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Attractors for the Generalized Benjamin–Bona–Mahony Equation

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We consider the periodic initial-boundary value problem for a multidimensional generalized Benjamin–Bona–Mahony equation. We show the existence of the global attractor with a finite fractal dimension and the existence of the exponential attractor for the corresponding semigroup. © 1999 Academic Press

Key Words: attractor; exponential attractor; fractal dimension.

1. INTRODUCTION

We consider the equation

$$u_t - a\,\Delta u_t - b\,\Delta u + \nabla \cdot \mathbf{F}(u) = h(x), \qquad x \in \mathbb{R}^n, \quad t \in \mathbb{R}^+ \tag{1}$$

with the initial condition

$$u(x, 0) = u_0(x), \qquad x \in \mathbb{R}^n \tag{2}$$

and the periodic boundary condition

$$u(x + L_i e_i, t) = u(x, t), \qquad x \in \mathbb{R}^n, \quad t > 0, \quad L_i > 0, \quad i = 1, 2, ..., n, \tag{3}$$

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where a and b are positive constants; $e_1, ..., e_n$ is the canonical basis of $\mathbb{R}^n$, $u_0(x)$ and $h(x)$ are given functions, $\nabla \cdot \mathbf{F} = \sum_{i=1}^{n} (\partial/\partial x_i) F_i$, and $\mathbf{F}(s) = (F_1(s), F_2(s), ..., F_n(s))$ is a given vector field satisfying the following properties:

(i) $F_k(0) = 0$, $k = 1, 2, ..., n$;

(ii) the functions F_k, $k = 1, 2, ..., n$ are twice continuously differentiable in R^1;

(iii) the functions $f_k(s) = (d/ds) F_k(s)$, $k = 1, 2, ..., n$, satisfy the growth conditions

$$|f_k(s)| \leqslant C(1 + |s|^m), \qquad k = 1, 2, ..., n,$$

where $0 \leqslant m < \infty$ if $n = 2$, $0 \leqslant m < 2$ if $n = 3$ and $m = 0$ if $n \geqslant 4$. No growth condition is required if $n = 1$.

Using the standard Faedo–Galerkin method, it is not difficult to prove that if $h \in \dot{L}_2(\Omega)$ and $u_0 \in \dot{H}^1_{per}(\Omega)$, then the problem (1)–(3) has a unique solution $u \in C(\mathbb{R}^+; \dot{H}^1_{per}(\Omega))$ in the sense of distributions, where $\Omega = \prod_{i=1}^{n} (0, L_i)$, $\dot{L}_2(\Omega)$ is the space of functions $v \in L_2(\Omega)$ such that $\int_\Omega v\, dx = 0$, and the space $\dot{H}^s_{per}(\Omega)$, $s \in \mathbb{R}^+$ is the space of functions $u \in L_2(\Omega)$ satisfying

$$\sum_{k \in \mathbb{Z}^n} (1 + |k|^2)^s\, |u_k|^2 < \infty, \qquad \int_\Omega u(x)\, dx = 0,$$

where u_k are the Fourier coefficients of u with respect to the system $\{\exp(2i\pi \sum_{j=1}^{N} k_j(x_j/L_j)),\ k = (k_1, k_2, ..., k_n) \in \mathbb{Z}^n\}$, $\dot{H}^{-s}_{per}(\Omega)$ is the dual of $\dot{H}^s_{per}(\Omega)$. So the problem (1)–(3) generates a semigroup $V_t : X^1 \to X^1$, $t \in \mathbb{R}^+$ where $X^1 := \dot{H}^1_{per}(\Omega)$. In this article we prove that the semigroup V_t has a global attractor, that is, a minimal closed set $\mathcal{M} \subset X^1$ which attracts each bounded subset of X^1. It will be shown that this attractor has a finite fractal dimension.

The Cauchy problem for the Benjamin–Bona–Mahony equation

$$u_t - u_{xxt} - \nu u_{xx} + u_x + u u_x = 0 \tag{4}$$

and some of its generalizations has been investigated by several authors, such as Amick *et al.* [2], Bona and Dougalis [6], and Karch [11]. In these articles the problem of global unique solvability and long time behaviour of solutions are studied. Kalantarov [10] has proved the existence of a global attractor for the semigroup generated by the initial-boundary value problem for the Kelvin–Voigt equations

$$\begin{aligned} \mathbf{v}_t - \alpha\, \Delta \mathbf{v}_t - \nu\, \Delta \mathbf{v} + \text{grad}\, p + v_k \mathbf{v}_{x_k} &= \mathbf{h}(x), \\ div\, \mathbf{v} &= 0. \end{aligned} \tag{5}$$

On the other hand Wang [16–18] using the technique of Ghidaglia [8] has proved the existence of a global attractor for the semigroup generated by (1)–(3) in one dimensional case, that is, the periodic initial-boundary value problem for the equation

$$u_t - u_{xxt} - \nu u_{xx} + f(u)\, u_x = g(x), \qquad x \in \mathbb{R}, \quad t \in \mathbb{R}^+ \tag{6}$$

In our studies, we have used the ideas of Hale [9] and Ladyzhenskaya [13] on attractors for asymptotically compact semigroups. In the sequel we will use the following theorems.

THEOREM 1 [9, 13]. *If a semigroup V_t, $t \in \mathbb{R}^+$ acts on a Banach space X, and V_t can be represented as a sum $W_t + Z_t$ in which W_t, $t \in R^+$, is a family of operators, such that*

$$\| W_t(B) \|_X \leqslant m_1(t)\, m_2(\| B \|_X), \tag{7}$$

where $m_1(\cdot)$ and $m_2(\cdot)$ are continuous functions on $\mathbb{R}^+$ and $m_1(t) \to 0$, as $t \to \infty$, $\|B\|_X = \sup_{v \in B} \|v\|_X$, while Z_t, $t \in \mathbb{R}^+$ maps bounded sets into precompact sets, then $V_t : t \in \mathbb{R}^+$ is asymptotically compact semigroup.

THEOREM 2 [9, 13]. *Let $V_t : X \to X$, $t \in \mathbb{R}^+$, be a continuous bounded point-dissipative asymptotically compact semigroup. Then for this semigroup there exists a non-empty global attractor $\mathscr{M}$. It is compact, invariant, and connected.*

THEOREM 3 [12]. *Let B be a bounded set in a Hilbert space X, and let there be defined a map $V : B \to X$ such that $B \subseteq V(B)$ and for all $v, \tilde{v} \in B$*

$$\| V(v) - V(\tilde{v}) \|_X \leqslant \ell \; \| v - \tilde{v} \|_X, \tag{8}$$

and

$$\| Q_N V(v) - Q_N V(\tilde{v}) \|_X \leqslant \delta \; \| v - \tilde{v} \|_X, \qquad \delta < 1, \tag{9}$$

where Q_N is the orthogonal projection of X onto the subspace $X_N^{\perp}$ of codimension N. Then for the fractal dimension of B the inequality

$$d_F(B) \leqslant N \log \left(\frac{8\kappa^2 \ell^2}{1-\delta^2} \right) \Big/ \log \frac{2}{1-\delta^2} \tag{10}$$

is true, where κ is the Gauss constant.

2. EXISTENCE OF THE GLOBAL ATTRACTOR

First let us show that the semigroup V_t is bounded dissipative in a phase space X^1; that is, it has an absorbing ball in X^1. Multiplying Eq. (1) by u in $L_2(\Omega)$ we get

$$\frac{1}{2}\frac{d}{dt}\left[\|u(\cdot,t)\|^2+a\,\|\nabla u(\cdot,t)\|^2\right]+b\,\|\nabla u(\cdot,t)\|^2=(h,u). \tag{11}$$

We will use the notations $\|\cdot\|$, $(\cdot,\cdot)$ for the norm and inner product in $L_2(\Omega)$, respectively. Using the *Poincaré–Friedrichs* inequality

$$\|u\|\leqslant\lambda_1^{-1/2}\,\|\nabla u\|, \tag{12}$$

which is valid for each $x\in X^1$, we can easily get

$$|(h,u)|\leqslant\frac{b}{2}\,\|\nabla u\|^2+\frac{\lambda_1^{-1}}{2b}\,\|h\|^2, \tag{13}$$

where λ_1 is the lowest eigenvalue of the periodic boundary value problem

$$\begin{aligned}&-\Delta\psi(x)=\lambda\psi(x),\\ &\psi(x+L_ie_i)=\psi(x),\qquad i=1,\dots,n,\\ &\int_\Omega\psi(x)\,dx=0.\end{aligned} \tag{E}$$

Due to (12) we have

$$\frac{b}{2}\,\|\nabla u(\cdot,t)\|^2+\frac{b\lambda_1}{2}\,\|u(\cdot,t)\|^2\leqslant b\,\|\nabla u(\cdot,t)\|^2. \tag{14}$$

By using (13), (14) we get from (11)

$$\frac{d}{dt}\left[\|u(\cdot,t)\|^2+a\,\|\nabla u(\cdot,t)\|^2\right]+\frac{b}{2}\,\|\nabla u(\cdot,t)\|^2+\frac{b\lambda_1}{2}\,\|u(\cdot,t)\|^2\leqslant\frac{1}{b\lambda_1}\,\|h\|^2$$

or

$$\frac{d}{dt}\left[\|u(\cdot,t)\|^2+a\,\|\nabla u(\cdot,t)\|^2\right]+K_0\left[\|u(\cdot,t)\|^2+a\,\|\nabla u(\cdot,t)\|^2\right]\leqslant\frac{1}{b\lambda_1}\,\|h\|^2, \tag{15}$$

where $K_0 = \min\{b\lambda_1/2, b/2a\}$. Integrating (15) we find

$$\|\nabla u(\cdot, t)\|^2 \leqslant \frac{1}{a} [\|u_0\|^2 + a \|\nabla u_0\|^2] e^{-K_0 t} + \frac{1}{bK_0\lambda_1} \|h\|^2.$$

From this inequality it follows that

$$B_0 := \left\{ u \in X^1 : \|u(\cdot, t)\|_{X^1} \leqslant \left(\frac{2}{\lambda_1 b K_0} \right)^{1/2} \|h\| \right\}$$

is an absorbing ball for the semigroup V_t in X^1.

Now, we will prove that the semigroup V_t is asymptotically compact, that is, for each sequence $t_k \to \infty$ and each bounded sequence $\{v_k\} \subset X^1$, the set $\{V_{t_k}(v_k)\}$ is precompact. To do this we will use Theorem 1. It is clear that the solution $u(x, t)$ of the problem (1)–(3) can be represented in the form

$$u(x, t) = w(x, t) + z(x, t),$$

where $w(x, t)$ is a solution of the problem

$$w_t - a\,\Delta w_t - b\,\Delta w = 0, \qquad x \in \mathbb{R}^n, \qquad t \in \mathbb{R}^+, \tag{16}$$

$$w(x, 0) = u_0(x), \qquad x \in \mathbb{R}^n, \tag{17}$$

$$w(x, t) = w(x + L_i e_i, t), \qquad i = 1, ..., n, \qquad t \in \mathbb{R}^+ \tag{18}$$

while $z(x, t)$ is a solution of the problem

$$z_t - a\,\Delta z_t - b\,\Delta z + \nabla \cdot \mathbf{F}(w + z) = h(x), \qquad x \in \mathbb{R}^n, \qquad t \in R^+ \tag{19}$$

$$z(x, 0) = 0, \qquad x \in \Omega \tag{20}$$

$$z(x, t) = z(x + L_i e_i, t), \qquad x \in \mathbb{R}^n\ i = 1, ..., n, \qquad t \in \mathbb{R}^+. \tag{21}$$

Thus, the semigroup V_t has the representation

$$V_t = W_t + Z_t, \tag{22}$$

where W_t is the semigroup generated by (16)–(18) and Z_t is a solution operator of the problem (19)–(21). Multiplying Eq. (16) by w in $L_2(\Omega)$, after some elementary operations we can easily get

$$\frac{d}{dt} [\|w(\cdot, t)\|^2 + a \|\nabla w(\cdot, t)\|^2] + k_1 [\|w(\cdot, t)\|^2 + a \|\nabla w(\cdot, t)\|^2] \leqslant 0. \tag{23}$$

Integrating (23) and then using Poincaré–Friedrichs inequality we obtain

$$\|\nabla w(\cdot, t)\|^2 \leqslant e^{-k_1 t}\left(\frac{1}{\lambda_1 a}+1\right)\|\nabla w(\cdot, 0)\|^2.$$

That is, the semigroup $W_t : X^1 \to X^1$ satisfies the condition (7) of Theorem 1 with $m_1(t)=e^{-k_1 t}(d/(\lambda_1 a)+1)$ and $m_2(t)=t$.

It remains now to show that $Z_t : X^1 \to X^1$ is precompact for each $t>0$, when $n=3$; the cases $n=1, 2$ and $n>3$ can be dealt with in a similar way. In order to see this property, let us rewrite Eq. (19) in the form

$$\begin{aligned} z_t - a\,\Delta z_t - b\,\Delta z &= h(x) - \sum_{i=1}^{n} f_i(u)\, u_{x_i} \\ &= g(x, t). \end{aligned} \tag{24}$$

Let $p=6/(m+3)$; using the Hölder's inequality and the condition (iii) we can easily get the estimate

$$\begin{aligned} \int_\Omega |f_i(u)\, u_{x_i}|^p\, dx &\leqslant \int_\Omega (C_1 + C_2\,|u|^{mp})\, |u_{x_i}|^p\, dx \\ &\leqslant C_3\left(1+\int_\Omega |u_{x_i}|^2\, dx\right) \\ &\quad + C_2\left(\int_\Omega |u_{x_i}|^2\, dx\right)^{p/2}\left(\int_\Omega |u|^{mp(2/(2-p))}\, dx\right)^{(2-p)/2}. \end{aligned}$$

Since $mp(2/(2-p))=6$, by using the well-known inequality [14, p. 45]

$$\|u(\cdot, t)\|_{L^6(\Omega)} \leqslant c\,\|\nabla u(\cdot, t)\|_{L_2(\Omega)}, \tag{25}$$

which is valid for each $u \in \dot{H}^1_{per}(\Omega)$, $\Omega \subset \mathbb{R}^3$ we obtain

$$\int_\Omega |f_1(u)\, u_{x_i}|^p\, dx \leqslant C_3\left(1+\int_\Omega |\nabla u|^2\, dx\right) + C_4\left(\int_\Omega |\nabla u|^2\, dx\right)^{3(2-p)/2}.$$

Since V_t: $X^1 \to X^1$ is bounded dissipative

$$\max_{t\in\mathbb{R}^+} \|\nabla u(\cdot, t)\|_{L^2(\Omega)} \leqslant C_5$$

and $h \in L_2(\Omega)$, we get $g \in C(\mathbb{R}^+; L_p(\Omega))$. By the embedding theorem (see Triebel [15, p. 327]) $L_p(\Omega) \subset \dot{H}^{-1+\sigma}_{per}(\Omega)$, $\sigma = 1 - (m/2)$, we have

$$g \equiv h + \sum_{i=1}^{n} f_i(u)\, u_{x_i} \in L_2(0, T; \dot{H}^{1+\sigma}_{per}(\Omega)), \qquad \forall T > 0$$

and the precompactness of the operator $W_t : X^1 \to X^1$ follows from

PROPOSITION 4. *If $g \in L^2(0, T; \dot{H}^s_{per}(\Omega))$ and $v_0 \in \dot{H}^{s+2}_{per}(\Omega)$, then the initial value problem*

$$\begin{aligned} v_t - a\,\Delta v_t - b\,\Delta v &= g(x, t), && x \in \mathbb{R}^n, \quad t \in (0, T) \\ v(x, 0) &= v_0(x), && x \in \mathbb{R}^n, \\ v(x, t) &= v(x + L_i e_i, t), && i = 1, ..., n, \quad t \in (0, T), \end{aligned}$$

has a solution $v(x, t)$ in $C(0, T; \dot{H}^{s+2}_{per}(\Omega))$ for $s \in \mathbb{R}$.

This proposition can be proved by using the standard Fourier method. Following the technique used in Babin and Vishik [4, Theorem 6.2] it can be proved that $\mathscr{M}$ is bounded in $X^2 = H^2_{per}(\Omega) \cap \dot{H}^1_{per}(\Omega)$. So we have obtained

THEOREM 5. *Suppose that the vector field* $\mathbf{F}$ *satisfies the conditions* (i)–(iii) *and $h \in \dot{L}_2(\Omega)$. Then the semigroup $V_t : X^1 \to X^1$ has a global attractor $\mathscr{M}$ which is compact, invariant and connected in X^1. $\mathscr{M}$ is included and bounded in X^2.*

3. ESTIMATE OF THE FRACTAL DIMENSIONS OF THE ATTRACTOR

Now we are going to show that for some $t_1 > 0$, the operator $V = V_{t_1}$ satisfies the conditions of Theorem 3, from which we get the estimate of the dimension of the global attractor. Let u and v be two solutions of the problem (1)–(3) with $u(x, 0) = u_0(x)$ and $v(x, 0) = v_0(x)$ in $\mathscr{M}$. Then from the Theorem 5, it follows that $u(\cdot, t)$, $v(\cdot, t) \in \mathscr{M}$, $\forall t \in \mathbb{R}^+$. Let us define $w = u - v$; then w will satisfy the equation

$$w_t - a\,\Delta w_t - b\,\Delta w + \nabla \cdot (\mathbf{F}(u) - \mathbf{F}(v)) = 0. \tag{26}$$

Taking the inner product with $w(x, t)$ in $L_2(\Omega)$, we obtain

$$\frac{1}{2}\frac{d}{dt}\left[\,\|w(\cdot, t)\|^2 + a\,\|\nabla w(\cdot, t\|^2\right] + b\,\|\nabla w(\cdot, t)\|^2 + (\nabla \cdot (\mathbf{F}(u) - \mathbf{F}(v)), w) = 0.$$

Now let us consider the last term,

$$|(\nabla\cdot(\mathbf{F}(u)-\mathbf{F}(v)),w)|=\left|\sum_{i=1}^{n}(F_i(u)-F_i(v),\nabla w)\right|$$
$$=\left|\sum_{i=1}^{n}\left(\int_0^1\frac{d}{d\theta}F_i(\theta u+(1-\theta)v)\,d\theta,\nabla w\right)\right|$$
$$\leqslant\sum_{i=1}^{n}\int_\Omega\left|\int_0^1 f_i(\theta u+(1-\theta)v)\,d\theta\right|\,|w|\,|\nabla w|\,dx.$$

Since

$$|f_i(\theta u+(1-\theta)v|\leqslant C_6(1+|u|^m+|v|^m),\qquad i=1,2,...,n,$$

using the Hölder's inequality and (25) we get

$$|(\nabla\cdot(\mathbf{F}(u)-\mathbf{F}(v)),w)|\leqslant C_6\sum_{i=1}^{n}\int_\Omega(1+|u|^m+|v|^m)\,|w|\,|\nabla w|\,dx$$
$$\leqslant C_7\,\|w\|\,\|\nabla w\|$$

and utilizing Young's inequality

$$|(\nabla\cdot(\mathbf{F}(u)-\mathbf{F}(v)),w)|\leqslant C_7\left[a\,\|\nabla w\|^2+\frac{1}{4a}\,\|w\|^2\right]$$
$$\leqslant\mu[\,\|w\|^2+a\,\|\nabla w\|^2],$$

where $\mu=C_7\max\{1,1/4a\}$. So we obtain

$$\frac{d}{dt}[\,\|w(\cdot,t)\|^2+a\,\|\nabla w(\cdot,t)\|^2]\leqslant\mu[\,\|w(\cdot,t)\|^2+a\,\|\nabla w(\cdot,t)\|^2].$$

Thus

$$\|w(\cdot,t)\|^2+a\,\|\nabla w(\cdot,t)\|^2\leqslant[\,\|w(\cdot,0)\|^2+a\,\|\nabla w(\cdot,0)\|^2]\,e^{\mu t}$$

and

$$\|\nabla w(\cdot,t)\|\leqslant(a+\lambda_1^{-1})^{1/2}\,\|\nabla w(\cdot,0)\|\,e^{\mu t/2}.\tag{27}$$

Now, let P_N denote the orthogonal projection to the subspace X^1_N of X^1 spanned by the first N basis elements of X^1, that is, the first N

eigenfunctions of the problem (E). Multiplying Eq. (26) in $L_2(\Omega)$ by $Q_N w := (I-P_N)w$, we obtain

$$\begin{aligned}
&(w_t(\cdot,t), Q_N w(\cdot,t)) - a(\Delta w_t(\cdot,t), Q_N w(\cdot,t)) + b\,\|\nabla Q_N w(\cdot,t)\|^2\\
&\quad = (\nabla\cdot\mathbf{F}(u) - \nabla\cdot\mathbf{F}(v), Q_N w)\\
&\quad = \left(\sum_{i=1}^n f_i(u)\,u_{x_i} - f_i(v)\,v_{x_i}, Q_N w\right)\\
&\quad = \left(\sum_{i=1}^n [f_i(u)\,w_{x_i} + (f_i(u)-f_i(v))\,v_{x_i}], Q_N w\right)\\
&\quad = \left(\sum_{i=1}^n f_i(u)\,w_{x_i}, Q_N w\right)\\
&\qquad + \left(\sum_{i=1}^n \int_0^1 f_i'(\theta u + (1-\theta)v)\,d\theta w v_{x_i}, Q_N w\right).
\end{aligned}\tag{28}$$

Since the attractor $\mathscr{M}$ is bounded in $H^2_{per}(\Omega)$ we have

$$\max_{x\in\Omega}|u|,\qquad \max_{x\in\Omega}|v|,\qquad \|u\|_{H^2(\Omega)},\qquad \|v\|_{H^2(\Omega)} \leqslant M_0. \tag{29}$$

Using the condition (iii), the Hölder inequality (29), (25) we can estimate the right hand side of (28) as

$$\begin{aligned}
&\left|\left(\sum_{i=1}^n f_i(u)\,w_{x_i}, Q_N w\right) + \left(\sum_{i=1}^n \int_0^1 f_i'(\theta u + (1-\theta)v)\,d\theta w v_{x_i}, Q_N w\right)\right|\\
&\leqslant C_8 \int_\Omega |\nabla w(x,t)|\,|Q_N w(x,t)|\,dx\\
&\quad + C_9 \int_\Omega |w(x,t)|\,|\nabla v(x,t)|\,|Q_N w(x,t)|\,dx\\
&\leqslant C_8\,\|\nabla w(\cdot,t)\|\,\|Q_N w(\cdot,t)\|\\
&\quad + C_9 \left(\int_\Omega |w(x,t)|^6\,dx\right)^{1/6} \left(\int_\Omega |\nabla v(x,t)|^3\,dx\right)^{1/3} \left(\int_\Omega |Q_N w(x,t)|^2\,dx\right)^{1/2}\\
&\leqslant C_8\,\|\nabla w(\cdot,t)\|\,\|Q_N w(\cdot,t)\|\\
&\quad + C_{10}\,\|\nabla w(\cdot,t)\|\,\|v(\cdot,t)\|_{H^2(\Omega)}\,\|Q_N w(\cdot,t)\|\\
&\leqslant C_{11}\,\|\nabla w(\cdot,t)\|\,\|Q_N w(\cdot,t)\|.
\end{aligned}$$

So (28) implies

$$\frac{1}{2}\frac{d}{dt}[\|Q_N w(\cdot,t)\|^2 + a\,\|\nabla Q_N w(\cdot,t)\|^2] + b\,\|\nabla Q_N w(\cdot,t)\|^2 \leqslant C_{11}\,\|\nabla w(\cdot,t)\|\,\|Q_N w(\cdot,t)\|. \tag{30}$$

By using the inequality

$$\|Q_N\psi\| \leqslant \lambda_{N+1}^{-1/2}\,\|\nabla Q_N \psi\|, \qquad \forall \psi \in (X_N^1)^{\perp},$$

where λ_N is the Nth eigenvalue of the problem (E), we can rewrite (30) as

$$\begin{aligned}\frac{d}{dt}[\|Q_N w(\cdot,t)\|^2 &+ a\,\|\nabla Q_N w(\cdot,t)\|^2] \\ &+ b\,\|\nabla Q_N w(\cdot,t)\|^2 + \lambda_1 b\,\|Q_N w(\cdot,t)\|^2 \\ &\leqslant 2C_{11}\,\|\nabla w(\cdot,t)\|\,\|\nabla Q_N w(\cdot,t)\|\,\lambda_{N+1}^{-1/2} \\ &\leqslant C_{11}\lambda_{N+1}^{-1/2}\,\|\nabla w(\cdot,t)\|^2 + C_{11}\lambda_{N+1}^{-1/2}\,\|\nabla Q_N w(\cdot,t)\|^2\end{aligned} \tag{31}$$

or

$$\begin{aligned}\frac{d}{dt}[\|Q_N w(\cdot,t)\|^2 + a\,\|\nabla Q_N w(\cdot,t)\|^2] + (b - C_{11}\lambda_{N+1}^{-1/2})\,\|\nabla Q_N w(\cdot,t)\|^2 \\ + b\lambda_1\,\|Q_N w(\cdot,t)\|^2 \leqslant C_{11}\lambda_{N+1}^{-1/2}\,\|\nabla w(\cdot,t)\|^2.\end{aligned}$$

Let us choose N large enough, so that $b - C_{11}\lambda_{N+1}^{-1/2} > 0$ and set

$$\mu_1 = \min\left\{\frac{b - C_{11}\lambda_{N+1}^{-1/2}}{a},\ \lambda_1 b\right\}.$$

From the last inequality we get

$$\begin{aligned}\frac{d}{dt}[\|Q_N w(\cdot,t)\|^2 &+ a\,\|\nabla Q_N w(\cdot,t)\|^2] \\ &+ \mu_1[\|Q_N w(\cdot,t)\|^2 + a\,\|\nabla Q_N w(\cdot,t)\|^2] \\ &\leqslant C_{11}\lambda_{N+1}^{-1/2}\,\|\nabla w(\cdot,t)\|^2 \\ &\leqslant (a + \lambda_1^{-1})\,C_{11}\lambda_{N+1}^{-1/2}\,\|\nabla w(\cdot,0)\|^2 \cdot e^{\mu t}\end{aligned}$$

by use of (27). Integrating this inequality, and after some elementary operations we obtain

$$\|Q_N w(\cdot,t)\|_{X^1}^2 \leqslant a^{-1}(a + \lambda_1^{-1})[C_{11}\lambda_{N+1}^{-1/2}(\mu + \mu_1)^{-1}\,e^{\mu t} + e^{-\mu_1 t}]\,\|\nabla w(\cdot,0)\|^2.$$

Now we can choose N and $t_0 > 0$ so that

$$a^{-1}(a+\lambda_1^{-1})[C_{11}\lambda_{N+1}^{-1/2}(\mu+\mu_1)^{-1} e^{\mu t_0}+e^{-\mu_1 t_0}] \leqslant \delta < 1.$$

Hence the conditions of the Theorem 3 are satisfied with $V = V_{t_0}$ and we obtain the estimate

$$d_F(\mathscr{M}) \leqslant N \frac{\log(8\kappa\ell^2/(1-\delta^2))}{\log(2/(1+\delta^2))}$$

for the fractal dimension of the global attractor.

So we have established the following theorem:

THEOREM 6. *Let all conditions of the Theorem 5 be satisfied. Then the attractor of the semigroup $V_t : X^1 \to X^1$ has a finite fractal dimension*

4. A REMARK ON THE EXISTENCE OF THE EXPONENTIAL ATTRACTOR

Consider now the one-dimensional version of the problem (1)–(3),

$$u_t - au_{xxt} - bu_{xx} + f(u)\, u_x = h(x), \qquad x \in \mathbb{R}, \quad t \in R^+, \tag{32}$$

$$u(x, 0) = u_0(x), \qquad x \in \mathbb{R}, \tag{33}$$

$$u(x, t) = u(x+L, t), \qquad x \in \mathbb{R}, \quad t \in R^+. \tag{34}$$

It follows from the Theorem 5, that the problem (32)–(34) has an absorbing ball $B_0 \subset X^1$ and a global attractor $\mathscr{M}$, which is compact.

Now, assume that u_0, v_0 are arbitrary two elements of B_0, then for $w(\cdot, t) = V_t(u_0) - V_t(v_0) = u(\cdot, t) - v(\cdot, t)$ the inequality (27) is valid:

$$\|w_x(\cdot, t)\|_{X^1} \leqslant (a+\lambda_1^{-1})^{1/2}\, \|\nabla w(\cdot, 0)\|\, e^{\mu t/2}. \tag{35}$$

It follows from (32) that w satisfies the equation

$$w_t - aw_{xxt} - bw_{xx} + \int_0^1 f'(\theta u + (1-\theta) v)\, d\theta \cdot w \cdot u_x + f(v)\, w_x = 0. \tag{36}$$

Let us multiply (36) by $Q_N w$ in $L_2(0, L)$,

$$\frac{1}{2}\frac{d}{dt}\|Q_N w\|^2 1 + \frac{a}{2}\frac{d}{dt}\|Q_N w_x\|^2 + b\,\|Q_N w_x\|^2$$
$$+ \int_0^L \int_0^1 f'(\theta u + (1-\theta) v)\, d\theta \cdot w \cdot u_x Q_N w\, dx + \int_0^L f(v)\, w_x Q_N w\, dx = 0. \tag{37}$$

Due to the Sobolev inequality

$$\max_{x\in[0,L]} |z(x)| \leqslant d_0 \|z'\|, \qquad \forall z \in \dot{H}^1_{per}(0, L)$$

we get from the relation (37)

$$\frac{1}{2}\frac{d}{dt}[\|Q_N w(\cdot, t)\|^2 + a\|Q_N w_x(\cdot, t)\|^2] + b\|Q_N w_x(\cdot, t)\|^2$$

$$\leqslant C_{12} \max_{x\in[0,L]} |w(x,t)| \, \|u_x\| \, \|Q_N w\| + C_{13}\|w_x\| \, \|Q_N w\|$$

$$\leqslant C_{14}\|w_x(\cdot, t)\| \, \|Q_N w(\cdot, t)\|$$

$$\leqslant \frac{1}{2} C_{14}\lambda_{N+1}^{-1/2}\|w_x(\cdot, t)\|^2 + \frac{1}{2} C_{14}\lambda_{N+1}^{-1/2}\|Q_N w_x(\cdot, t)\|^2.$$

So we have got the inequality similar to (31). Therefore the following inequality holds:

$$\|Q_N w(\cdot, t)\| \leqslant a^{-1}(a+\lambda_1^{-1})[C_{14}\lambda_{N+1}^{-1/2}(\mu+\mu_1)^{-1} e^{\mu t} + e^{-\mu_1 t}] \, \|\nabla w(\cdot, 0)\|^2.$$

It follows from the last estimate that the semigroup $V_t : X^1 \to X^1_1$, $t \in \mathbb{R}^+$ satisfies the discrete squeezing property (see [7]), that is, there exists N_0 and t_1 such that the operator $T := V_{t_1}$ satisfies the conditions

$$\|Tx - Ty\|_{X^1} \leqslant \ell_0 \|x - y\|_{X^1}, \qquad \forall x, y \in B_0$$

and for some $\delta \in (0, 1/\sqrt{2})$

$$\|(I - P_{N_0})(Tx - Ty)\|_{X^1} \leqslant \delta \|x - y\|_{X^1}, \qquad \forall x, y \in B_0.$$

Therefore the semigroup $V_t : X^1 \to X^1$, $t \in \mathbb{R}^+$ has an exponential attractor $\mathcal{M}_e$, (see [3, 7]), that is a compact set $\mathcal{M}_e$ such that

(i) $\mathcal{M} \subseteq \mathcal{M}_e \subseteq B_0$,

(ii) $V_t \mathcal{M}_e \subseteq \mathcal{M}_e$,

(iii) $\mathcal{M}_e$ has finite fractal dimension,

(iv) there exist C_1 and C_2, which does not depend on x such that $\forall x \in B$ and each $t > 0$

$$dist(V_t x, \mathcal{M}_e) \leqslant C_1 \exp\{-C_2 t\}.$$

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