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Rings Whose Modules Are ⊕-Supplemented

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We prove that a ring R is serial if and only if every finitely presented right and left R-module is ⊕-supplemented, and that R is artinian serial if and only if every right and left R-module is ⊕-supplemented.

1. INTRODUCTION AND PRELIMINARIES

Throughout this paper we assume that R is an associative ring with identity and all R-modules are unitary right R-modules, unless otherwise specified. The (Jacobson) radical of R is denoted by J.

An R-module M is *uniserial* if its submodules are linearly ordered by inclusion and it is *serial* if it is a direct sum of uniserial submodules. The ring R is *right* (*left*) *serial* if the right (left) R-module R_R ($_RR$) is serial and it is *serial* if it is both right and left serial.

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Let M be an R-module. A submodule N of M is *superfluous* (*small*) if $N + L \neq M$ for every proper submodule L of M. The notation $N \ll M$ means that N is a superfluous submodule of M. M is called *lifting* (or *satisfies* (D1)) if for every submodule N of M there are submodules K' and K of M such that $M = K \oplus K'$, $K' \leq N$, and $N \cap K \ll K$. It is easy to see that every uniserial module is lifting.

Vanaja and Purav [VP] proved that a ring R has the property that all right R-modules are lifting if and only if R is an artinian serial ring with $J^2 = 0$. Also, Oshiro and Wisbauer obtained this result as a corollary in [OW].

As a generalization of lifting modules, Mohamed and Müller [MM] call an R-module M $\oplus$-*supplemented* if for every submodule N of M there is a summand K of M such that $M = N + K$ and $N \cap K \ll K$. It was shown in [KHS, Theorem 1.4] that a finite direct sum of $\oplus$-supplemented modules is $\oplus$-supplemented.

In this paper we study rings whose modules are $\oplus$-supplemented. In Section 2, we show that every f.g. (finitely generated) right R-module is $\oplus$-supplemented if and only if every cyclic right R-module is $\oplus$-supplemented and every f.g. right R-module is a direct sum of cyclic modules. Arbitrary direct sums of lifting right R-modules over a right perfect ring R are shown to be $\oplus$-supplemented. In Section 3, we prove that R is serial if and only if every f.p. (finitely presented) right R-module and f.p. left R-module is $\oplus$-supplemented. Rings whose f.g. right and f.g. left modules are $\oplus$-supplemented are also characterized. This class of rings properly contains noetherian serial rings. As stated in the Abstract, we show that R is artinian serial if and only if every right and left R-module is $\oplus$-supplemented. However, we note that an artinian left serial ring R need not be right serial although every right R-module is $\oplus$-supplemented.

For characterizations of $\oplus$-supplemented modules and lifting modules we refer to [MM] and [Wi]. Also, for the other definition and notation in this paper we refer to [AF].

2. $\oplus$-SUPPLEMENTED MODULES

THEOREM 2.1. *The following statements are equivalent for a ring R with radical J.*

(1) *R is semiperfect.*

(2) *Every f.g. free R-module is $\oplus$-supplemented.*

(3) *R_R is $\oplus$-supplemented.*

(4) *For every maximal right ideal A of R there exists an idempotent $e \in R - A$ such that $A \cap eR \subseteq J$.*

(5) *Any of the left-handed versions of* (2), (3) *or* (4).

Proof. (1) ⇒ (2) Let R be a semiperfect ring. Let F be a f.g. free R-module. By [MM, Theorem 4.41(2) and Proposition 4.8], F is $\oplus$-supplemented.

(2) ⇒ (3) Clear.

(3) ⇒ (4) Let A be a maximal right ideal of R. There exists a direct summand K of R such that $R = A + K$ and $A \cap K \ll K$. There exists an idempotent e in R such that $K = eR$. Clearly $e \notin A$. Moreover, $A \cap K \ll R_R$ so that $A \cap K \subseteq J$.

(4) ⇒ (1) Let U be any simple R-module. Let $0 \neq u \in U$ and $B = \{r \in R \mid ur = 0\}$. Then B is a maximal right ideal of R and $U \cong R/B$. By (4), there exists an idempotent $f \in R - B$ such that $B \cap fR \subseteq J$. Clearly $R = B + fR$. Moreover, $B \cap fR \ll R$ implies that $B \cap fR \ll fR$. Now $fR/(B \cap fR) \cong (B + fR)/B = R/B \cong U$. It follows that U has a projective cover. By [AF, Theorem 27.6], R is semiperfect.

(1) ⇔ (5) By symmetry. ∎

COROLLARY 2.2. *A commutative ring R is semiperfect if and only if every cyclic R-module is $\oplus$-supplemented.*

Proof. (⇐) By Theorem 2.1.

(⇒) Let I be any ideal of R. Then the factor ring $\overline{R} = R/I$ is still a semiperfect ring. By Theorem 2.1, $\overline{R}$ is $\oplus$-supplemented as an $\overline{R}$-module and hence $\oplus$-supplemented as an R-module. Thus every cyclic R-module is $\oplus$-supplemented. ∎

THEOREM 2.3. *Let R be any ring and let M be a f.g. R-module such that every direct summand of M is $\oplus$-supplemented. Then M is a direct sum of cyclic modules.*

Proof. Suppose that $M = m_1R + \cdots + m_kR$ for some positive integer k and elements $m_i \in M$ $(1 \leq i \leq k)$. If $k = 1$ then there is nothing to prove. Suppose that $k > 1$ and that the result holds for $(k-1)$-generated modules with the stated condition. There exist submodules K, K' of M such that $M = K \oplus K'$, $M = m_1R + K$, and $m_1R \cap K \ll K$. Note that $K' \cong (M/K) = (m_1R + K)/K \cong m_1R/(m_1R \cap K)$, so that K' is cyclic. On the other hand, $K/(m_1R \cap K) \cong (m_1R + K)/m_1R = M/m_1R$, so that $K/(m_1R \cap K)$ is $(k-1)$-generated. Since $m_1R \cap K \ll K$ it follows that K is $(k-1)$-generated. By induction, K is a direct sum of cyclic modules. Thus $M = K \oplus K'$ is a direct sum of cyclic modules. ∎

Using the proof of Theorem 2.3, we have

COROLLARY 2.4. *Let R be a ring. Then every 2-generated $\oplus$-supplemented R-module is a direct sum of cyclic modules.*

COROLLARY 2.5. *Let R be a ring and let n be a positive integer. Then every n-generated R-module is ⊕-supplemented if and only if*

(i) *every cyclic R-module is ⊕-supplemented, and*

(ii) *every n-generated R-module is a direct sum of cyclic modules.*

Proof. $(\Rightarrow)$ By Theorem 2.3, since every direct summand of an n-generated module is n-generated.

$(\Leftarrow)$ By [KHS, Theorem 1.4]. ∎

COROLLARY 2.6. *Let R be a ring. Then every f.g. R-module is ⊕-supplemented if and only if*

(i) *every cyclic R-module is ⊕-supplemented, and*

(ii) *every f.g. R-module is a direct sum of cyclic modules.*

A commutative ring R is called an *FGC ring* if every f.g. R-module is a direct sum of cyclic modules. FGC rings are discussed by Brandal where he gives a complete characterization [B, Theorem 9.1].

It is easy to give an example of a semiperfect ring which is not FGC. Let F be any field and $R = F[[X, Y]]$, the ring of formal power series over F in the indeterminates X, Y. Then R is a commutative noetherian local domain and thus is semiperfect. However, the ideal $J = RX + RY$ is the unique maximal ideal of R and is uniform, so is not a direct sum of cyclic modules. Thus J is not a ⊕-supplemented R-module (Corollary 2.4) and R is not an FGC ring.

The following definitions are given in [B], and we recall them for the convenience of the reader:

A family of sets is said to have the *finite intersection property* if the intersection of every finite subfamily is non-empty. An R-module M is *linearly compact* if whenever $\{m_i + M_i\}_{i \in I}$ is a family of cosets of submodules of M ($m_i \in M$ and $M_i \leq M$ for each $i \in I$) with the finite intersection property, then $\bigcap_{i \in I}(m_i + M_i)$ is non-empty. A commutative ring R is a *maximal ring* if R is a linearly compact R-module; R is an *almost maximal ring* if R/I is a linearly compact R-module for all non-zero ideals I of R, and R is a *valuation ring* if R is a uniserial R-module.

EXAMPLE 2.7. Let $R = A(p)$, the ring of *p-adic integers* [AF, p. 54]. The ring R is a maximal valuation ring, which is an FGC ring by [B, Theorem 9.1], and hence every f.g. R-module is ⊕-supplemented. Consider the countably generated R-module $M = Z(p^{\infty})$ (the Prüfer p-group). Clearly M is a hollow module and so is ⊕-supplemented but is not a direct sum of cyclic R-modules. Thus it is not the case that every countably generated module with every direct summand ⊕-supplemented is a direct sum of cyclic modules (cf. Theorem 2.3.).

PROPOSITION 2.8. *Let R be a commutative ring. Then the following statements are equivalent.*

(1) *Every f.g. R-module is $\oplus$-supplemented.*

(2) *R is a semiperfect FGC ring.*

(3) *R is a direct sum of almost maximal valuation rings.*

Proof. (1) $\Leftrightarrow$ (2) follows from Corollaries 2.2 and 2.6.

(2) $\Rightarrow$ (3) Since R is semiperfect we have $R = R_1 \oplus \cdots \oplus R_n$ where every R_i is a local FGC ring for all $1 \leq i \leq n$. By [B, Theorem 4.5], every R_i is an almost maximal valuation ring.

(3) $\Rightarrow$ (2) Let $R = R_1 \oplus \cdots \oplus R_n$ where every R_i is an almost maximal valuation ring for all $1 \leq i \leq n$. By [B, Theorem 4.5] again, every R_i is an FGC ring. Being a valuation ring, every R_i is local and so R is semiperfect. ∎

COROLLARY 2.9. *A commutative indecomposable ring R is an almost maximal valuation ring if and only if every f.g. R-module is $\oplus$-supplemented.*

Now we come back to rings that are not necessarily commutative.

The following fact is known from [K]. We give a different proof in this study.

THEOREM 2.10. *A ring R is right perfect if and only if $R^{(N)}$ is a $\oplus$-supplemented R-module.*

Proof. ($\Rightarrow$) By [MM, Theorem 4.41 and Proposition 4.8].

($\Leftarrow$) Let F denote the countably generated free R-module $R^{(N)}$. Let N be any proper submodule of F. There exists a direct summand G of F such that $F = G + N$ and $N \cap G \ll G$. Then $F/N = (G+N)/N \cong G/(N \cap G)$. By [AF, Proposition 17.14], there exists a maximal submodule H of G. Since $N \cap G \ll G$ it follows that $N \cap G \leq H$. Thus N is contained in a maximal submodule L of F. Clearly, $FJ \leq L$ and hence $N + FJ \leq L$. It follows that $FJ \ll F$. By [AF, Lemma 28.3], J is right T-nilpotent. Since F is $\oplus$-supplemented, F/FJ is semisimple, and so, we have that R/J is semisimple. Hence by [AF, Theorem 28.4], R is right perfect. ∎

COROLLARY 2.11. *The following statements are equivalent for a ring R.*

(1) *R is right perfect.*

(2) *The R-module $R^{(N)}$ is $\oplus$-supplemented.*

(3) *Every countably generated free right R-module is $\oplus$-supplemented.*

(4) *Every free right R-module is $\oplus$-supplemented.*

Proof. (1) $\Leftrightarrow$ (2) By Theorem 2.10.

(1) $\Leftrightarrow$ (4) By [MM, Theorem 4.41] and [KHS, Lemma 1.2].

(4) $\Rightarrow$ (3) $\Rightarrow$ (2) Clear. ∎

THEOREM 2.12. *Let R be any ring and let M be an R-module such that $M = \bigoplus_{i \in I} M_i$ where M_i is a lifting module for each $i \in I$. Suppose further that $\mathrm{Rad}(M) \ll M$. Then M is $\oplus$-supplemented.*

Proof. Let N be a submodule of M. For each $i \in I$ let $J_i = \mathrm{Rad}(M_i)$. If $T = \mathrm{Rad}(M)$ then $T = \bigoplus_{i \in I} J_i$ by [AF, Proposition 9.19]. For each $i \in I$, $J_i = T \cap M_i$ and hence $M_i/J_i \cong (M_i + T)/T$ and so is semisimple. Now $M/T = \sum_{i \in I}(M_i + T)/T$. By [AF, Lemma 9.2], $M/T = ((N + T)/T) \oplus \{\bigoplus_{\alpha \in \Lambda}((L_\alpha + T)/T)\}$ for some submodule L_α of M_α ($\alpha \in \Lambda$) and an index set $\Lambda \subseteq I$. By [MM, Proposition 4.8(2)], for each $\alpha \in \Lambda$ there exists a direct summand K_α of M_α such that $K_\alpha \subseteq L_\alpha \subseteq K_\alpha + J_\alpha$. Let $K = \bigoplus_{\alpha \in \Lambda} K_\alpha$. Then K is a direct summand of M. Note that $M = N + (\sum_{\alpha \in \Lambda} L_\alpha) + T \subseteq N + K + T$, so that $M = N + K + T$ and hence $M = N + K$ since $\mathrm{Rad}(M) \ll M$. Next, $N \cap K \subseteq (N + T) \cap (\sum_{\alpha \in \Lambda} L_\alpha + T) \subseteq T \ll M$. It follows that $N \cap K \ll K$. Therefore M is $\oplus$-supplemented. ∎

COROLLARY 2.13. *Let R be a right perfect ring and let M be an R-module such that $M = \bigoplus_{i \in I} M_i$ where M_i is a lifting module for each $i \in I$. Then M is $\oplus$-supplemented.*

THEOREM 2.14. *Let R be a ring. Then R is right perfect if and only if R is semiperfect and every R-module $M = \bigoplus_{i \in I} M_i$, where M_i is a lifting module for all $i \in I$, is $\oplus$-supplemented.*

Proof. The necessity follows from Corollary 2.13. Conversely, let F be a free right R-module. By [MM, Corollary 4.42], R is lifting and hence F is $\oplus$-supplemented by hypothesis. Hence R is right perfect from Corollary 2.11. ∎

It is well known that an artinian serial ring R is of finite representation type, and every right and left R-module is a direct sum of uniserial R-modules (see [AF, Theorem 32.3]). Since every uniserial module is lifting the following result follows from Corollary 2.13.

COROLLARY 2.15. *If R is an artinian serial ring then every right and left R-module is $\oplus$-supplemented.*

In fact the converse of the above result is also true, and we shall establish it in the next section.

It is known that R is an artinian serial ring with $J^2 = 0$ if and only if every right R-module is lifting (see [VP, Proposition 2.13] or [OW, Corollary 2.5]). Since the ring R of 3×3 upper triangular matrices over a field is artinian serial with $J^2 \neq 0$, we see that every R-module is $\oplus$-supplemented but not every R-module is lifting.

We now aim to prove the following result in a series of lemmas. The injective envelope of an R-module M is denoted by $E(M)$.

THEOREM 2.16. *Let R be a right nonsingular right perfect ring such that any submodule of $E(R_R)$ is $\oplus$-supplemented. Then R is right artinian.*

LEMMA 2.17. *Let R be any ring and let U be a uniform R-module such that every 2-generated submodule of U is $\oplus$-supplemented. Then U is uniserial.*

Proof. Let $x, y \in U$ and let $V = xR + yR$. Let W be a maximal submodule of V. By hypothesis, there exist submodules V', V'' of V such that $V = V' \oplus V'' = W + V'$ and $W \cap V' \ll V'$. Clearly $V' \neq 0$ so that $V'' = 0$, $V = V'$, and $W \ll V$. Either $x \notin W$ or $y \notin W$ so that $V = xR$ or $V = yR$. Thus $yR \subseteq xR$ or $xR \subseteq yR$. It follows that U is uniserial. ∎

LEMMA 2.18. *Let R be a right perfect ring and let U be a uniform right R-module such that every submodule of U is $\oplus$-supplemented. Then U is noetherian.*

Proof. Let V be any non-zero submodule of U. Let W be a maximal submodule of V. By the proof of Lemma 2.17 we have $W \ll V$. Let $v \in V - W$. Then $V = W + vR$ and hence $V = vR$. Hence every submodule of U is cyclic and hence U is noetherian. ∎

LEMMA 2.19. *Let R be a right perfect ring and let E be a nonsingular $\oplus$-supplemented injective right R-module. Then E has an indecomposable decomposition.*

Proof. Without loss of generality, we can assume that $E \neq 0$. Because R is right perfect, E has a maximal submodule N. By hypothesis, $E = E' \oplus E'' = E' + N$ and $N \cap E' \ll E'$ for some submodules E', E''. Note that $E'/(N \cap E') \cong (E' + N)/N = E/N$ so that $N \cap E'$ is a maximal submodule of E'. It follows that E' is hollow and hence E' is an indecomposable injective summand of E.

By Zorn's lemma, there exists a maximal collection $\{E_i \mid i \in I\}$ of independent indecomposable injective summands of E. Suppose that $E \neq \bigoplus_{i \in I} E_i$. Because R is right perfect, E has a maximal submodule L such that $\bigoplus_{i \in I} E_i \subseteq L$. Now E is $\oplus$-supplemented, so that $E = F' \oplus F'' = F' + L$ and $L \cap F' \ll F'$ for some submodules F', F''. By the above argument, F' is an indecomposable injective summand of E. By the choice of $\{E_i \mid i \in I\}$ we see that $F' \cap (\bigoplus_{i \in I} E_i) \neq 0$. There exists a finite subset J of I such that $F' \cap (\bigoplus_{i \in J} E_i) \neq 0$. Note that $G = \bigoplus_{i \in J} E_i$ is injective and hence $E = G \oplus G'$ for some submodule G'. Now $E/G \cong G'$ so that E/G is nonsingular.

Next observe that F' is uniform so that $(F' + G)/G$ is singular because $(F' + G)/G \cong F'/(F' \cap G)$. It follows that $F' \subseteq G \subseteq L$ and $E = F' + L = L$, a contradiction. Thus $E = \bigoplus_{i \in I} E_i$, as required. ∎

COROLLARY 2.20. *Let R be a right perfect ring and let E be a nonsingular injective R-module such that every submodule of E is $\oplus$-supplemented. Then E is a direct sum of noetherian modules.*

Proof. By Lemmas 2.18 and 2.19. ∎

A module M is called *locally noetherian* if every f.g. submodule of M is noetherian.

COROLLARY 2.21. *Let R be a right perfect ring and let M be a nonsingular R-module such that every submodule of $E(M)$ is $\oplus$-supplemented. Then M is locally noetherian.*

Proof. Because M is nonsingular and M is essential in $E(M)$, the module $E(M)$ is nonsingular [G, Proposition 1.22] and injective. By Corollary 2.20, $E(M) = \bigoplus_{i \in I} E_i$ for some index set I and noetherian submodules E_i $(i \in I)$. Let N be any f.g. submodule of M. There exists a finite subset J of I such that $N \subseteq \bigoplus_{i \in J} E_i$. But $\bigoplus_{i \in J} E_i$ is noetherian and hence so too is N. Thus M is locally noetherian. ∎

Proof of Theorem 2.16. Let R be a right nonsingular right perfect ring such that all submodules of $E(R_R)$ are $\oplus$-supplemented. Then Corollary 2.21 gives R_R is locally noetherian and hence R is right noetherian. Therefore, R is right artinian (since R is right perfect). ∎

The crux of the above proof is Lemma 2.19 and the reason the argument works is because if E is a nonsingular injective R-module then every submodule of E has a unique injective envelope in E. This is not true for injective modules which are not nonsingular.

3. SERIAL RINGS AND $\oplus$-SUPPLEMENTED MODULES

Recall that an R-module M is *local* [AF, p. 357] if $\mathrm{Rad}(M)$ is a superfluous maximal submodule of M; equivalently, M is cyclic and has a unique maximal submodule. Hence an R-module M over a semiperfect ring R is local if and only if it is an epimorphic image of eR for some primitive idempotent e of R.

LEMMA 3.1. *Every indecomposable $\oplus$-supplemented R-module M with maximal submodules is local.*

Proof. Let N be a maximal submodule of M. Since M is indecomposable and $\oplus$-supplemented, N is superfluous. Hence N is the unique maximal submodule of M. ∎

The indecomposable injective $A(p)$-module $Z(p^\infty)$ in Example 2.7 is uniserial, and so is $\oplus$-supplemented but it is not local. Hence the condition "with maximal submodules" in the above lemma is necessary.

An element m in a left R-module is *primitive* [I] if $m = em$ for some primitive idempotent e of R. The next lemma should be known, but we include it here for completeness.

LEMMA 3.2. *Let R be a semiperfect ring. The following statements are equivalent for a left R-module M.*

(1) *M is uniserial.*

(2) *Every f.g. submodule of M is local.*

(3) *Every 2-primitive generated submodule of M is local.*

Proof. Let $e_1, \dots, e_n$ be a complete set of orthogonal primitive idempotents of R.

(1) ⇒ (2) Since M is uniserial, every f.g. submodule N of M is cyclic. Let $N = Rm$. Then $N = Re_1m + \cdots + Re_nm$. We must have $N = Re_im$ for some i since again M is uniserial.

(2) ⇒ (3) Obvious.

(3) ⇒ (1) If M is not uniserial then there are $x, y \in M$ such that $Rx \not\subseteq Ry$ and $Ry \not\subseteq Rx$. Now $Rx = \sum_{i=1}^{n} Re_ix$ and $Ry = \sum_{i=1}^{n} Re_iy$. Using $Rx \not\subseteq Ry$ and $Ry \not\subseteq Rx$, we must have $Re_ix \not\subseteq Re_jy$ and $Re_jy \not\subseteq Re_ix$ for some i and j. Hence both Re_ix and Re_jy are proper submodules of the 2-primitive generated submodule $Re_ix + Re_jy$ of M. So $Re_ix + Re_jy$ is local by the assumption, and then both Re_ix and Re_jy are superfluous in $Re_ix + Re_jy$. This is impossible. ∎

Let M be a f.p. R-module with no non-zero projective summands. We call M a 2-f.p. *module* if there are primitive idempotents e, e_1, and e_2 of R and there is a minimal projective presentation (see [AF]).

$$eR \to e_1R \oplus e_2R \to M \to 0.$$

Hence a 2-f.p. module is both 2-primitive generated and f.p.

PROPOSITION 3.3. *The following statements are equivalent for a semiperfect ring R.*

(1) *R is left serial.*

(2) *Every f.g. left ideal $L \subseteq Re$ is local for each primitive idempotent e of R.*

(3) *Every* 2-*primitive generated left ideal* $L \subseteq Re$ *is local for each primitive idempotent* e *of* R.

(4) *Every* 2-$f.p.$ *right* R-*module is* $\oplus$-*supplemented.*

(5) *There are no* 2-$f.p.$ *right* R-*modules*.

Proof. By Lemma 3.2 we have (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3). Clearly we have (5) $\Rightarrow$ (4).

(4) $\Rightarrow$ (3) Suppose there is a primitive idempotent e of R and there is a 2-primitive generated left ideal $L \subseteq Re$ such that L is not local. Then we have a projective cover $Re_1 \oplus Re_2 \xrightarrow{f} L$ where e_1 and e_2 are two primitive idempotents of R. Then we have a minimal projective presentation

$$Re_1 \oplus Re_2 \xrightarrow{f} Re \to Re/L \to 0,$$

which induces a minimal projective presentation

$$eR \xrightarrow{f^*} e_1R \oplus e_2R \to T(Re/L) \to 0$$

by [AF, Theorem 32.13] where $T(Re/L) = \operatorname{Coker} f^*$ is the transpose of Re/L. Now ${}_R Re/L$ is indecomposable with no non-zero projective summand and hence so too is $T(Re/L)_R$ by [AF, Corollary 32.14]. Hence $T(Re/L)_R$ is 2-f.p. and then is $\oplus$-supplemented. By Lemma 3.1, $T(Re/L)_R$ is local, which contradicts the above minimal projective presentation.

(3) $\Rightarrow$ (5) Suppose M is a 2-f.p. right R-module and

$$eR \xrightarrow{f} e_1R \oplus e_2R \to M \to 0$$

is a minimal projective presentation, where e, e_1, and e_2 are primitive idempotents. Then we have a minimal projective presentation

$$Re_1 \oplus Re_2 \xrightarrow{f^*} RE \to T(M) \to 0.$$

Now we have a 2-primitive generated left ideal $\operatorname{Im}(f^*) \subseteq Re$ and hence $\operatorname{Im}(f^*)$ is local by (3). We obtain a contradiction since $Re_1 \oplus Re_2 \xrightarrow{f^*} \operatorname{Im}(f^*)$ is a projective cover. ∎

Since the ring R is semiperfect if and only if the right R-module R_R is $\oplus$-supplemented by Theorem 2.1, we have the following corollary by Proposition 3.3(4).

COROLLARY 3.4. *If every* 2-*generated* $f.p.$ *right* R-*module is* $\oplus$-*supplemented*, *then* R *is left serial*.

The converse of Corollary 3.4 is false, even for artinian rings which are commutative modulo their radicals (see Example 3.17). However, the converse of Corollary 3.4 is true for commutative rings using the following characterizations of serial rings.

THEOREM 3.5. *The following statements are equivalent for a ring R.*

(1) *R is serial.*

(2) *Every f.p. right R-module and f.p. left R-module is $\oplus$-supplemented.*

(3) *Every 2-generated f.p. right R-module and 2-f.p. left R-module is $\oplus$-supplemented.*

(4) *R is semiperfect, and every 2-f.p. right R-module and 2-f.p. left R-module is $\oplus$-supplemented.*

Proof. Warfield [W, Corollary 3.4] proved that every f.p. right R-module (and left R-module) over a serial ring R is a finite direct sum of local submodules. Since every local module is $\oplus$-supplemented, and a finite direct sum of $\oplus$-supplemented modules is $\oplus$-supplemented by [KHS, Theorem 1.4], we have the first implication (1) $\Rightarrow$ (2). Clearly we have (2) $\Rightarrow$ (3) $\Rightarrow$ (4). Finally (4) $\Rightarrow$ (1) follows from Proposition 3.3. ∎

Using Proposition 3.3, we see that the condition "2-f.p. left (right) R-module is $\oplus$-supplemented' in Theorem 3.5(3)(4) can be replaced by that "there are no 2-f.p. left (right) R-modules." This is the case for the rest of the theorems.

The equivalence (1) $\Leftrightarrow$ (2) of the next result was established by Ivanov in [I, Theorem 2].

THEOREM 3.6. *The following statements are equivalent for a ring R.*

(1) *R is serial and every indecomposable injective right R-module is uniserial.*

(2) *Every f.g. right R-module is serial.*

(3) *Every f.g. right R-module and f.p. left R-module is $\oplus$-supplemented.*

(4) *Every f.g. right R-module and 2-f.p. left R-module is $\oplus$-supplemented.*

(5) *Every 2-generated right R-module and 2-f.p. left R-module is $\oplus$-supplemented.*

(6) *R is semiperfect, and every 2-primitive generated right R-module and 2-f.p. left R-module is $\oplus$-supplemented.*

Proof. (1) $\Leftrightarrow$ (2) By [I, Theorem 2].

(2) $\Rightarrow$ (3) Since every f.g. uniserial right R-module is $\oplus$-supplemented, every f.g. right R-module, being a finite direct sum of f.g. uniserial modules, is $\oplus$-supplemented by [KHS, Theorem 1.4]. Since we already have (2) $\Rightarrow$ (1), the rest of (3) follows from Theorem 3.5.

(3) $\Rightarrow$ (4) $\Rightarrow$ (5) $\Rightarrow$ (6) Clear.

(6) $\Rightarrow$ (1) By Theorem 3.5, R is a serial ring. Let U be an indecomposable injective right R-module with a 2-primitive generated submodule $V = xR + yR$. Modifying the proof of Lemma 2.17, we see that $V = xR$ or $V = yR$. Hence V is a local R-module. By Lemma 3.2, U is uniserial. ∎

Using Theorem 3.6, we have the following characterizations of rings whose f.g. right and f.g. left modules are $\oplus$-supplemented.

THEOREM 3.7. *The following statements are equivalent for a ring R.*

(1) *R is serial, and every indecomposable injective right R-module and indecomposable injective left R-module is uniserial.*

(2) *Every f.g. right R-module and f.g. left R-module is serial.*

(3) *Every f.g. right R-module and f.g. left R-module is $\oplus$-supplemented.*

(4) *Every 2-generated right R-module and 2-generated left R-module is $\oplus$-supplemented.*

(5) *R is semiperfect, and every 2-primitive generated right R-module and 2-primitive generated left R-module is $\oplus$-supplemented.*

By Theorem 3.5, serial right noetherian rings belong to the class of rings in Theorem 3.6. Consequently, noetherian serial rings belong to the class of rings in Theorem 3.7. However, by Proposition 2.8, even commutative rings whose f.g. modules are $\oplus$-supplemented need not be noetherian.

Let S be a noetherian valuation domain with non-zero radical J. According to [H], S is said to be *complete* if it is complete with respect to its J-adic topology. Let Q be the quotient ring of S. Then the ring

$$R = \begin{bmatrix} S & Q \\ 0 & Q \end{bmatrix}$$

is serial and right noetherian but not left noetherian. Hence R satisfies the conditions of Theorem 3.6.

Suppose S is not complete (e.g., the ring $Z_{(p)} = \{a/b \mid a, b \in Z, b \notin pZ\}$, the localization of Z at the prime ideal pZ). Then by [H, Corollary 2.4], not every f.g. left R-module is serial, and so R is not one of the rings in

Theorem 3.7. Consequently, $R \times R^{\text{op}}$ is a serial ring but this ring does not belong to the class of rings in Theorem 3.6. Hence we have the following strict containments of classes of serial rings:

$$\begin{aligned} \{\text{serial rings}\} &\supset \{\text{serial rings in Theorem 3.6}\} \\ &\supset \{\text{serial rings in Theorem 3.7}\} \\ &\supset \{\text{noetherian serial rings}\}. \end{aligned}$$

Suppose S is complete (e.g., the ring $A(p)$ of p-adic integers). Then by [H, Corollary 2.4], R is one of the rings in Theorem 3.7, and so is the ring $R \times R^{\text{op}}$ which is not right noetherian. Consequently, $R \times R^{\text{op}}$ is one of the rings in Theorem 3.6. So we also have the following strict containments of classes of serial rings.

$$\begin{aligned} \{\text{serial rings}\} &\supset \{\text{serial rings in Theorem 3.6}\} \\ &\supset \{\text{serial right noetherian rings}\} \\ &\supset \{\text{noetherian serial rings}\}. \end{aligned}$$

Moreover, we have

$$\{\text{serial rings in Theorem 3.7}\} \not\supseteq \{\text{serial right noetherian rings}\},$$
$$\{\text{serial right noetherian rings}\} \not\supseteq \{\text{serial rings in Theorem 3.7}\}.$$

Next we consider artinian serial rings, which are properly contained in the class of noetherian serial rings.

It is known that a semiprimary left serial ring is left artinian. We generalize this as follows. (This may be known.)

PROPOSITION 3.8. *Let R be a left serial ring. If R is either right perfect or left perfect then R is left artinian.*

Proof. To show R is a semiprimary ring, we may assume (see [AF, Proposition 28.11]) that R is a local ring. Then the left R-module ${}_R R$ is uniserial.

(1) Let R be left perfect. We show R is left noetherian. Suppose there is a left ideal A which is not f.g. Since R is left perfect A has a maximal submodule B. For each $a \in A$, Ra is a proper submodule of A. Hence $Ra \subseteq B$ since ${}_R R$ is uniserial and B is a maximal submodule of A. It follows that $A \subseteq B$, a contradiction.

(2) Let R be right perfect but no semiprimary. Then there is an infinite chain $J \supset J^2 \supset J^3 \supset \cdots$ where J is the radical of R. Let $B = \bigcap_{n=1}^{\infty} J^n$. Since R is right perfect there is a left ideal $A \supset B$ such that

A/B is simple. Since $_R R$ is uniserial and for each n we have $B \subset J^n$ we must have $A \subseteq J^n$. Then $A \subseteq \bigcap_{n=1}^{\infty} J^n = B$. This is a contradiction. ∎

COROLLARY 3.9. *Let R be a serial ring. If R is either right perfect or left perfect then R is an artinian serial ring.*

Recall that R is right perfect if and only if every countably generated free right R-module is $\oplus$-supplemented (see Corollary 2.11).

COROLLARY 3.10. *If every countably generated right R-module is $\oplus$-supplemented, then R is left artinian and left serial.*

Proof. By the assumption, R is right perfect. By Corollary 3.4, R is left serial. Hence R is left artinian by Proposition 3.8. ∎

The converse of Corollary 3.10 is false, even for artinian left serial rings which are commutative modulo the radicals (see Example 3.17).

The next theorem gives characterizations of artinian serial rings via their $\oplus$-supplemented modules.

THEOREM 3.11. *The following statements are equivalent for a ring R.*

(1) *R is artinian serial.*

(2) *Every right R-module and left R-module is $\oplus$-supplemented.*

(3) *Every right R-module and f.g. left R-module is $\oplus$-supplemented.*

(4) *Every countable generated right R-module and f.g. left R-module is $\oplus$-supplemented.*

(5) *Every countably generated right R-module and f.p. left R-module is $\oplus$-supplemented.*

(6) *Every countably generated right R-module and 2-f.p. left R-module is $\oplus$-supplemented.*

(7) *R is right (left) perfect, and every 2-f.p. right R-module and 2-f.p. left R-module is $\oplus$-supplemented.*

Proof. (1) $\Rightarrow$ (2) is Corollary 2.15. Clearly we have (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5) $\Rightarrow$ (6) $\Rightarrow$ (7), and (7) $\Rightarrow$ (1) follows from Theorem 3.5 and Corollary 3.9. Finally, using the equivalence (1) $\Leftrightarrow$ (7), we see that (1) is also equivalent to the parenthetical version of (7). ∎

COROLLARY 3.12. *The following statements are equivalent for a ring R.*

(1) *R is artinian serial.*

(2) *Every right R-module and indecomposable injective left R-module is $\oplus$-supplemented.*

(3) *Every countably generated right R-module and indecomposable injective left R-module is $\oplus$-supplemented.*

(4) *R is right (left) perfect, and every 2-f.p. right R-module and indecomposable injective left R-module is $\oplus$-supplemented.*

Proof. It remains to prove (4) $\Rightarrow$ (1). By Proposition 3.3, R is left serial. Then by Proposition 3.8, R is left artinian in either case. Let U be an indecomposable injective left R-module. Since U is $\oplus$-supplemented, it is local by Lemma 3.1, so, U is an epimorphic image of an indecomposable projective left R-module which is uniserial. Hence U is uniserial. By [AF, Theorem 32.3], which was due to Fuller [F], R is artinian serial. ∎

COROLLARY 3.13. *The following statements are equivalent for a commutative ring R.*

(1) *R is artinian serial.*

(2) *Every R-module is $\oplus$-supplemented.*

(3) *Every countably generated R-module is $\oplus$-supplemented.*

(4) *R is perfect, and every f.p. R-module is $\oplus$-supplemented.*

(5) *R is perfect, and every 2-f.p. R-module is $\oplus$-supplemented.*

We do not know if every right R-module is $\oplus$-supplemented, provided that every countably generated right R-module is $\oplus$-supplemented (in this case, R is left artinian and left serial by Corollary 3.10).

THEOREM 3.14. *The following statements are equivalent for a right serial ring R.*

(1) *R is artinian serial.*

(2) *Every right R-module is $\oplus$-supplemented.*

(3) *Every countably generated right R-module is $\oplus$-supplemented.*

(4) *R is right (left) perfect, and every f.g. right R-module is $\oplus$-supplemented.*

(5) *R is right (left) perfect, and every f.p. right R-module is $\oplus$-supplemented.*

(6) *R is right (left) perfect, and 2-f.p. right R-module is $\oplus$-supplemented.*

(7) *R is right (left) perfect, and every indecomposable injective right R-module is $\oplus$-supplemented.*

Proof. We clearly have (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5) $\Rightarrow$ (6). The implication (6) $\Rightarrow$ (1) follows from Proposition 3.3 and Corollary 3.9.

Let (i′) denote the parenthetical version of (i) for $i = 4, 5, 6, 7$. Then we have (1) $\Rightarrow$ (4′) $\Rightarrow$ (5′) $\Rightarrow$ (6′). By Proposition 3.3 and Corollary 3.9 again we have (6′) $\Rightarrow$ (1).

By Theorem 3.11, we have (1) ⇒ (7), (7′). Using Proposition 3.8, we know that R is right artinian and right serial in either case of (7) or (7′). Then we have (7) ⇒ (1) and (7′) ⇒ (1) by modifying the proof of Corollary 3.12. ∎

Recall that R is an artinian serial ring with $J^2 = 0$ if and only if every right R-module is lifting (see [VP, Proposition 2.13] or [OW, Corollary 2.5]). We generalize the direction "⇐" as follows.

THEOREM 3.15. *The following statements are equivalent for a ring R with radical J.*

(1) *R is artinian serial and $J^2 = 0$.*

(2) *Every right R-module is lifting.*

(3) *Every f.g. right R-module is lifting.*

(4) *Every 2-generated right R-module is lifting.*

(5) *R is semiperfect, and every 2-primitive generated right R-module is lifting.*

Proof. We first have (1) ⇒ (2) ⇒ (3) ⇒ (4). Since every lifting module is ⊕-supplemented we have (4) ⇒ (5) by Theorem 2.1.

It remains to prove (5) ⇒ (1). By Proposition 3.3, R is left serial.

Let U be an indecomposable injective right R-module. Every 2-primitive generated submodule of U is lifting and hence ⊕-supplemented. Using the proof of Theorem 3.6(6) ⇒ (1), we see that U is uniserial. If $c(U) \geq 3$ we can produce a uniserial R-module M with the composition series $0 \subset V \subset N \subset M$. Then the 2-primitive generated right R-module $M \oplus (N/V)$ is not lifting by [OW, Lemma 2.3]. Hence $c(U) \leq 2$. Since $UJ^2 = 0$ for every indecomposable injective right R-module U we must have $J^2 = 0$, so R is semiprimary.

Now it suffices to show that R is right serial; i.e., the right R-module eJ is simple or 0 for each primitive idempotent e of R. Suppose $c(eJ) \geq 2$ for some primitive idempotent e. Then eR has a submodule N such that eR/N has length 3 and $\mathrm{Soc}(eR/N) = S_1 \oplus S_2$ where S_1 and S_2 are simple R-modules. Then $eR/N \leq E(eR/N) = E(S_1 \oplus S_2) = E(S_1) \oplus E(S_2)$. Since the indecomposable module eR/N has length 3 and each $c(E(S_i)) \leq 2$, we must have each $c(E(S_i)) = 2$. Since the 2-primitive generated module $E(S_1) \oplus E(S_2) = E(eR/N)$ is lifting, we have $E(eR/N) = K' \oplus K$ with $K' \subseteq (eR/N)$ and $eR/N) \cap K \ll K$. Since the indecomposable non-injective module eR/N has length 3 and K' is injective, we have $K' = 0$. Then $K = E(eR/N)$, and $(eR/N) = (eR/N) \cap E(eR/N) \ll E(eR/N)$. It follows that $(eR/N) \subseteq \mathrm{Rad}(E(eR/N)) = S_1 \oplus S_2$ which is a contradiction since $c(eR/N) = 3$ and $c(S_1 \oplus S_2) = 2$. ∎

If every right R-module is $\oplus$-supplemented then R is left artinian and left serial by Corollary 3.10. But R need not be right serial by the following example.

EXAMPLE 3.16 [DR1]. Let R be a local artinian ring with radical W such that $W^2 = 0$, $Q = R/W$ is commutative, $\dim({}_QW) = 1$, and $\dim(W_Q) = 2$. Then R is left serial but not right serial. Let $W = uR \oplus vR$. By [DR1, Proposition 3], there are three isomorphism types of indecomposable right R-modules, namely, $A_1 = R/W$ (the simple module), $A_2 = R/uR$ (the injective module), and $A_3 = R_R$. Moreover every right R-module is a direct sum of indecomposables (so R is of finite representation type). It is easy to see that each of the A_i's is a lifting module. We conclude by Corollary 2.13 that every right R-module is $\oplus$-supplemented.

The concluding example shows that the converse of Corollary 3.4 or Corollary 3.10 is false, even for local artinian rings which are commutative modulo their radicals.

EXAMPLE 3.17 [DR2]. Let R be a local artinian ring with radical W such that $W^2 = 0$, $Q = R/W$ is commutative, $\dim({}_QW) = 1$, and $\dim(W_Q) = 3$. Then R is left serial but not right serial. Let $W = w_1R \oplus w_2R \oplus w_3R$. By [DR 2, Proposition 4.9], there are five isomorphism types of indecomposable right R-modules defined in [DR2, Lemmas 4.1 and 4.2], where $X_5 = (R_R \oplus R_R)/((w_1, 0)R + (0, w_1)R + (w_2, w_3)R)$ is an indecomposable right R-module of length 5 and it is not local. Hence X_5 is not $\oplus$-supplemented by Lemma 3.1. Clearly, X_5 is 2-generated and f.p. (even 2-primitive generated).

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