}$, hence $\mathsf{C}(X)$ has the weak selectively $(\tau_k,\tau_p)$-Reznichenko property at $\underline{0}$.

**Case 2.** $M$ is finite.

There is $m_0\in\mathbb{N}$ such that for each $m\geqslant m_0$ and each $n\in\mathbb{N}$, the set $\mathcal{U}_{n,m}$ is a $k$-cover of $X$. Without loss of generality one may suppose $m_0=1$. Further, we can consider only $k$-covers $\mathcal{U}_{n,n}$, $n\in\mathbb{N}$. By (2) choose for each $n\in\mathbb{N}$ a finite set

$\mathcal{V}_{n,n} \subset \mathcal{U}_{n,n}$ so that each finite subset $F$ of $X$ belongs to some $V \in \mathcal{V}_{n,n}$ for all but finitely many $n$. Choose the corresponding functions $f_{K_V,1/n,1/n}$, $V \in \mathcal{V}_{n,n}$, and put $B_n = \{f_{K_V,1/n,1/n}: V \in \mathcal{V}_{n,n}\}$. Then each $B_n$ is a finite subset of $A_n$. Let $W(\underline{0}, F, 1/i)$ be a neighborhood of $\underline{0}$. Let $n_0$ be such that $i/n < 1/i$ and for each $n > n_0$ there is $V_n \in \mathcal{V}_{n,n}$ containing $F$. Choose a corresponding $f_n \in B_n$. Since this can be done for all $n > n_0$, we conclude that for all $n > n_0$ we have $f_n \in W(\underline{0}, F, 1/i)$, i.e., $B_n \cap W(\underline{0}, F, 1/i) \neq \emptyset$ for all $n > n_0$.

**Claim 2.** *$\mathsf{C}(X)$ is $\mathsf{H}_{(\tau_k,\tau_p)}$-separable.*

Since $w(X) = \omega$, and $\tau_1 \leqslant \tau_2$, there is a countable $d$-dense subset $A = \{a_n: n \in \mathbb{N}\}$ of $\mathsf{C}(X)$. Let $\langle D_n: n \in \mathbb{N}\rangle$ be a sequence of dense subsets of $(\mathsf{C}(X), \tau_k)$. Fix $m \in \mathbb{N}$. Since $a_m \in \mathrm{Cl}_{\tau_k}(D_n)$ for each $n \in \mathbb{N}$, and $\mathsf{C}(X)$ has the weak selectively $(\tau_k, \tau_p)$-Reznichenko property, there are finite sets $F_{n,m}$, such that for each $n$, $F_{n,m} \subset D_n$ and each $\tau_p$-neighbourhood of $a_m$ intersects all but finitely many $F_{n,m}$. For each $n$ put $F_n = \cup\{F_{n,m}: m \leqslant n\}$. The sequence $\langle F_n: n \in \mathbb{N}\rangle$ witnesses for $\langle D_n: n \in \mathbb{N}\rangle$ that $\mathsf{C}(X)$ is $\mathsf{H}_{(\tau_k,\tau_p)}$-separable. Indeed, let $W$ be an open set in $(\mathsf{C}(X), \tau_p)$. Then there is $a_m \in W$, hence $W$ meets all but finitely many $F_n$. □

**Theorem 11.** *If $\mathsf{C}(X)$ is $\mathsf{GN}_{(\tau_k,\tau_p)}$-separable, then $X$ satisfies $\mathsf{S}_{fin}(\mathcal{K}, \Omega^{gp})$.*

**Proof.** Let $\langle \mathcal{U}_n: n \in \mathbb{N}\rangle$ be a sequence of $k$-covers of $X$. As in the proof of Theorem 10 define dense sets $A_n$ in $(\mathsf{C}(X), \tau_k)$, hence $d$-dense in $\mathsf{C}(X)$. Since $\mathsf{C}(X)$ is $\mathsf{GN}_{(\tau_k,\tau_p)}$-separable, there are $f_n \in A_n$, $n \in \mathbb{N}$, such that $A = \{f_n: n \in \mathbb{N}\}$ is $\tau_p$-groupable, i.e. $A = \bigcup_{m\in\mathbb{N}} B_m$, where each $B_m = \{f_m^{k_1}, \ldots, f_m^{k_m}\}$ is a finite subset of $A$ and each $\tau_p$-open set meets all but finitely many $B_m$. For each $m \in \mathbb{N}$, let

$$\mathcal{V}_m = \{U_m^{k_i}: f_m^{k_i}(X \setminus U_m^{k_i}) = \{1\},\ i \leqslant m\}.$$

We prove that each finite subset of $X$ is contained in some $V \in \mathcal{V}_m$ for all but finitely many $m$. Let $F$ be a finite subset of $X$. Then the $\tau_p$-open set $W(\underline{0}, F, 1)$ intersects $B_m$ for all $m$ bigger than some $m_0 \in \mathbb{N}$. Let $f_m^{k_j} \in W(\underline{0}, F, 1) \cap B_m$, $m \geqslant m_0$. Then $F \subset U_m^{k_j}$, $m \geqslant m_0$. This means that $X$ satisfies $\mathsf{S}_{fin}(\mathcal{K}, \Omega^{gp})$. □

From Theorems 4 and 11 we obtain

**Theorem 12.** *If $\mathsf{C}(X)$ is $\mathsf{GN}_{(\tau_k,\tau_p)}$-separable, then $X$ satisfies $\mathsf{S}_1(\mathcal{K}, \Omega)$ and $\mathsf{S}_{fin}(\mathcal{K}, \Omega^{gp})$.*

A direct consequence is the following assertion.

**Corollary 13.** *If $\mathsf{C}(X)$ is $\mathsf{GN}_{(\tau_k,\tau_p)}$-separable, then $\mathsf{C}(X)$ is $\mathsf{R}_{(\tau_k,\tau_p)}$-separable as well as $\mathsf{H}_{(\tau_k,\tau_p)}$-separable.*

## 6. Concluding remarks

It would be interesting to investigate relations between bitopological selective versions of separability and games which naturally correspond to them (see [3] for M-separability in topological spaces). For example, we conjecture that the following two assertions are not equivalent:

(1) TWO has a winning strategy in the game $\mathsf{G}_1(\mathcal{D}_i, \mathcal{D}_j)$ (resp. $\mathsf{G}_1(\mathcal{D}_i, \mathcal{D}_j^{gr})$);
(2) $(X, \tau_1, \tau_2)$ is $\mathsf{R}_{(\tau_i,\tau_j)}$-separable (resp. $\mathsf{GN}_{(\tau_i,\tau_j)}$-separable).

Similarly for the $\mathsf{H}_{(\tau_i,\tau_j)}$-separability and the corresponding game.

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