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Dynamic critical index of the Swendsen–Wang algorithm by dynamic finite-size scaling

M. Dilaver [a,*], S. Gündüç [a], M. Aydın [b], Y. Gündüç [a]

[a] *Hacettepe University, Physics Department, 06800 Beytepe, Ankara, Turkey*
[b] *Bilkent University, Physics Department, 06800 Bilkent, Ankara, Turkey*

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Abstract

In this work we have considered the dynamic scaling relation of the magnetization in order to study the dynamic scaling behavior of 2- and 3-dimensional Ising models. We have used the literature values of the magnetic critical exponents to observe the dynamic finite-size scaling behavior of the time evolution of the magnetization during early stages of the Monte Carlo simulation. In this way we have calculated the dynamic critical exponent Z for 2- and 3-dimensional Ising Models by using the Swendsen–Wang cluster algorithm. We have also presented that this method opens the possibility of calculating z and x_0 separately. Our results show good agreement with the literature values. Measurements done on lattices with different sizes seem to give very good scaling.

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Keywords: Ising model; Dynamic scaling; Time evolution of the magnetization

1. Introduction

Fortuin and Kasteleyn's solution of the Potts model by using percolating clusters [1] has been an inspiration for the Swendsen–Wang cluster algorithm [2]. This algorithm uses the Hamiltonian of the Potts model in order to identify the clusters of the spins with the same orientations. In defining a cluster, starting from a seed spin, a new spin is added to the already growing cluster with the probability $P = 1 - e^{-\beta}$, where β is the inverse temperature. After obtaining all possible clusters on the lattice, clusters are flipped with equal probability. Immediately after the work by Swendsen and Wang, Wolff proposed an new algorithm [3], which is basically a modification to the Swendsen–Wang algorithm. Despite the fact that the Wolff algorithm is an alternative method of updating clusters, decorrelation times have shown to be very different between Wolff and Swendsen–Wang algorithms. Following these two cluster update algorithms many alternative cluster update algorithms are introduced with decorrelation times always higher than that of the Wolff algorithm. For 2-dimensional Ising model Heerman and Burkitt [4] suggested that the autocorrelation data are consistent with a logarithmic divergence, but it is very difficult to distinguish between the logarithm and a small power [5].

With the introduction of cluster algorithms, a great improvement in the simulations of the magnetic spin systems has been possible since it has been shown that the dynamic critical exponents of these algorithms are much less than that of local algorithms such as Metropolis and Heat Bath. The efficiencies (dynamic behavior and the dynamic critical exponents) have been discussed by many authors by using various spin systems at thermal equilibrium [2–8].

Recently we have studied the dynamic behavior of 2-, 3-, and 4-dimensional Ising models by using the Wolff cluster algorithm [9]. We based our work on the dynamic scaling which exists in the early stages of the quenching process in the system [10]. The efficiency of the Wolff algorithm is directly related to the size of the updated clusters, hence the efficiency increases during the quenching process as the number of iterations increases. In our calculations, we have observed that

* Corresponding author.
E-mail address: dilaver@hacettepe.edu.tr (M. Dilaver).

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our results are consistent with vanishing dynamic critical exponent.

In this work we aimed to discuss the dynamic critical exponent of the Swendsen–Wang algorithm by using dynamic finite-size scaling. This will give us an opportunity to compare the efficiencies of cluster algorithms.

2. The method

In this work we have employed 2-, 3-, and 4-dimensional Ising models which are described by the Hamiltonian

$$-\beta H = K\sum_{\langle ij\rangle} S_i S_j. \quad (1)$$

Here $\beta = 1/kT$ and $K = J/kT$, where k is the Boltzmann constant, T is the temperature and J is the magnetic interaction between the spins. In the Ising model the spin variables take the values $S_i = \pm 1$.

To observe the critical behavior of the systems exhibiting second-order phase transitions we use dynamic scaling which exists in the early stages of the quenching process in the system. For the kth moment of the magnetization of a system, dynamic finite-size relation can be written as [10]

$$M^{(k)}(t,\epsilon,m_0,L) = L^{(-k\beta/\nu)}\mathcal{M}^{(k)}\big(tL^{-z},\epsilon L^{1/\nu}, m_0 L^{x_0}\big), \quad (2)$$

where L is the spatial size of the system, β and ν are the well-known critical exponents, t is the simulation time and $\epsilon = (T - T_c)/T_c$ is the reduced temperature. In Eq. (2), z is the dynamic critical exponent and x_0 is an independent exponent which is the anomalous dimension of the initial magnetization m_0.

In order to discuss time evolution of the Swendsen–Wang algorithm, we have selected wide range of thermodynamic quantities. Eq. (2) implies that magnetization ($\langle S\rangle$) and its higher moments are good candidates for observing dynamic finite size scaling behavior. For this reason we have considered $\langle S\rangle$, $\langle S^2\rangle$ and $\langle S^4\rangle$ where, nth moment of magnetization is given by,

$$\langle S^n\rangle = \frac{1}{L^d}\left\langle\left(\sum_i s_i\right)^n\right\rangle. \quad (3)$$

Since the efficiency of cluster algorithms is related to the average cluster size ($\langle C\rangle$),

$$\langle C\rangle = \frac{1}{N_c}\sum_i^{N_c}\frac{1}{L^d}(C_i), \quad (4)$$

this quantity has also been considered in our calculations.

All of the above quantities have their own anomalous dimensions and using such quantities, in order to obtain dynamic exponent, one may expect some ambiguities due to correction to scaling. Since our calculations are done in the early stages of the simulations, the correlation length is expected to be less than the lattice size; hence use of infinite lattice critical exponents in Eq. (2) can be sufficient to explain the critical behavior of the system. For this reason, critical exponents are taken as the Onsager solution for the 2-dimensional Ising model. For the 3-dimensional case, the critical exponent values are taken from the literature [11,12]. The 4-dimensional case is the critical dimension for the Ising model, and above 4-dimension the critical exponents are the mean-field critical exponents.

Following closely our previous calculations [9], two different scaling functions are also used. The first such quantity is Binder's cumulant [13–15]. Binder's cumulant is widely used in order to obtain the critical parameters as well as to determine the type of the phase transition. The second quantity is the scaling function (F) based on the surface renormalization. This function is studied in detail for the Ising model [16–18] and q-state Potts model [19–21]. We propose that the dynamic finite size scaling relation also holds for the scaling functions and the scaling relation can be written similarly to the moments of the magnetization,

$$O(t,\epsilon,m_0,L) = \mathcal{O}^{(k)}\big(t/\tau,\epsilon L^{1/\nu}, m_0 L^{x_0}\big). \quad (5)$$

Our aim is to study dynamic finite size scaling behavior of the scaling functions by using Eq. (5).

Binder's cumulant involves the ratio of the moments of the magnetization or energy. In this work we have used Binder's cumulant ($B_2(t)$) for $n = 2$ by using the relation

$$B_2(t) = \frac{\langle S^2\rangle(t)}{\langle|S|\rangle^2(t)}. \quad (6)$$

In order to calculate surface renormalization function F, one considers the direction of the majority of spins of two parallel surfaces which are $L/2$ distance away from each other [18]. Similar to the calculations of Binder's cumulant, iteration-dependent calculation of F requires the configuration averages which are obtained for each iteration yielding a Monte Carlo time-dependent expression,

$$F(t) = \big\langle \mathrm{sign}[S_i]\mathrm{sign}[S_{i+L/2}]\big\rangle(t). \quad (7)$$

$F(t)$ can be used in calculating the dynamic finite size scaling relation given in Eq. (5).

3. Results and discussion

We have studied 2-, 3-, and 4-dimensional Ising Models evolving in time by using the Swendsen–Wang algorithm. Following our previous work [9], we have prepared lattices with vanishing magnetization and total random initial configurations are quenched at the corresponding infinite lattice critical temperature. We have used the lattices $L = 256, 384, 512, 640$ and $L = 32, 48, 64, 80$, $L = 16, 20, 24$ for 2-, 3-, and 4-dimensional Ising models, respectively. Twenty bins of two thousand runs have been performed for 2-, 3-, and 4-dimensional models. Errors are calculated from the average values for each iteration obtained in different bins.

In Fig. 1 we have presented the magnetization data ($\langle S\rangle(t)$) before and after the dynamic finite size scaling for 2-, 3-, and 4-dimensional Ising models for the lattice sizes considered. Fig. 1 (a), (c) and (e) shows the time evolution of $\langle S\rangle(t)$ during the relaxation of the system until a plateau is reached for 2-, 3-, and 4-dimensional Ising models, respectively. It is seen from

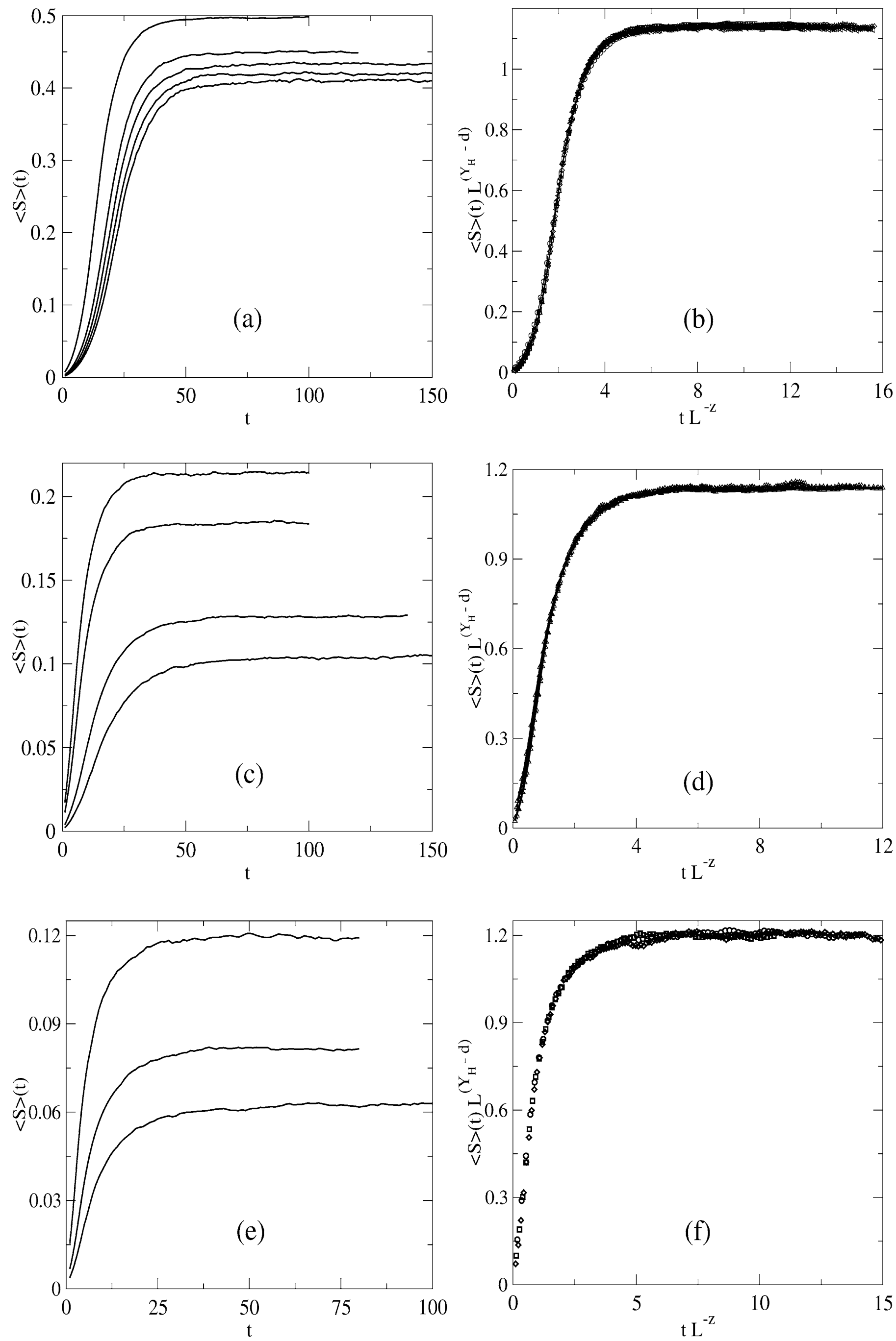

Fig. 1. (a) Magnetization data $\langle S\rangle(t)$ for the 2-dimensional Ising Model for linear lattice sizes $L = 256$, 384, 512, 640 as a function of simulation time t, (b) scaling of $\langle S\rangle(t)$ data given in (a). (c) Simulation data for $\langle S\rangle(t)$ as a function of simulation time t for the 3-dimensional Ising model for linear lattice sizes $L = 32$, 48, 64, 80, (d) scaling of $\langle S\rangle(t)$ data given in (c). (e) Simulation data for $\langle S\rangle(t)$ as a function of simulation time t for the 4-dimensional Ising model for linear lattice sizes $L = 16$, 20, 24, (f) scaling of $\langle S\rangle(t)$ data given in (e).

these figures that time to reach the plateau is proportional to the linear size (L) of the system. As it is seen from Eq. (2), in the dynamic finite size scaling, $\langle S\rangle(t)$ scales with a factor $L^{(Y_H-d)}$ and t scales as t/L^z. Fig. 1 (b), (d) and (f) shows scaling of the time-dependent magnetization. For scaling of the magnetization, literature values of infinite lattice critical exponents are used. Y_H is taken as $Y_H = \frac{15}{8}$ (Onsager solution), $Y_H = 2.4808$ [11,12], $Y_H = 3$ (mean-field solution) for the 2-, 3-, and 4-dimensional models, respectively.

In Fig. 2 scaling of Binder's cumulant ($B_2(t)$) has been presented. Fig. 2 (a), (c) and (e) shows the time evolution of $B_2(t)$ during the relaxation of the system for 2-, 3-, and 4-dimensional Ising models, respectively. Fig. 2 (b), (d) and (f) shows scaling of $B_2(t)$.

In Fig. 3 we have presented the surface renormalization function data ($F(t)$) before and after the dynamic finite size scaling for 2-, 3-, and 4-dimensional Ising models for the lattice sizes considered. Fig. 3 (a), (c) and (e) shows the time evolu-

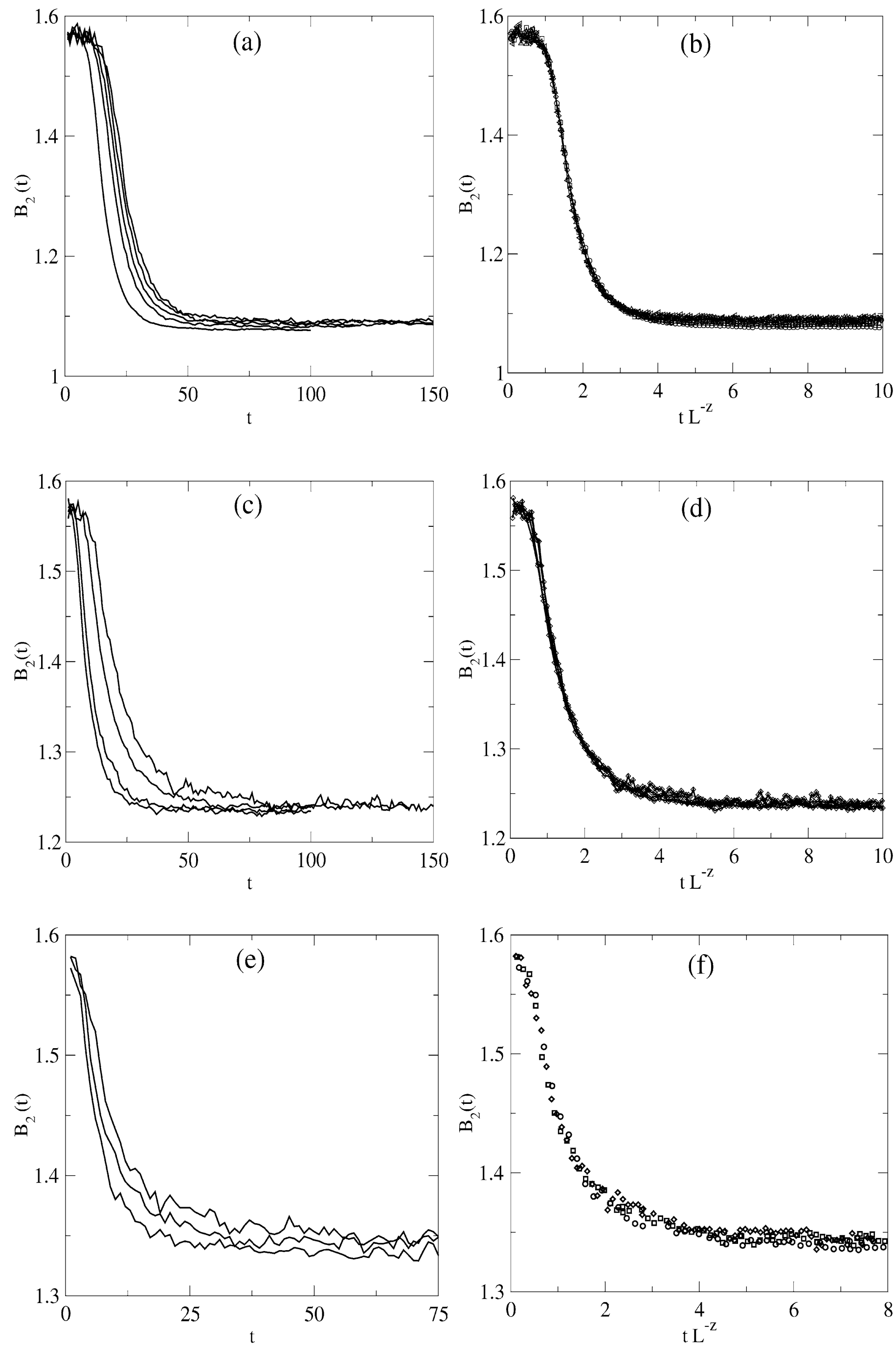

Fig. 2. (a) Binder cumulant data ($B_2(t)$) for the 2-dimensional Ising model for linear lattice sizes $L = 256$, 384, 512, 640 as a function of simulation time t, (b) scaling of $B_2(t)$ data given in (a). (c) Simulation data for $B_2(t)$ as a function of simulation time t for the 3-dimensional Ising model for linear lattice sizes $L = 32$, 48, 64, 80, (d) scaling of $B_2(t)$ data given in (c). (e) Simulation data for $B_2(t)$ as a function of simulation time t for the 4-dimensional Ising model for linear lattice sizes $L = 16$, 20, 24, (f) scaling of $B_2(t)$ data given in (e).

tion of $F(t)$ during the relaxation of the system until a plateau is reached for 2-, 3-, and 4-dimensional Ising models, respectively. It is seen from these figures that time to reach the plateau is proportional to the linear size (L) of the system. Fig. 3 (b), (d) and (f) shows scaling of $F(t)$.

In all these figures scaling is very good for functions $S(t)$, $B_2(t)$ and $F(t)$. The errors in the values of z are obtained from the largest fluctuations in the simulation data for $B_2(t)$ and $F(t)$. The values of the dynamic critical exponent z obtained for 2-, 3-, and 4-dimensional Ising models are given in Table 1.

Coddington and Baillie introduced a conjecture for the dynamic critical exponent for both Swendsen–Wang and Wolff cluster algorithms. The conjecture suggests that for both Swendsen–Wang and Wolff cluster algorithms, dynamic evolution is governed by the dynamic critical exponents, which are

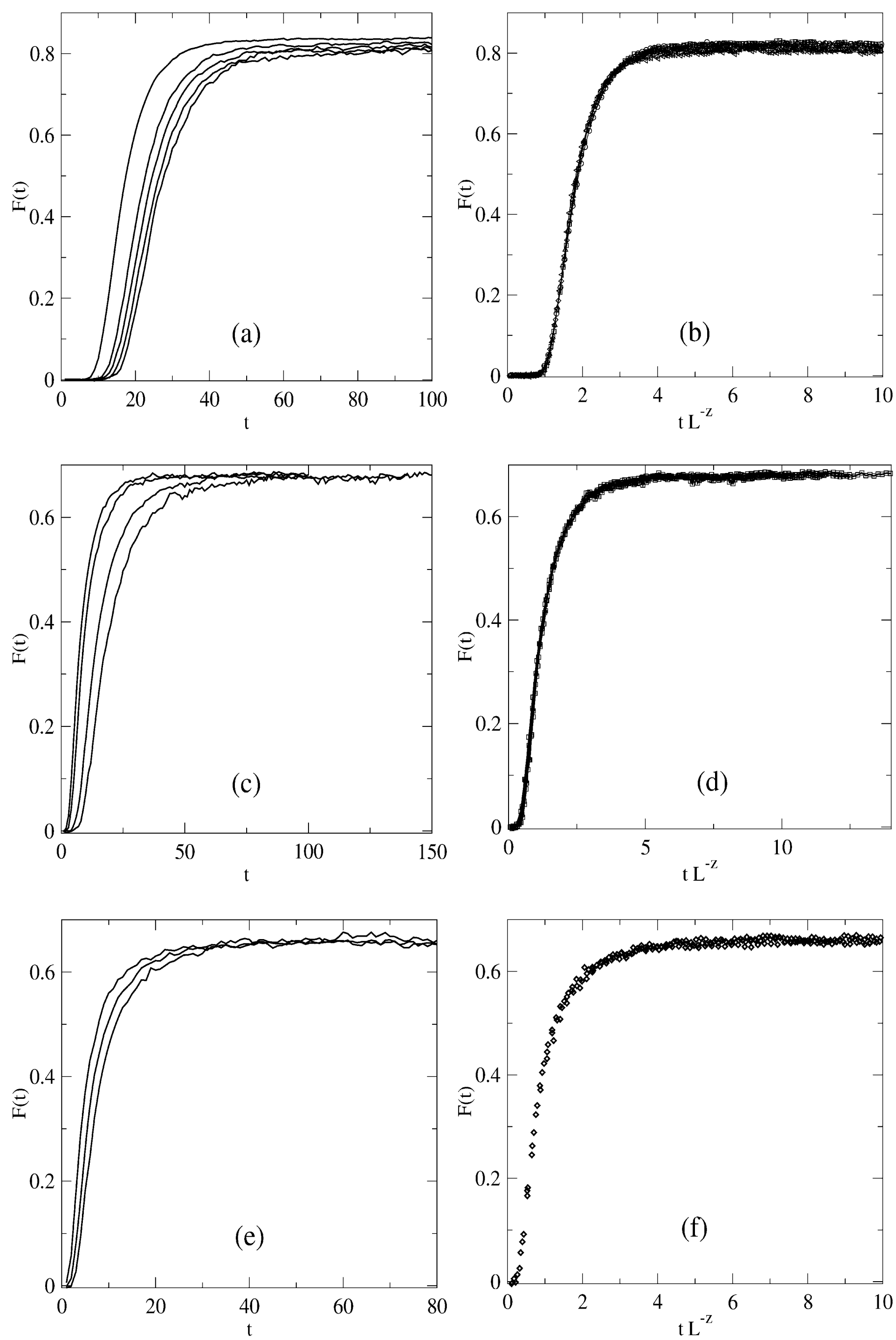

Fig. 3. (a) Simulation data for the renormalization function ($F(t)$) as a function of simulation time t for the 2-dimensional Ising model for linear lattice sizes $L = 256$, 384, 512, 640, (b) scaling of $F(t)$ data given in (a). (c) Simulation data for $F(t)$ as a function of simulation time t for the 3-dimensional Ising model for linear lattice sizes $L = 32$, 48, 64, 80, (d) scaling of $F(t)$ data given in (c). (e) Simulation data for $F(t)$ as a function of simulation time t for the 4-dimensional Ising model for linear lattice sizes $L = 16$, 20, 24, (f) scaling of $F(t)$ data given in (e).

Table 1
The values of calculated dynamic critical exponents (z) for 2-, 3-, and 4-dimensional Ising models

d	$z(S)$	$z(B_2)$	$z(F)$	β/ν
2	0.36 ± 0.05	0.40 ± 0.05	0.40 ± 0.05	0.25
3	0.60 ± 0.05	0.60 ± 0.09	0.60 ± 0.09	0.5185
4	0.74 ± 0.05	0.75 ± 0.19	0.70 ± 0.19	1.0

First three columns are the values obtained from scaling functions $S(t)$, $B_2(t)$ and $F(t)$, respectively, and the fourth column includes the literature values.

related to thermodynamic critical exponents of the d-dimensional Ising model. The difference between these two cluster algorithms comes from the updating procedure. In the Wolff cluster algorithm at each step one cluster is picked among the existing clusters which indicates that a Wolff cluster update is related to the average cluster size. Average cluster size is a quantity which is related to specific heat. Hence Baillie and Coddington suggested that the dynamic critical exponent of the Wolff algorithm is related to α/ν. In case of the Swendsen–

Wang algorithm all clusters are defined and flipped according to a given probability. In this sense dynamic behavior is related to average magnetization. Hence the dynamic exponent is related to β/ν. This conjecture suggests that the Wolff algorithm for the Ising model in all three dimensions must have vanishing critical dynamic exponent while for the Swendsen–Wang algorithm dynamic critical exponent has a value of 0.25, 0.5185 and 1.0 for 2-, 3-, and 4-dimensional Ising models. Literature values are in good agreement with the conjectured behavior of both Wolff and Swendsen–Wang cluster algorithms [2–9,22–24].

4. Conclusion

In this work we have considered the dynamic scaling behavior of moments of magnetization, ($\langle S^n \rangle$), Binder's cumulant ($B_2(t)$) and the renormalization function ($F(t)$) for 2-, 3-, and 4-dimensional Ising models using the Swendsen–Wang algorithm. The values of dynamic critical exponent (z) obtained using this algorithm vary between 0.25 and 1.00, depending on the dimension, as shown in Table 1. One can see from the results of dynamic scaling that scaling is very good, the errors are very small, and these values are in good agreement with the literature values [2–9,22–24]. In our previous work [9], we calculated the dynamic critical exponent of the Wolff Algorithm using the same method, and we have observed that our results are consistent with vanishing dynamic critical exponent. Considering the finite size effects and time consuming iterations necessary for the calculations using autocorrelation times, one can say that it is more advantageous to use this method and it serves as a powerful method to study the dynamic critical behavior of spin models.

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