

## Some Properties of Fuzzy Groups

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### 1. INTRODUCTION

The concept of fuzzy sets was introduced by Zadeh in [7]. In [2], Rosenfeld defined fuzzy subgroupoid and fuzzy subgroups in the following way.

**DEFINITION 1.1.** Let  $G$  be a group. A fuzzy set  $\mu$  of  $G$  is said to be a fuzzy subgroup of  $G$  if for all  $x, y$  in  $G$

- (i)  $\mu(xy) \geq \min\{\mu(x), \mu(y)\}$
- (ii)  $\mu(x^{-1}) \geq \mu(x)$ .

Recently fuzzy subgroups have been studied by Anthony and Sherwood in [1, 2], by Sivaramakrishna Das in [6] and by Sherwood in [5].

We begin by giving some conventions, definitions, and propositions. Although some of these definitions can be found elsewhere, they are repeated here to help to the reader. Following this we define fuzzy normal subgroups, fuzzy level normal subgroups and their homomorphisms, and give some of their properties. After this, we extend some theorem of Sivaramakrishna Das [6, Theorem 3.4 and Theorem 4.2]. Finally we extend a theorem of Sherwood [5, Theorem 4.1].

### 2. PRELIMINARIES AND SOME PROPERTIES

Let  $I = [0, 1]$  be the unit closed interval and  $G$  be a group. If  $\mu$  is a fuzzy subgroup of  $G$  then for all  $x$  in  $G$  we obtain  $\mu(x^{-1}) = \mu(x)$  by [4, Proposition 5.4].

**PROPOSITION 2.1.** *Let  $Y$  be the set of all fuzzy subgroups of  $G$ . Consider*

the relation “ $\sim$ ” in  $Y$  defined by  $\mu \sim \mu'$  if and only if for all  $x, y \in G$ ,  $\mu(x) > \mu(y) \Leftrightarrow \mu'(x) > \mu'(y)$ . Then “ $\sim$ ” is an equivalence relation in  $Y$ .

*Proof.* Clear. ■

Let  $[\mu]$  denote the equivalence class of  $\mu$  in  $Y$  and  $e$  be the unit element of  $G$ .

**PROPOSITION 2.2.** *Let  $[\mu] \neq [0]$ . There exists a fuzzy subgroup  $\lambda$  of  $G$  in  $[\mu]$  such that  $\lambda(e) = 1$ .*

*Proof.* [2, Lemma 1]. ■

**DEFINITION 2.3.** Let  $\mu$  be a fuzzy subset of  $G$ . For any  $t$  in  $I$ , the set  $G_t = \{x \in G \mid \mu(x) \geq t\}$  is called a level subset of the fuzzy subset  $\mu$ .

Note that, if  $\mu$  is a fuzzy subgroup of  $G$  then  $\mu(x) \leq \mu(e)$  for every  $x$  in  $G$  and  $G_t$  is a subgroup of  $G$  for any  $t \in I$  with  $t \leq \mu(e)$ .

**DEFINITION 2.4.** The subgroups  $G_t$  are called level subgroups of  $G$  for any  $t \in I$ .

Clearly, if  $t < t'$  iff  $G_t > G_{t'}$  for any  $t, t'$  in  $\mu(G)$ . So every fuzzy subgroup of a finite group  $G$  gives a chain with subgroups of  $G$ .

$$\{e\} = G_{t_0} < G_{t_1} < \dots < G_{t_r} = G,$$

where  $t_i \in \text{Im } \mu$  and  $\mu(e) = t_0 > t_1 > \dots > t_r$ . We denote this chain of level subgroups by  $\Gamma_\mu(G)$ . As known, all subgroups of a group  $G$  usually do not form a chain. It follows that not all subgroups are level subgroups of a fuzzy subgroup.

**PROPOSITION 2.5.** *Let  $\mu, \mu'$  be two fuzzy subgroups of a finite group  $G$ .  $\mu \sim \mu'$  if and only if  $\Gamma_\mu(G) \equiv \Gamma_{\mu'}(G)$ .*

*Proof.* Let  $G_t \in \Gamma_\mu(G)$  and take  $t' = \inf_{x \in G_t} \mu(x)$ . Then  $G_t = G_{t'}$  and similarly if  $G_{t'} \in \Gamma_{\mu'}(G)$  and  $t = \inf_{x \in G_{t'}} \mu(x)$  then  $G_{t'} = G_t$ .

Conversely for any  $x, y$  in  $G$  if  $\mu(x) > \mu(y)$  then  $y \notin G_{\mu(x)} = G_{t'}$  and  $\mu'(y) < t' \leq \mu(x)$  it follows that  $\mu'(x) > \mu'(y)$  and similarly,  $\mu'(x) > \mu'(y)$  which implies that  $\mu(x) > \mu(y)$ . ■

If  $[\mu]$  is the equivalence class of  $\mu$  then every fuzzy subgroup in  $[\mu]$  has similar properties. Hence, in addition to Definition 1.1, without loss of generality we may assume that  $\mu(e) = 1$ . From now on a fuzzy group  $\mu$  of  $G$  is understood to satisfy the following definition.

**DEFINITION 2.6.** Let  $G$  be a group, A fuzzy set  $\mu$  of  $G$  is said to be a fuzzy subgroup of  $G$ , if for all  $x, y$  in  $G$ :

- (i)  $\mu(xy) \geq \min(\mu(x), \mu(y))$
- (ii)  $\mu(x) = \mu(x^{-1})$
- (iii)  $\mu(e) = 1$ .

**PROPOSITION 2.7.** Let  $G$  be a group. If  $\mu$  is a fuzzy subgroup of  $G$  then  $\mu(xy) = \min\{\mu(x), \mu(y)\}$  for each  $x, y$  in  $G$  with  $\mu(x) \neq \mu(y)$ .

*Proof.* Assume that  $\mu(x) > \mu(y)$ , then

$$\begin{aligned} \mu(y) &= \mu(x^{-1}xy) \geq \min\{\mu(x^{-1}), \mu(xy)\} \\ &= \min\{\mu(x), \mu(xy)\} = \mu(xy) \geq \min\{\mu(x), \mu(y)\} \\ &= \mu(y). \end{aligned}$$

Thus  $\mu(xy) = \mu(y) = \min\{\mu(x), \mu(y)\}$ . ■

### 3. FUZZY NORMAL SUBGROUP AND FUZZY LEVEL NORMAL SUBGROUPS

**THEOREM 3.1.** If  $\mu$  is a fuzzy subgroup of  $G$ , then the following conditions are equivalent:

- (i)  $\mu(xy) = \mu(yx)$  for all  $x, y$  in  $G$ ;
- (ii)  $\mu(xyx^{-1}) = \mu(y)$  for all  $x, y$  in  $G$ .

*Proof.* Let  $x$  and  $y$  be in  $G$ :

- (i)  $\Rightarrow$  (ii)  $\mu(xyx^{-1}) = \mu((xy)x^{-1}) = \mu(x^{-1}(xy)) = \mu(x^{-1}xy) = \mu(y)$ .
- (ii)  $\Rightarrow$  (i) Since  $xy = x(yx)x^{-1}$ ,  $\mu(xy) = \mu(x(yx)x^{-1}) = \mu(yx)$ . ■

**DEFINITION 3.2.** A fuzzy subgroup of  $G$  which satisfies the equivalent conditions of Theorem 3.1 is said to be a fuzzy normal subgroup of  $G$ .

**DEFINITION 3.3.** A fuzzy subgroup  $\mu$  of a group  $G$  is a fuzzy level normal subgroup if and only if the number of the level subgroups are finite and  $G_{t_i} \triangleleft G_{t_{i+1}}$ , for each  $G_{t_i}$  in  $\Gamma_\mu(G)$ .

**THEOREM 3.4.** Let  $G$  be a finite group. A fuzzy normal subgroup of  $G$  is a fuzzy level normal subgroup of  $G$ .

*Proof.* Let  $G = \{x \in G \mid \mu(x) \geq t_i, t_i \in \mu(G)\}$ , then  $G_{t_i} \triangleleft G$  and hence  $G_{t_i} \triangleleft G_{t_{i+1}}$ . ■

EXAMPLE 3.5. If  $G$  is a commutative group then every fuzzy subgroup of  $G$  is a fuzzy normal subgroup and if  $G$  also is finite then it is a fuzzy level normal subgroup.

EXAMPLES 3.6. A fuzzy level normal subgroup may not be a fuzzy normal subgroup. For the alternating group

$$A_4 = \{e, (12)(34), (13)(24), (14)(23), (123), (132), (142), (124), (234), (243), (134), (143)\}$$

define  $\mu(e) = 1$ ,  $\mu((12)(34)) = \frac{1}{2}$ ,  $\mu\{(14)(23), (13)(24)\} = \{\frac{1}{3}\}$ ,  $\mu(ijk) = 0$ . Then  $\mu$  is a fuzzy level normal subgroup of  $A_4$ . Hence  $\Gamma_\mu(A_4): \{e\} < \langle (12)(34) \rangle = G_{1/2} < \{e, (12)(34), (13)(24), (14)(32)\} = G_{1/3} < G_0 = A_4$  and  $\{e\} < G_{1/2} < G_{1/3} < A_4$ . Hence  $\mu$  is a fuzzy level normal subgroup of  $A_4$ . But  $\mu$  is not a fuzzy normal subgroup of  $A$  since  $G_{1/2}$  is not normal in  $A_4$ .

EXAMPLE 3.7. A fuzzy subgroup  $\mu$  of a finite group  $G$  may not be a fuzzy level normal subgroup of  $G$ . Let  $G = \{a, b \mid a^2 = b^3 = e, ab^2 = ba\} = \{e, a, b, b^2, ab, ab^2\}$ . If  $\mu(a) = \mu(e) = 1$ ,  $\mu(b) = \mu(b^2) = \mu(ab) = \mu(ab^2) = 0$  then  $\Gamma_\mu(G): \{e, a\} = G_1 < G_0 = G$ .  $G_1$  is not normal in  $G$  since  $bab^{-1} = bab^2 = ab^2 \cdot b^2 = ab$ . Hence  $\mu$  is not a fuzzy level normal subgroup of  $G$ .

DEFINITION 3.8. Let  $\lambda, \mu$  be two fuzzy subgroups of the groups  $G$  and  $H$ , respectively. If there is a group homomorphism  $\phi: G \rightarrow H$  such that  $\lambda = \mu \circ \phi$ , then we say  $\lambda$  is homomorphic to  $\mu$ . If  $\phi$  is an isomorphism then we say  $\mu$  and  $\lambda$  are isomorphic.

THEOREM 3.9. Let  $G$  and  $H$  be two groups,  $\phi: G \rightarrow H$  a homomorphism and  $\mu$  a fuzzy (normal) subgroup of  $H$ . Then  $\lambda = \mu \circ \phi$  is a fuzzy (normal) subgroup of  $G$ .

*Proof.* Clear. ■

THEOREM 3.10. If  $H$  is a subgroup of  $G$ ,  $\mu$  is a fuzzy (normal, level normal), subgroup of  $G$  and  $\eta$  is the restriction of  $\mu$  to  $H$ , then  $\eta$  is a fuzzy (normal, level normal) subgroup of  $H$ .

*Proof.* Clear. ■

THEOREM 3.11. Let  $G$  be a (finite) group,  $\mu$  is a fuzzy (level) normal subgroup of  $G$ . If  $N = \{x \in G \mid \mu(x) = 1\}$  then the function  $\bar{\mu}: G/N \rightarrow I$  defined by  $\bar{\mu}(gN) = \mu(g)$  is a fuzzy (level) normal subgroup of the quotient group  $G/N$ . Furthermore, if  $\theta: G \rightarrow G/N$  is the natural homomorphism then  $\mu = \bar{\mu} \circ \theta$ .

*Proof.*  $\mu$  is well defined: Take  $x, y$  in  $G$  with  $xN = yN$ , then  $x = yn$  for some  $n \in N$  and

$$\bar{\mu}(xN) = \mu(x) = \mu(yn) \geq \min\{\mu(y), \mu(n)\} = \mu(y) = \bar{\mu}(yN).$$

So  $\bar{\mu}(xN) \geq \bar{\mu}(yN)$  and similarly  $\bar{\mu}(yN) \geq \bar{\mu}(xN)$ . Hence  $\bar{\mu}(xN) = \bar{\mu}(yN)$ . It is easy to see the leaving parts. ■

**THEOREM 3.12.** *If  $G$  is a group,  $\mu$  is a fuzzy (normal, level normal) subgroup of  $G$ , then there exists a free group  $F$  and a surjective homomorphism  $\theta: F \rightarrow G$  such that  $\lambda = \mu \circ \theta$  is a fuzzy (normal, level normal) subgroup of  $F$ .*

*Proof.* This is an immediate consequence of Theorem 3.8. ■

#### 4. CHARACTERIZATION OF FUZZY GROUPS OF FINITE GROUPS

Let  $G$  be a finite group and  $\mu$  be one of its fuzzy subgroups, then the level subgroup chain  $\Gamma_\mu(G)$  is completely determined by  $\mu$ . Conversely, for any finite group  $G$  and the subgroups chain

$$\{e\} = G_0 < G_1 < \dots < G_r = G \tag{1}$$

there exists an equivalence class  $[\mu]$  of fuzzy subgroups of  $G$  such that  $\Gamma_\mu(G)$  is the chain (1). We shall prove this fact.

**THEOREM 4.1.** *Let  $G$  be a finite group. Then there exists a fuzzy subgroup  $\mu$  of  $G$  such that  $\Gamma_\mu(G)$  is a maximal chain of all subgroups of  $G$ .*

*Proof.* Let  $G$  be a finite group; then the number of the subgroups of  $G$  is finite. So there exists some maximal chain of subgroups of  $G$ . Take

$$G_0 = \{e\} < G_1 < G_2 < \dots < G_{r-1} < G_r = G. \tag{2}$$

Define  $\mu(G_0) = \{1\}$  and  $\mu(G_{i+1} \setminus G_i) = \{1/(i+1)\}$  for any  $i, 0 \leq i < r$ . Clearly,  $\mu$  is a fuzzy subgroup of  $G$  and  $\Gamma_\mu(G)$  is the chain (2). ■

**THEOREM 4.2.** *Let  $G$  be a finite group. Then there exists a fuzzy normal subgroup  $\mu$  such that  $\Gamma_\mu(G)$  is a maximal chain of all normal subgroups of  $G$ .*

*Proof.* Proof is similar to the proof of the previous theorem. ■

**THEOREM 4.3.** *Let  $G$  be a finite group. Then there exists a fuzzy level normal subgroup  $\mu$  such that  $\Gamma_\mu(G)$  is a composition series of  $G$ .*

*Proof.* Every finite group has a composition series

$$\{e\} = G_0 \triangleleft G_1 \triangleleft \cdots \triangleleft G_n = G \quad (3)$$

(see [3]). By Theorem 4.1, there exists a fuzzy subgroup  $\mu$  of  $G$  with  $\Gamma_\mu(G)$  equal to (3) and so  $\mu$  is a fuzzy level normal subgroup of  $G$ . ■

## 5. PRODUCTS OF FUZZY SUBGROUPS

**DEFINITION 5.1.** Let  $\mu_i$  be a fuzzy subgroup of the group  $G_i$  for each  $i = 1, 2, \dots, n$ . The product of  $\mu_i$  ( $i = 1, 2, \dots, n$ ) is the function  $\mu_1 \times \mu_2 \times \cdots \times \mu_n$  defined on  $G_1 \times G_2 \times \cdots \times G_n$  by

$$\mu_1 \times \cdots \times \mu_n(x_1, \dots, x_n) = \min\{\mu_1(x_1), \dots, \mu_n(x_n)\}$$

and  $\mu = \mu_1 \times \cdots \times \mu_n$  is a fuzzy subgroup of  $G = G_1 \times \cdots \times G_n$  (see [5]).

**LEMMA 5.2.** Let  $\mu$  be a fuzzy subgroup of a finite group  $G$ . For any integer  $k$  and  $x, y, z$  in  $G$ ,

- (i)  $\mu(x^k) \geq \mu(x)$ .
- (ii)  $|z|$  divides  $|y|$  implies  $\mu(y) \leq \mu(z)$ , for  $y, z$  in  $\langle x \rangle$ .
- (iii) If  $(|x|, k) = 1$ , then  $\mu(x^k) = \mu(x)$ .
- (iv) If  $G = G_1 \times \cdots \times G_n$ ,  $(|G_i|, |G_j|) = 1$  for all  $i \neq j$ ,  $x = (x_1, \dots, x_{l-1}, x_l, x_{l+1}, \dots, x_n)$  then

$$\mu(x) = \mu(x_1, \dots, x_n) \leq \mu(e_1, \dots, e_{l-1}, x_l, e_{l+1}, \dots, e_n)$$

for all  $l$ ,  $1 \leq l \leq n$ .

(ii) and (iii) are an improvement of [6, Theorem 4.1].

*Proof.* (i) We use induction on  $k$ , for  $k$  nonnegative. The result is clear for  $k = 0, 1$ . If  $k = 2$ , then

$$\mu(x^2) \geq \mu(x \cdot x) \geq \min\{\mu(x), \mu(x)\} = \mu(x).$$

Make the hypothesis  $\mu(x^s) \geq \mu(x)$  ( $s \geq 2$ ). Then

$$\mu(x^{s+1}) = \mu(x^s \cdot x) \geq \min\{\mu(x^s), \mu(x)\} = \mu(x)$$

which completes the induction. If  $k < 0$  then  $\mu(x^k) = \mu(x^k)^{-1} = \mu(x^{-k}) \geq \mu(x)$ .

(ii) Let  $|z|$  divide  $|y|$  for  $y, z$  in  $\langle x \rangle$ . By [3, Theorem 3.1]  $z$  is in  $\langle y \rangle$  and (ii) follows from (i).

(iii)  $\langle x \rangle = \langle x^k \rangle$  iff  $(|x|, k) = 1$ . So  $x = (x^k)^r$  and hence  $\mu(x) = \mu(x^k)^r \geq \mu(x^k) \geq \mu(x)$ . Thus  $\mu(x) = \mu(x^k)$ .

(iv) Let  $x = (x_1, \dots, x_n) \in G$ ,  $r = |G_1| \cdots |G_{l-1}| |G_{l+1}| \cdots |G_n|$  then  $(r, |G_l|) = 1$  and  $(r, |x_l|) = 1$ . Hence

$$\begin{aligned} \mu(x) &\leq \mu(x^r) \\ &= \mu[(x_1, e_2, \dots, e_n)^r \cdots (e_1, \dots, e_{l-1}, x_l, e_{l+1}, \dots, e_n)^r \cdots (e_1, \dots, e_{n-1}, x_n)^r] \\ &= \mu(e_1, \dots, e_{l-1}, x_l^r, e_{l+1}, \dots, e_n) \\ &= \mu(e_1, \dots, e_{l-1}, x_l, e_{l+1}, \dots, e_n). \quad \blacksquare \end{aligned}$$

Let  $G$  be a finite group and  $\mu$  be a fuzzy subgroup of  $G$ . The following theorem extends [5, Theorem 4.1].

**THEOREM 5.3.** *Let  $G = G_1 \times G_2 \times \cdots \times G_n$ , where  $(|G_i|, |G_j|) = 1$  for all  $i \neq j$ . Then  $\mu$  can be written as a product of fuzzy subgroups of  $G_l$ .*

*Proof.* For each  $l = 1, 2, \dots, n$  and each  $a$  in  $G_l$  let us define

$$\mu_l(a) = \mu(e_1, \dots, e_{l-1}, a, e_{l+1}, \dots, e_n).$$

It is clear by Theorems 3.9 and 3.10 that  $\mu_l$  is a fuzzy subgroup of  $G_l$ . Now let us prove that  $\mu = \mu_1 \times \mu_2 \times \cdots \times \mu_n$ . Notice that for all  $x = (x_1, \dots, x_n)$  in  $G$

$$\begin{aligned} \mu(x_1, \dots, x_n) &= \mu[(x_1, \dots, e_n) \cdot (e_1, x_2, \dots, x_n)] \\ &\geq \min\{\mu_1(x_1), \mu(e_1, x_2, \dots, x_n)\} \\ &\geq \min\{\mu_1(x_1), \dots, \mu_n(x_n)\}. \end{aligned}$$

Conversely, by Lemma 5.2(iv),

$$\mu(x_1, \dots, x_n) \leq \mu(e_1, \dots, e_{l-1}, x_l, e_{l+1}, \dots, e_n) = \mu_l x_l.$$

So  $\mu(x_1, \dots, x_n) \leq \min\{\mu_1(x_1), \dots, \mu_n(x_n)\}$ . Thus

$$\mu(x_1, \dots, x_n) = \min\{\mu_1(x_1), \dots, \mu_n(x_n)\}$$

which means  $\mu = \mu_1 \times \cdots \times \mu_n$ .  $\blacksquare$

**COROLLARY 5.4.** *If  $G$  is a nilpotent group then  $\mu$  can be written as a product of fuzzy subgroups of Sylow subgroups of  $G$ .*

*Proof.* By [3, Theorem 10.3.4]  $G$  is the direct product of this Sylow subgroups. Now we can use the Theorem 4.3.  $\blacksquare$

COROLLARY 5.5. *If  $G$  is an abelian group, then  $\mu$  can be written as a product of fuzzy subgroups of  $G$ .*

*Proof.*  $G$  is the direct product of its Sylow subgroups. ■

COROLLARY 5.6 [5, Theorem 4.1]. *If  $G$  is a cyclic group, then  $\mu$  can be written as a product of fuzzy subgroups of Sylow subgroups of  $G$ .* ■

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