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Semi-interior and Semi-closure of a Fuzzy Set*

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INTRODUCTION

We generally follow the terminology of Azad [1] and Ming and Ming [3].

Azad defined the fuzzy semi-open, fuzzy semi-closed, fuzzy regular-open, and fuzzy regular-closed sets and the fuzzy semi-continuous, fuzzy weakly-continuous, fuzzy semi-open, and fuzzy open functions in [1].

We defined the fuzzy H. almost-continuous and fuzzy W. almost-open functions in [8].

Continuing the work in [8], we define the semi-interior and semi-closure of a fuzzy set in a manner similar to that used in ordinary topological spaces. At the same time, the definition of almost-open function defined by Singal (cf. [6]) and irresolute, pre-semi-open functions and semi-homeomorphism defined by Crossley *et al.* [2] are extended to fuzzy sets. Some results are obtained in the functions of fuzzy topological spaces defined by Azad [1] and those are defined in [8] and here.

1. BASIC NOTATION AND DEFINITIONS

Fuzzy sets of a non-empty set X will be denoted by the capital letters A, B, C, etc. The value of a fuzzy set A at the element x of X will be denoted by $A(x)$ and a fuzzy point will be denoted by p.

If we write $p \in A$, then the definitions of a fuzzy point and being an element of a fuzzy set are understood as in [5] or [3], i.e., $p \in A$ either means that fuzzy point p takes its non-zero value in $(0, 1)$ at the support x_p and $p(x_p) < A(x_p)$ [5] or fuzzy point p takes its non-zero value in $(0, 1]$ and $p(x_p) \leqslant A(x_p)$ [3]. Furthermore, if we say only "fuzzy point p," then p

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will be considered as in [5] or [3]. If we write $p \in_1 A$ then the definition of fuzzy point–fuzzy elementhood will be the same as Srivastava *et al.* used in [5].

If for a fuzzy point p and a fuzzy set A, we have $p(x_p) + A(x_p) > 1$, then this case, which is defined by Ming and Ming [3] as "p quasi coincident with A," will be denoted by pqA.

In this work, X and Y denote fuzzy topological spaces with fuzzy topology τ and ϑ, respectively, and by f: $X \to Y$ we denote a function f of a fuzzy space X into a fuzzy space Y.

A^o, $\bar{A}$ and A' will denote respectively the interior, closure, and complement of the fuzzy set A.

2. Fuzzy Semi-interior and Fuzzy Semi-closure

Definition 2.1. Let $A \subset X$ be a fuzzy set and define the following sets:

$$\underline{A} = \bigcap \{B \mid A \subset B, B \text{ fuzzy semi-closed}\}$$

$$A_o = \bigcup \{B \mid B \subset A, B \text{ fuzzy semi-open}\}.$$

We call $\underline{A}$ the fuzzy semi-closure of A and A_o, the fuzzy semi-interior of A.

Obviously A_o is the greatest fuzzy semi-open set which is contained in A and $\underline{A}$ is the lowest fuzzy semi-closed set which contains A, and we have

$$A \subset \underline{A} \subset \bar{A} \qquad \text{and} \qquad A \supset A_o \supset A^o.$$

These are easely seen from [1, Theorem 4.3 and Remark 4.4] and the definitions of $\underline{A}$ and A_o.

In addition to these facts, if $A, B \subset X$ then

$$A \text{ is fuzzy semi-open} \Leftrightarrow A = A_o$$

$$A \text{ is fuzzy semi-closed} \Leftrightarrow A = \underline{A}$$

$$A \subset B \Rightarrow \underline{A} \subset \underline{B} \text{ and } A_o \subset B_o.$$

Theorem 2.2. *Let f: $X \to Y$. f is fuzzy semi-continuous iff $f(\underline{A}) \subset \overline{f(A)}$ for every $A \subset X$.*

Proof. Let $A \subset X$. Since $\overline{f(A)}$ is a fuzzy closed set, $f^{-1}(\overline{f(A)})$ is a fuzzy semi-closed set [8, Theorem 4.5].

Clearly $f^{-1}(\overline{f(A)}) = \underline{f^{-1}(\overline{f(A)})}$. From [4, Lemma 1.1], step by step we get

$$A \subset f^{-1}(f(A))$$

$$\underline{A} \subset \underline{f^{-1}(f(A))} \subset \underline{f^{-1}(\overline{f(A)})} = f^{-1}(\overline{f(A)})$$

$$f(\underline{A}) \subset f(f^{-1}(\overline{f(A)})) \subset \overline{f(A)}.$$

Conversely, let $B \subset Y$ be a fuzzy closed set. From the hypothesis we have

$$f(\underline{f^{-1}(B)}) \subset \overline{f(f^{-1}(B))} \subset \bar{B} = B.$$

So $\underline{f^{-1}(B)} \subset f^{-1}(f(\underline{f^{-1}(B)})) \subset f^{-1}(B)$.

Since $\underline{f^{-1}(B)} \subset f^{-1}(B)$ and $f^{-1}(B) \subset \underline{f^{-1}(B)}$, we get $f^{-1}(B) = \underline{f^{-1}(B)}$.

Hence $f^{-1}(B)$ is a fuzzy semi-closed set and f is a fuzzy semi-continuous function. ∎

3. Fuzzy Irresolute, Fuzzy Almost-open, and Fuzzy Pre-semi-open Functions

Definition 3.1. Let f be a function from a fuzzy topological space X to a fuzzy topological space Y.

(i) If for any fuzzy semi-open set B in Y, $f^{-1}(B)$ is a fuzzy semi-open set in X, then we say that f is a fuzzy irresolute function.

(ii) If for any fuzzy semi-open set A in X, $f(A)$ is a fuzzy semi-open set in Y, then we say that f is a fuzzy pre-semi-open function.

(iii) If for any fuzzy regular-open set A in X, $f(A)$ is a fuzzy open set in Y, then we say that f is a fuzzy almost-open function.

(iv) If f is one-to-one, onto, fuzzy pre-semi-open, and fuzzy irresolute, then we say that f is a fuzzy semi-homeomorphism.

Remark 3.2. For the function $f\colon X \to Y$, the following statements are valid:

f, fuzzy continuous $\not\Rightarrow$ f, fuzzy irresolute,

f, fuzzy irresolute $\not\Rightarrow$ f, fuzzy weakly-continuous,

f, fuzzy irresolute $\not\Rightarrow$ f, fuzzy H. almost-continuous,

f, fuzzy irresolute $\Rightarrow$ f, fuzzy semi-continuous.

EXAMPLE 3.3. Let $X = \{a, b, c\}$, $Y = \{x, y, z\}$ and $T_1 \subset X$, $T_2 \subset X$, $U_1 \subset Y$, $U_2 \subset Y$ be defined as follows:

$$T_1(a) = 0, \quad T_1(b) = 0,3, \quad T_1(c) = 0,2$$
$$T_2(a) = 0, \quad T_2(b) = 0,2, \quad T_2(c) = 0,2$$
$$U_1(x) = 0, \quad U_1(y) = 0,4, \quad U_1(z) = 0,2$$
$$U_2(x) = 0, \quad U_2(y) = 0,2, \quad U_2(z) = 0,7.$$

(a) Let $\tau = \{X, \phi, T_2\}$, $\vartheta = \{Y, \phi, U_2\}$.

If we define $f: X \to Y$ satisfying $f(a) = x$, $f(b) = y$, $f(c) = y$, then f is fuzzy continuous but not fuzzy irresolute. Because if we define the fuzzy set A in Y being $A(x) = 0,1$, $A(y) = 0,9$, $A(z) = 0,8$, then A is a fuzzy semi-open set since $U_2 \subset A \subset \overline{U_2}$. But $f^{-1}(A)$ is not a fuzzy semi-open set.

Clearly f is fuzzy continuous.

(b) Let $\tau = \{X, \phi, T_1\}$, $\vartheta = \{Y, \phi, U_1\}$.

If we define $f: X \to Y$ satisfying $f(a) = x$, $f(b) = y$, $f(c) = z$, then f is fuzzy irresolute but not fuzzy weakly-continuous and not fuzzy H. almost-continuous.

Remark 3.4. For the function $f: X \to Y$ the following statements are valid:

f, fuzzy pre-semi-open $\Rightarrow$ f, fuzzy semi-open

f, fuzzy pre-semi-open $\not\Leftarrow$ f, fuzzy almost-open

f, fuzzy pre-semi-open $\not\Rightarrow$ f, fuzzy W. almost-open

f, fuzzy open $\not\Rightarrow$ f, fuzzy pre-semi-open

f, fuzzy open $\Rightarrow$ f, fuzzy almost-open

f, fuzzy almost-open $\not\Rightarrow$ f, fuzzy semi-open

f, fuzzy almost-open $\not\Rightarrow$ f, fuzzy W. almost-open

f, fuzzy W. almost-open $\not\Rightarrow$ f, fuzzy almost-open.

EXAMPLE 3.5. Let $X = \{a, b, c\}$, $Y = \{x, y, z\}$ and $T_1 \subset X$, $T_2 \subset X$, $T_3 \subset X$, $U_1 \subset Y$, $U_2 \subset Y$ be defined as follows:

$$T_1(a) = 0, \quad T_1(b) = 0,3, \quad T_1(c) = 0,2$$
$$T_2(a) = 0,9, \quad T_2(b) = 0,6, \quad T_2(c) = 0,7$$
$$T_3(a) = 0, \quad T_3(b) = 0,8, \quad T_3(c) = 0,9$$
$$U_1(x) = 0, \quad U_1(y) = 0,3, \quad U_1(z) = 0,2$$
$$U_2(x) = 0, \quad U_2(y) = 0,2, \quad U_2(z) = 0,1.$$

(a) Let $\tau = \{X, \phi, T_1, T_2\}$, $\vartheta = \{Y, \phi, U_1\}$.

If we define $f\colon Y \to X$ satisfying $f(x) = a$, $f(y) = b$, $f(z) = c$, then f is fuzzy open, but not fuzzy pre-semi-open.

(b) Let $\tau = \{X, \phi, T_1\}$, $\vartheta = \{Y, \phi, U_2\}$.

If we define $f\colon X \to Y$ satisfying $f(a) = x$, $f(b) = y$, $f(c) = z$, then f is fuzzy pre-semi-open, but neither fuzzy almost-open nor fuzzy W. almost-open.

(c) Let $\tau = \{X, \phi, T_1, T_3\}$, $\vartheta = \{Y, \phi, U_1\}$.
If we define f as in (b), then f is fuzzy almost-open, but not fuzzy semi-open.

(d) If we define τ and ϑ as in (b), and f as in (a), then f is fuzzy W. almost-open, but not fuzzy almost-open.

(e) Let $\tau = \{X, \phi, T_2\}$, $\vartheta = \{Y, \phi, U_1\}$.
If we define f as in b), then f is fuzzy almost-open, but not fuzzy W. almost-open.

THEOREM 3.6. *Let $f\colon X \to Y$. The following are equivalent*:

(1) *f is fuzzy irresolute.*

(2) *For every $p \in X$ and for every fuzzy semi-open set O in Y containing $f(p)$ there exists a fuzzy semi-open set O^* in X such that $p \in O^* \subset f^{-1}(O)$.*

(3) *For every $p \in X$ and for every fuzzy semi-open set O in Y containing $f(p)$ there exists a fuzzy semi-open set O^* in X such that $p \in O^*$ and $f(O^*) \subset O$.*

(4) *For every $p \in X$ and for every fuzzy semi-open set O in Y satisfying $f(p)\, qO$ there exists a fuzzy semi-open set O^* in X such that $pqO^* \subset f^{-1}(O)$.*

(5) *For every $p \in X$ and for every fuzzy semi-open set O in Y satisfying $f(p)\, qO$ there exists a fuzzy semi-open set O^* in X such that pqO^* and $f(O^*) \subset O$.*

(6) *For every fuzzy semi-closed set F in Y, $f^{-1}(F)$ is a fuzzy semi-closed set in X.*

(7) *For every fuzzy semi-open set O in Y, $f^{-1}(O) \subset \overline{f^{-1^o}(O)}$.*

(8) *For every fuzzy semi-closed set F in Y, $f^{-1}(F) \supset \overline{f^{-1}(F)}^{o}$.*

Proof. (1) $\Rightarrow$ (2): Let $p \in X$ and O be any fuzzy semi-open set such that $f(p) \in O$.
Since f is fuzzy irresolute, $f^{-1}(O)$ is a fuzzy semi-open set and we have $p \in f^{-1}(O) = O^* \subset f^{-1}(O)$

$(2) \Rightarrow (3)$ and $(3) \Rightarrow (2)$ can be easily seen.

$(2) \Rightarrow (1)$: Let $O \subset Y$ be a fuzzy semi-open set and $p \in f^{-1}(O)$ be any fuzzy point. This implies $f(p) \in f(f^{-1}(O)) \subset O$. From hyphothesis there exists a fuzzy semi-open set O^* in X such that $p \in O^* \subset f^{-1}(O)$.
Hence, $f^{-1}(O)$ is a fuzzy semi-open set [8, Theorem 3.5].

$(4) \Rightarrow (5)$ and $(5) \Rightarrow (4)$ can be easily seen.

$(1) \Rightarrow (4)$: Let $p \in X$ and O be any fuzzy semi-open set such that $f(p)\, qO$. Clearly $f^{-1}(O)$ is a fuzzy semi-open set and $pqf^{-1}(O) = O^* \subset f^{-1}(O)$ [8, Proposition 4.2].

$(4) \Rightarrow (1)$: Let $O \subset Y$ be any fuzzy semi-open set. Let $p \in_1 f^{-1}(O)$. Clearly $f(p) \in_1 O$. Choose the fuzzy point p' as $p'(x_p) = 1 - p(x_p)$. For this p', we have $f(p')\, qO$ [8, Proposition 2.5]. From (4), there exists a fuzzy semi-open set such that $p'qO^* \subset f^{-1}(O)$.
Since $p'qO^*$,

$$p'(x_p) + O^*(x_p) = 1 - p(x_p) + O^*(x_p) > 1 \Rightarrow O^*(x_p) > p(x_p) \Rightarrow p \in_1 O^*.$$

Hence we have $p \in_1 O^* \subset f^{-1}(O)$. From [8, Theorem 3.5], $f^{-1}(O)$ is a fuzzy semi-open set.

$(1) \Rightarrow (6)$: Let F be any fuzzy semi-closed set in Y. F' is a fuzzy semi-open set. From (1), $f^{-1}(F')$ is a fuzzy semi-open set and from known equality $f^{-1}(F') = (f^{-1}(F))'$, $(f^{-1}(F))'$ is a fuzzy semi-open set and hence $f^{-1}(F)$ is a fuzzy semi-closed set [1, Theorem 4.2].

$(6) \Rightarrow (1)$ can be proved in the same way as $(1) \Rightarrow (6)$.

$(6) \Rightarrow (8)$, $(8) \Rightarrow (6)$, $(1) \Rightarrow 7)$, $(7) \Rightarrow (1)$ can be easily proved by using Theorem 4.2 in [1]. ▮

PROPOSITION 3.7. *Let (Z, W) be a fuzzy topological space and $f\colon X \to Y$, $g\colon Y \to Z$. Then the following statements are valid:*

(1) *If f and g are fuzzy pre-semi-open functions then $g \circ f$ is too.*

(2) *If f and g are fuzzy irresolute functions then $g \circ f$ is too.*

(3) *If f is fuzzy irresolute and g is fuzzy semi-continuous then $g \circ f$ is a fuzzy semi-continuous function.*

(4) *If f is fuzzy semi-open and g is fuzzy pre-semi-open then $g \circ f$ is a fuzzy semi-open function.*

(5) *If f is fuzzy almost-open and g is fuzzy open then $g \circ f$ is a fuzzy almost-open function.*

Proof. It is easy since we have $(g \circ f)(A) = g(f(A))$ for $A \subset X$ and $(g \circ f)^{-1}(B) = f^{-1}(g^{-1}(B))$ for $B \subset Z$. ▮

THEOREM 3.8. *If $f\colon X \to Y$ is fuzzy almost-open and fuzzy semi-continuous then f is a fuzzy irresolute function.*

Proof. It can be easily shown as in ordinary topological spaces [6, Theorem 1.12]. ▮

THEOREM 3.9. *If $f\colon X \to Y$ is fuzzy H. almost-continuous and fuzzy semi-open then f is a fuzzy pre-semi-open function.*

Proof. It can be proved as in the proof of Theorem 2.5 in [6] by using Theorem 4.17 and Proposition 3.4 in [8]. ▮

COROLLARY 3.10. *If $f\colon X \to Y$ is fuzzy continuous and fuzzy open then f is both fuzzy irresolute and fuzzy pre-semi-open.*

COROLLARY 3.11. *Eevery fuzzy homeomorphism (i.e., one-to-one, onto, fuzzy continuous, and fuzzy open function) is a fuzzy semi-homeomorphism.*

THEOREM 3.12. *If $f\colon X \to Y$ is fuzzy semi-continuous and fuzzy W. almost-open then f is a fuzzy irresolute function.*

Proof. Let $B \subset Y$ be any fuzzy semi-open set in Y. There exists a fuzzy open-set U in Y such that $U \subset B \subset \bar{U}$. From [8, Definition 4.13(b)], we have $f^{-1}(U) \subset f^{-1}(B) \subset f^{-1}(\bar{U}) \subset \overline{f^{-1}(U)}$.
Since f is fuzzy semi-continuous, $f^{-1}(U)$ is a fuzzy semi-open set.
Hence $f^{-1}(B)$ is a fuzzy semi-open set [8, Proposition 3.4]. ▮

THEOREM 3.13. *$f\colon X \to Y$ is a fuzzy irresolute function iff for every $A \subset X$, $f(\underline{A}) \subset \underline{f(A)}$.*

Proof. It can be easily proved as in ordinary topological space [2, Theorem 1.5]. ▮

THEOREM 3.14. *$f\colon X \to Y$ is a fuzzy irresolute function iff for every $B \subset Y$, $\underline{f(B)} \subset f^{-1}(\underline{B})$.*

Proof. It is similar to the proof of Theorem 1.6 in [2]. ▮

PROPOSITION 3.15. *Let $f\colon X \to Y$ be one-to-one and onto. f is a fuzzy semi-homeomorphism iff f and f^{-1} are fuzzy irresolute functions iff f and f^{-1} are fuzzy pre-semi-open functions.*

Proof. Obvious. ▮

COROLLARY 3.16. *Let $f\colon X \to Y$ be one-to-one and onto. f is a fuzzy semi-homeomorphism iff for every $A \subset X$, $f(\underline{A}) = \underline{f(A)}$.*

Proof. It can be seen from Proposition 3.15, Theorem 3.13, Theorem 3.14, and the fact $(f^{-1})^{-1}=f$. ∎

COROLLARY 3.17. *Let f be one-to-one and onto. f is a fuzzy semi-homeomorphism iff for every $B\subset Y$, $f^{-1}(\underline{B})=\underline{f(B)}$.*

THEOREM 3.18. *f: $X\rightarrow Y$ is a fuzzy irresolute function iff for every $B\subset Y$, $f^{-1}(B_o)\subset(f^{-1}(B))_o$.*

Proof. Let $B\subset Y$. B_o is a fuzzy semi-open set. Clearly $f^{-1}(B_o)$ is a fuzzy semi-open set and we have $f^{-1}(B_0)=(f^{-1}(B_o))_o\subset(f^{-1}(B))_o$.

Conversely, let B be any fuzzy semi-open set in Y. Then $B_o=B$ and $f^{-1}(B)=f^{-1}(B_o)\subset(f^{-1}(B))_o$.

Hence we have $f^{-1}(B)=(f^{-1}(B))_o$. This shows that $f^{-1}(B)$ is a fuzzy semi-open set. ∎

THEOREM 3.19. *Let f: $X\rightarrow Y$ be one-to-one and onto. f is fuzzy irresolute function iff for every $A\subset X$, $(f(A))_o\subset f(A_o)$.*

Proof. Let $A\subset X$. $(f(A))_o$ is a fuzzy semi-open set. Clearly $f^{-1}((f(A))_o)$ is a fuzzy semi-open set. $f^{-1}(f(A))=A$ [4, Lemma 1.1]. We have

$$f^{-1}((f(A))_o)\subset(f^{-1}(f(A)))_o=A_o \qquad \text{(Theorem 3.18)}$$

$$f(f^{-1}((f(A))_o))\subset f(A_o).$$

Since f is onto

$$(f(A))_o=f(f^{-1}((f(A))_o))\subset f(A_o), \qquad \text{[4, Lemma 1.1]}.$$

Conversely, let $B\subset Y$ be any fuzzy semi-open set. Immediately $B=B_o$. From hypothesis

$$f((f^{-1}(B))_o)\supset(f(f^{-1}(B)))_o=B_o=B.$$

This implies that $f^{-1}(f((f^{-1}(B))_o))\supset f^{-1}(B)$. Since f is one-to-one we have $(f^{-1}(B))_o\supset f^{-1}(B)$. Hence $f^{-1}(B)=(f^{-1}(B))_o$, i.e., $f^{-1}(B)$ is a fuzzy semi-open set. ∎

COROLLARY 3.20. *Let f: $X\rightarrow Y$ be one-to-one and onto. f is a fuzzy semi-homeomorphism iff for every $A\subset X$, $f(A_o)=(f(A))_o$.*

COROLLARY 3.21. *Let f: $X\rightarrow Y$ be one-to-one and onto. f is a fuzzy semi-homeomorphism iff for every $B\subset Y$, $f^{-1}(B_o)=(f^{-1}(B))_o$.*

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