DERIVATIONS AND AUTOMORPHISMS OF CERTAIN SUBRINGS OF MATRIX RINGS

MATRİS HALKALARININ BAZI ALTHALKALARININ TÜREVLERİ VE OTOMORFİZMALARI

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UMUT SAYIN

ABSTRACT

DERIVATIONS AND AUTOMORPHISMS OF CERTAIN SUBRINGS OF MATRIX RINGS

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Let K be an arbitrary associative ring with identity. We denote by $M_n(K)$ the ring of all $n \times n$ matrices over K. Say $K = F$ for some field F. Then it is a consequence of Skolem-Noether theorem that every automorphism of $M_n(F)$ is inner.

Recall that a derivation of a ring K is an additive map $D : K \to K$ satisfying $D(ab) = D(a)b + aD(b)$ for all $a, b \in K$.

The problem of describing all derivations of a ring is an interesting topic for many researchers. Many papers are concerned with the study of derivations of matrix rings and their subrings.

As a result of Skolem-Noether theorem, every derivation of $M_n(F)$ is inner.

In 1982, S.A. Amitsur proved that any derivation of $M_n(K)$ is the sum of an inner derivation and a derivation arising from a derivation on K where K is an arbitrary ring.

Let $NT_n(K)$ be the ring of all niltriangular $n \times n$ matrices over K whose entries are all zeros on and above the main diagonal. V.M. Levchuk characterized the automorphisms of $NT_n(K)$ in 1983.

In 2006, J.H. Chun and J.W. Park proved that every derivation of $NT_n(K)$ is a sum of a certain diagonal, trivial extension and a strongly nilpotent derivation.

The set defined by $R_n(K, J) = NT_n(K) + M_n(J)$ is a ring with usual matrix addition and multiplication where K is a unital ring and J is an ideal of K. The automorphisms of the ring $R_n(K, J)$ were described by F. Kuzucuoğlu and V.M. Levchuk under certain spesific properties.

In the first section of this thesis, we give the historical background of derivations and automorphisms of some certain matrix rings and algebras.

In the second section, we characterize all derivations of $R_n(K, J)$.

Recall that the Jordan multiplication on a ring K is given with $a \circ b = ab + ba$ for any $a, b \in K$. An additive map Ω of K satisfying $\Omega(a \circ b) = \Omega(a) \circ b + a \circ \Omega(b)$ is called a Jordan derivation of K.

Every derivation is a Jordan derivation but there are Jordan derivations which are not derivations.

All Jordan derivations of the ring $NT_n(K)$ are described by F. Kuzucuoğlu in 2011.

For an arbitrary associative and 2-torsion free ring K with identity and an ideal J of K, we describe all Jordan derivations of $R_n(K, J)$ in the third section.

Key words: niltriangular matrix, automorphism, derivation, Jordan derivation.

ÖZET

MATRİS HALKALARININ BAZI ALTHALKALARININ TÜREVLERİ VE OTOMORFİZMALARI

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K birimli ve birleşmeli herhangi bir halka olmak üzere K üzerinde tanımlı $n \times n$ tipindeki tüm matrislerin kümesi $M_n(K)$ ile gösterilir. Bu küme, matrislerin bilinen toplama ve çarpma işlemlerine göre bir halka yapısı oluşturur. F herhangi bir cisim olmak üzere, Skolem-Noether teoreminin bir sonucu olarak, $M_n(F)$ halkasının her otomorfizmasının bir iç otomorfizma olduğu görülür.

D dönüşümü, K halkası üzerinde tanımlı toplamsal bir dönüşüm olmak üzere eğer her $a, b \in K$ için $D(ab) = D(a)b + aD(b)$ koşulunu sağlıyorsa bu dönüşüme $K - nın$ bir türevi denir. Yine Skolem-Noether teoreminin bir sonucu olarak, F bir cisim olmak üzere $M_n(F)$ halkasının her türev dönüşümü bir iç türev dönüşümüdür.

K birimli ve birleşmeli herhangi bir halka olmak üzere tüm nilüçgensel matrislerin halkası $NT_n(K)$ olsun. Bu halkanın otomorfizmaları V.M. Levchuk tarafından 1983 yılında belirlenmiştir.

2006 yılında, J.H. Chun ve J.W. Park, $NT_n(K)$ halkasının her türev dönüşümünün köşegen(diagonal), halka(trivial extension) ve kuvvetli nilpotent(strongly nilpotent) türev dönüşümlerinin toplamı olarak yazılabileceğini göstermişlerdir.

Birimli ve birleşmeli bir halka K ve bu halkanın bir ideali J olsun. $R_n(K, J) = NT_n(K) + M_n(J)$ ile tanımlı küme, bilinen matris toplamı ve çarpımı ile bir halka olur. 2001 yılında F. Kuzucuoğlu ve V.M. Levchuk tarafından, $R_n(K, J)$ halkasının otomorfizmaları bazı özel şartlar altında belirlenmiştir.

Bu tezin ilk bölümünde, bazı matris halkalarının ve cebirlerinin türev dönüşümleri ve otomorfizmaları ile ilgili gelişmeler, tarihleriyle birlikte verilmiştir. Daha sonra ikinci bölümde, birimli ve birleşmeli bir K halkası ile bu halkanın bir ideali J için $R_n(K,J)$ halkasının türev dönüşümleri karakterize edilmiştir.

Herhangi bir halkadan alınan keyfi a ve b elemanları için $a \circ b = ab + ba$ ile tanımlı çarpıma Jordan çarpımı denir. Bu halka üzerinde tanımlı toplamsal bir Δ dönüşümü $\Delta(a \circ b) = \Delta(a) \circ b + a \circ \Delta(b)$ koşulunu sağlıyorsa bu dönüşüme halkanın bir Jordan türev dönüşümü denir. Her türev dönüşümü bir Jordan türev dönüşümü iken bunun tersi her zaman doğru değildir. $NT_n(K)$ halkasının tüm Jordan türev dönüşümleri F. Kuzucuoğlu tarafından 2011 yılında verilmiştir.

Son olarak, keyfi birimli ve birleşmeli K halkası ile bu halkanın herhangi bir J ideali üzerinde tanımlı $R_n(K,J)$ halkasının bütün Jordan türevleri üçüncü bölümde karakterize edilmiştir.

Anahtar Kelimeler: nilüçgensel matris, otomorfizma, türev, Jordan türev.

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CONTENTS

1 INTRODUCTION

Historical Background of Automorphisms and Derivations on Rings and Algebras

A central simple algebra over a field F is a finite dimensional associative $F-algebra$ without nontrivial two-sided ideals and whose center is F. The easiest example of a central simple algebra is the matrix algebra over F. For any natural number n, the $F-algebra$ $M_n(F)$ of $n \times n$ matrices with coefficients in F is a central simple algebra.

An inner automorphism of an algebra $\mathbb{A}$ is defined as $x \longrightarrow a^{-1}xa$ for an invertible element $a \in \mathbb{A}$.

Skolem-Noether Theorem characterizes the automorphisms of simple rings and it is a fundamental result in the theory of central simple algebras.

Theorem 1.1 *[1, Theorem 4.3.1] (Skolem-Noether) Every automorphism of a finite dimensional central simple algebra is inner.*

A consequence of Skolem-Noether theorem is that any automorphism of $M_n(F)$ is inner. An easy proof of this fact is given by Semrl:

Theorem 1.2 *[2, Theorem 1.1] Let F be a field and Φ be a bijective linear map of $M_n(F)$ satisfying $\Phi(PQ) = \Phi(P)\Phi(Q)$ for all $P, Q \in M_n(F)$. Then there is an invertible matrix $H \in M_n(F)$ so that $\Phi(G) = HGH^{-1}$ for every G in $M_n(F)$.*

Proof. Let $\phi : M_n(F) \to M_n(F)$ be a bijection satisfying $\phi(PQ) = \phi(P)\phi(Q)$ for arbitrary matrices $P, Q \in M_n(F)$ and $u, y \in F^n$ be nonzero column vectors. Then there must be an element $z \in F^n$ such that $\phi(uy^t)z \neq 0$ since ϕ is a bijection. Now choose $H : F^n \to F^n$ with $x \to \phi(xy^t)z$. Clearly, H is a linear map as ϕ is linear. Besides, H is nonzero since Hu is nonzero. It can be seen that $HP = \phi(P)H$ by

$$\begin{aligned} HPx &= \phi(Pxy^t)z \\ &= \phi(P)\phi(xy^t)z \\ &= \phi(P)Hx \end{aligned}$$

for arbitrary $x \in F^n$ and $P \in M_n(F)$. As $Hu \neq 0$ and ϕ is surjective, we can find $Q \in M_n(F)$ for any $w \in F^n$ such that $\phi(Q)Hu = w = HQu$. Hence H is onto and hence invertible. This completes the proof. ■

In 1987, Isaacs proved the theorem given below:

Theorem 1.3 *[3, Corollary 15] Let F be a unique factorization domain(UFD). Then every automorphism of $M_n(F)$ is inner.*

Automorphisms of certain subalgebras of matrix algebras have been actively studied since 1950s.

Jondrup showed that if A is a simple algebra, finite dimensional over its center K, then all K-automorphisms of the algebra of upper triangular matrices of A are inner (see [4]).

Let $NT_n(F)$ be the set of all $n \times n$ matrices over a field F with zeros on and above the main diagonal. In 1951, Dubisch and Perlis described the algebra automorphisms of $NT_n(F)$ as follows:

Theorem 1.4 *[5, Theorem 5] Every automorphism of $NT_n(F)$ is equal to a product of a certain diagonal, an inner and a nil automorphism.*

Let $UT_n(K)$ be the set of all matrices with entries above the main diagonal zero and with the entries on the main diagonal all the identity element of K. The automorphism group of $UT_n(F)$ over a field F was studied by many authors. The first paper was published by Pavlov in 1952. Pavlov ([6]) described the automorphism group of the group of unitriangular matrices over a finite field of odd prime order. In 1955, Weir characterized the automorphism group of unitriangular matrices over a finite field of odd characteristic (see [7]).

Let $NT_n(K)$ be the set of all (lower) niltriangular $n \times n$ matrices over any associative ring K with identity.

Definition 1.5 *A ring R is called a nilpotent ring if there is a positive integer n such that $R^n = 0$.*

Proposition 1.6 *$NT_n(K)$ is a nilpotent ring with usual matrix addition and multiplication.*

It is clear that $NT_n(K)$ is an adjoint group with adjoint multiplication $a \bullet b = a + b + ab$ for all $a, b \in NT_n(K)$.

The unitriangular group $UT_n(K)$ and adjoint group of $NT_n(K)$ are isomorphic. The map $a \to a + I$ is a well-known isomorphism from $NT_n(K)$ to $UT_n(K)$ where I is the $n \times n$ identity matrix.

Definition 1.7 *[8, p9-10] A Lie ring is a nonassociative ring without identity and its multiplication $*$ satisfies the following conditions*

i) $a * a = 0$ *(anticommutativity)*

ii) $(a * b) * c + (b * c) * a + (c * a) * b = 0$ *(the Jacobi identity).*

One can easily see that $(NT_n(K), +, *)$ is a Lie ring with $a * b = ab - ba$ as obviously both anticommutativity and the Jacobi identity properties hold.

There are examples of Lie rings which are not rings;

Example 1.8 *Let $\mathbb{A}$ be a set of all $n \times n$ matrices over a field F admitting $A^T = -A$ for all $A \in \mathbb{A}$. It is easy to show that $\mathbb{A}$ is a Lie ring under commutator but not a ring under usual matrix multiplication.*

Let $(R, +, *)$ be a Lie ring, $S \subseteq R$ and s, t be arbitrary elements in S. Then the subset S is called a Lie ideal of the Lie ring $(R, +, *)$ if and only if $s + t \in S$ and $r * s \in S$ where r is an arbitrary element of R.

In 1983, Levchuk characterized the group of automorphisms of $NT_n(K)$ as a ring, Lie ring and an adjoint group where K is an associative ring with identity ([9]).

The fundamental ring automorphisms of $NT_n(K)$ can be defined as follows:

- If $d \in M_n(K)$ is an invertible diagonal matrix, then $x \to d^{-1}xd$ is an automorphism of $NT_n(K)$ which is called diagonal.

- Let a be an invertible element of $M_n(K)$. Then $x \to a^{-1}xa$ is an automorphism of $NT_n(K)$ called inner automorphism.

- Every automorphism θ of K can be extended to a ring automorphism $\bar{\theta}$ of $NT_n(K)$ with $[x_{i,j}] \to [\bar{\theta}(x_{i,j})]$.

- Let λ be a group endomorphism of K^+. Then $[x_{i,j}] \to [x_{i,j}] + \sum_{i=1}^{n-1} \lambda(x_{i+1,i})e_{n,1}$ defines an annihilator automorphism of $NT_n(K)$.

Theorem 1.9 *[9, Theorem 1] If K is unitary ring , then any automorphism of $NT_n(K)$ can be written as a product of certain diagonal, inner, ring and annihilator automorphisms of $NT_n(K)$ for n>2.*

Levchuk proved this theorem by taking advantage of the structural relations between the Lie ring and the adjoint group of $NT_n(K)$:

Theorem 1.10 *[10, Theorem 1] The class of all normal subgroups of the adjoint group of $NT_n(K)$ matches with the class of all ideals of the Lie ring of $NT_n(K)$ for an arbitrary ring K with identity.*

For any associative ring R, the operation $*$ is a Lie product with $x * y = xy - yx$ for all $x, y \in R$.

Definition 1.11 *An additive map Δ of a ring R is called a Lie automorphism of R if $\Delta(x * y) = \Delta(x) * \Delta(y)$.*

Proposition 1.12 *Let $R = NT_n(K)$. Then the automorphism group of R is the intersection of the automorphism group of the adjoint group G(R) and the automorphism group of the Lie ring $\Lambda(R)$ of R.*

Proof. We need to show that $Aut(R) = Aut(G(R)) \cap Aut(\Lambda(R))$. Let $\Psi \in Aut(R)$ and $x, y \in R$. Then $\Psi(x + y) = \Psi(x) + \Psi(y)$ and $\Psi(xy) = \Psi(x)\Psi(y)$.

First, for $x, y \in R$, we have

$$\begin{aligned}\Psi(x \bullet y) &= \Psi(x + y + xy) \\ &= \Psi(x) + \Psi(y) + \Psi(xy) \\ &= \Psi(x) + \Psi(y) + \Psi(x)\Psi(y) \\ &= \Psi(x) \bullet \Psi(y).\end{aligned}$$

This implies that $\Psi \in Aut(G(R))$.
Secondly,

$$\begin{aligned}\Psi(x * y) &= \Psi(xy - yx) \\ &= \Psi(xy) - \Psi(yx) \\ &= \Psi(x)\Psi(y) - \Psi(y)\Psi(x) \\ &= \Psi(x) * \Psi(y).\end{aligned}$$

Therefore $\Psi \in Aut(\Lambda(R))$. On the other hand, let $\Psi \in Aut(G(R)) \cap Aut(\Lambda(R))$. We know that $\Psi(x + y) = \Psi(x) + \Psi(y)$ since $\Psi \in Aut(\Lambda(R))$. Then we have $\Psi \in Aut(R)$ considering that

$$\begin{aligned}\Psi(xy) &= \Psi((x \bullet y) - (x + y)) \\ &= \Psi(x \bullet y) - \Psi(x + y) \\ &= \Psi(x) + \Psi(y) + \Psi(x)\Psi(y) - \Psi(x) - \Psi(y) \\ &= \Psi(x)\Psi(y).\end{aligned}$$

■

Levchuk also described the Lie automorphisms and adjoint group automorphisms of $NT_n(K)$ in 1983 (see [9, Theorem 1]).

Definition 1.13 *An ideal J of the associative ring K is called quasi-regular if J is a group with respect to the adjoint multiplication $x \bullet y = x + y + xy$ (see [11]).*

The adjoint multiplication in an associative ring is always a semigroup operation.

An element r of a ring is nilpotent if there is a natural number n such that $r^n = 0$.

Definition 1.14 *An arbitrary associative ring K is called a nil ring if every element of K is nilpotent.*

Definition 1.15 *Let $(K, \bullet)$ be a group. Then K is called a radical ring.*

It is clear that every nilpotent ring is nil.

Proposition 1.16 *Any nil ring is radical.*

Proof. Let $a \in R$. Then there can be found a natural number n such that $a^{n+1} = 0$. Consider the element $b = -a + a^2 - a^3 + ... + (-1)^n a^n$. Then $ab + b = -a$ and $a + b + ab = 0$. Therefore R is radical. ∎

Next example demonstrates that the converse of Proposition 1.16 is not true:

Example 1.17 *[12] Let* $S = \left\{ \frac{2x}{2y+1} : x, y \in \mathbb{Z} \text{ and } (2x, 2y+1) = 1 \right\} \subset \mathbb{Q}$. *This set is a commutative ring under the usual addition and multiplication. The ring* S *has no nilpotent element and so* S *is not nil. But every element of* S *is quasi-regular;*

$$\begin{aligned} \frac{2x}{2y+1} \bullet \frac{-2x}{2(x+y)+1} &= \frac{2x}{2y+1} - \frac{2x}{2(x+y)+1} - \frac{2x}{2y+1} . \frac{2x}{2(x+y)+1} \\ &= \frac{2x[2x+2y+1] - 2x(2y+1) - 4x^2}{(2y+1)(2x+2y+1)} \\ &= \frac{4x^2 + 4xy + 2x - 4xy - 2x - 4x^2}{(2y+1)(2x+2y+1)} \\ &= 0. \end{aligned}$$

Therefore, S *is a radical ring.*

The characterization of all associative radical rings R with the property that the class of all ideals of the associated Lie ring matches with the class of all normal subgroups of the adjoint group is still an open question (See [13], Question 10-19).

Let $e_{i,j}$ denote the $n \times n$ matrix whose $(i, j)-$projection is equal to 1 and the other projections are 0. The matrices $xe_{i,j}$ $(x \in K)$ are called elementary matrices. Multiplication, Lie multiplication and adjoint multiplication of elementary matrices are given as

$$(xe_{i,j})(ye_{k,m}) = \delta_{j,k} xye_{i,m}$$

$$xe_{i,j} * ye_{k,m} = \delta_{j,k} xye_{i,m} - \delta_{m,i} yxe_{k,j}$$

and

$$xe_{i,j} \bullet ye_{k,m} = xe_{i,j} + ye_{k,m} + \delta_{j,k} xye_{i,m}$$

respectively. Note that $\delta_{i,j}$ is Kronecker delta function.

Let K be a unitary ring. If J is an ideal of K then $R = R_n(K, J) = NT_n(K) + M_n(J)$ is a ring with usual matrix addition and multiplication.

Note that $R_n(K, J)$ is generated by the sets $Ke_{i+1,i}$ $(i < n)$ and $Je_{1,n}$.

Proposition 1.18 *If J is a quasi-regular ideal of the ring K then $R_n(K,J)$ is radical.*

Proof. Let $(J,\bullet)$ be a group. An arbitrary matrix $\alpha = [a_{i,j}] \in R_n(K,J)$ can be represented uniquely as follows:

$\alpha = \beta \bullet \delta \bullet \gamma$ where $\beta = \sum\limits_{i>j} x_{i,j}e_{i,j}$, $\delta = \sum\limits_{i=1}^{n} y_{i,i}e_{i,i}$ and $\gamma = \sum\limits_{i<j} z_{i,j}$ ($x_{i,j} \in K$, $y_{i,i} \in J$, $z_{i,j} \in J$).

Taking into account the relations

$$\begin{aligned} y_{i,i}e_{i,i} \bullet y'_{i,i}e_{i,i} &= y_{i,i}e_{i,i} + y'_{i,i}e_{i,i} + y_{i,i}y'_{i,i}e_{i,i} \\ &= (y_{i,i} + y'_{i,i} + y_{i,i}y'_{i,i})e_{i,i} \\ &= 0 \quad (y_{i,i} \in J), \end{aligned}$$

$$x_{i,j}e_{i,j} \bullet (-x_{i,j})e_{i,j} = 0 \quad (x_{i,j} \in K,\ i > j)$$

and

$$z_{i,j}e_{i,j} \bullet (-z_{i,j}e_{i,j}) = 0 \quad (z_{i,j} \in J, \quad i < j),$$

we can say that β, γ, δ have adjoint inverses β', γ', δ' where $\beta' = \sum\limits_{i>j} x'_{i,j}e_{i,j}$, $\gamma' = \sum\limits_{i=1}^{n} y'_{i,i}e_{i,i}$, $\delta' = \sum\limits_{i<j} z'_{i,j}e_{i,j}$. Hence

$$\begin{aligned} \beta \bullet \beta' &= \beta' \bullet \beta = 0, \\ \delta \bullet \delta' &= \delta' \bullet \delta = 0, \\ \gamma \bullet \gamma' &= \gamma' \bullet \gamma = 0. \end{aligned}$$

Therefore, $R_n(K,J)$ is a radical ring forasmuch as

$$\begin{aligned} \alpha \bullet \gamma' \bullet \delta' \bullet \beta' &= \beta \bullet \delta \bullet \gamma \bullet \gamma' \bullet \delta' \bullet \beta' \\ &= 0 \end{aligned}$$

and $\alpha' = \gamma' \bullet \delta' \bullet \beta'$. ■

For the ring $NT_n(K)$, the following are equivalent;

1. Any subgroup H of the adjoint group $NT_n(K)$ is normal.

2. H is a Lie ideal of the associated Lie ring $(NT_n(K), +, *)$.

The following two examples show that this equivalence doesn't hold for the ring $R_n(K, J)$.

Example 1.19 *Let K be a commutative ring, $R = R_n(K, J)$ and $L = \{C \in R : tr(C) = 0\}$ be a subset of R. If we choose any $P \in L$ and $Q \in R$, then we get*

$$\begin{aligned} tr(P * Q) &= tr(PQ - QP) \\ &= tr(PQ) - tr(QP) \\ &= tr(PQ) - tr(PQ) \\ &= 0 \end{aligned}$$

and obviously $tr(C_1 + C_2) = 0$ for all $C_1, C_2 \in L$. Therefore L is a Lie ideal of R. Now we will show that L is not even a subgroup of the adjoint group unless $J = 0$. Let $e_{2,1}$ and $ye_{1,2}$ $(y \neq 0)$ be two elements in L. Then the adjoint product $e_{2,1} \bullet ye_{1,2}$ is not in L because $e_{2,1} \bullet ye_{1,2} = e_{2,1} + ye_{1,2} + ye_{2,2}$ and $tr(e_{2,1} \bullet ye_{1,2}) = y \neq 0$.

Example 1.20 *Let K be a commutative ring with identity and J be a quasi-regular ideal of K. Define a map*

$$\begin{array}{cccc} \varphi : & (R_n(K, J)\ ,\ \bullet\ ) & \rightarrow & (GL_n(K)\ ,\ .\ ) \\ & A & \rightarrow & A + I \end{array}$$

where I is the $n \times n$ identity matrix. As a result of that

$$\begin{aligned} \varphi(A \bullet B) &= \varphi(A + B + AB) \\ &= A + B + AB + I \\ &= (A + I)(B + I) \\ &= \varphi(A)\varphi(B), \end{aligned}$$

φ is a group homomorphism. In fact, φ is a monomorphism. Now assume that $H = \varphi^{-1}(\varphi(R) \cap SL_n(K))$. Obviously H is normal in $(R_n(K, J), \bullet)$. Now let's see that H is not a Lie ideal of the Lie ring $R_n(K, J)$:

If $xe_{1,1} + ye_{2,2} \in H$, then $1 = \det(\varphi(xe_{1,1} + ye_{2,2}))$ and $1 = (1+x)(1+y) = 1+(x \bullet y)$. This means $x \bullet y = 0$. So y is the adjoint inverse of x. Now $xe_{1,2} * e_{2,1} = xe_{1,1} - xe_{2,2} = x(e_{1,1} - e_{2,2}) \in H$ $(x \in J)$ holds only if $y = -x$ and hence $x^2 = 0$. Therefore, if the ideal J contains an element a satisfying $a^2 \neq 0$, then it is not a Lie ideal.

Definition 1.21 *Let K be a ring. Then annihilator of K and annihilator of an ideal J in K are defined by*

$$Ann(K) = \{a \in K : \; aK = Ka = 0\}$$
$$Ann_K(J) = \{a \in K : \; aJ = Ja = 0\}.$$

Definitions of the automorphisms of $R_n(K, J)$ are given as follows:

- $[x_{i,j}] \to [x_{i,j}] + \left(\lambda_n(x_{1,n}) + \sum_{i=1}^{n-1} \lambda_i(x_{i+1,i}) \right) e_{n,1}$ is an **annihilator automorphism** of R with the conditions $\lambda_n : J \to Ann_K(J)$, $\lambda_n(J^2) = 0$, $\lambda_i : K \to Ann_K(J)$, $\lambda_i(J) = 0 \; (i < n)$.

- Let $\alpha, \beta : J \to J$, $\gamma : J \to K$ be additive maps satisfying

$$\begin{aligned} i)&\ \alpha(xy) = x\alpha(y) \\ ii)&\ \beta(yx) = \beta(y)x \\ iii)&\ y\beta(z) + \alpha(y)z = 0 \\ iv)&\ \gamma(zy) = \beta(z)\alpha(y) \\ v)&\ \beta(y)\gamma(z) + \gamma(y)\alpha(z) = y\gamma(z) + \alpha(y)\alpha(z) = \gamma(y)z + \beta(y)\beta(z) = 0 \end{aligned}$$

  where $x \in K$, $y, z \in J$. Then

$$\begin{aligned} \Delta : R_n(K, J) &\longrightarrow R_n(K, J) \\ ye_{1,n} &\longrightarrow ye_{1,n} + \alpha(y)e_{1,1} + \beta(y)e_{n,n} + \gamma(y)e_{n,1} \\ ye_{i,n} &\longrightarrow ye_{i,n} + \alpha(y)e_{i,1} \quad , \quad 1 < i \leq n \\ ye_{1,j} &\longrightarrow ye_{1,j} + \beta(y)e_{n,j} \quad , \quad 1 \leq j < n \\ x_{i,j}e_{i,j} &\longrightarrow x_{i,j}e_{i,j} \quad , \quad i > 1 \text{ and } j < n \end{aligned}$$

  is an automorphism of R called **almost annihilator**.

- If d is an invertible diagonal matrix in $M_n(K)$ then $x \to d^{-1}xd$ is an automorphism of $R_n(K, J)$ called a **diagonal automorphism**.

- Let $\theta \in Aut(K)$. Then $\bar{\theta} : [a_{i,j}] \to [\theta(a_{i,j})]$ is an automorphism of $R_n(K, J)$ if $\theta(J) = J$. Such automorphisms will be called **ring automorphisms** of R.

Theorem 1.22 *[14, Theorem 2.1] Let J be an ideal of K such that a one-sided or two-sided annihilator of J^t in K matches with J for a nonnegative integer t. Then any automorphism of $R_n(K, J)$ can be written as a product of annihilator, almost annihilator, inner, diagonal and ring automorphisms for $n > 2$.*

Definition 1.23 *An additive map Ψ of a ring R is called a derivation of R if $\Psi(ab) = \Psi(a)b + a\Psi(b)$ for all $a, b \in R$.*

The set of all derivations of a ring R is denoted by $Der(R)$. Lie product of two derivations is a derivation again;

Let $k, r \in R$.

$$\begin{aligned}
(d_1 * d_2)(kr) &= (d_1 d_2 - d_2 d_1)(kr) \\
&= d_1 d_2(kr) - d_2 d_1(kr) \\
&= d_1 [d_2(k)r + k d_2(r)] - d_2[d_1(k)r + k d_1(r)] \\
&= d_1[d_2(k)]r + d_2(k)d_1(r) + d_1(k)d_2(r) + k d_1[d_2(r)] \\
&\quad - d_2[d_1(k)]r - d_1(k)d_2(r) - d_2(k)d_1(r) - k d_2[d_1(r)] \\
&= d_1[d_2(k)]r - d_2[d_1(k)]r + k d_1[d_2(r)] - k d_2[d_1(r)] \\
&= \{d_1 d_2(k) - d_2 d_1(k)\} r + k \{d_1 d_2(r) - d_2 d_1(r)\} \\
&= (d_1 * d_2)(k)r + k(d_1 * d_2)(r).
\end{aligned}$$

Hence $Der(R)$ is a Lie Ring.

Let a is an arbitrary element of a ring K. An inner derivation of K is defined by $x \longrightarrow ax - xa$.

Theorem 1.24 *[1, Proposition(p100)] Let A be a simple algebra finite dimensional over its center F. Then any derivation of A is inner.*

Proof. Let A_2 be the ring of all 2×2 matrices over A. Obviously A_2 is simple and has F as its center and is finite dimensional over F. Let

$$B = \left\{ \begin{bmatrix} a & \delta(a) \\ 0 & a \end{bmatrix} : a \in A \right\}$$

where δ is a derivation of A and let

$$C = \left\{ \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} : a \in A \right\}.$$

It is trivial to show that $\delta(\alpha) = 0$ for $\alpha \in F$. Hence the mapping $\psi : C \to B$ defined by

$$\psi \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} = \begin{bmatrix} a & \delta(a) \\ 0 & a \end{bmatrix}$$

is easily shown to be an isomorphism of C onto B leaving F elementwise fixed. Also $C \approx A$. All the conditions of Skolem-Noether Theorem are satisfied. Thus there is an invertible matrix

$$\begin{bmatrix} x & y \\ z & w \end{bmatrix}$$

such that

$$\begin{bmatrix} a & \delta(a) \\ 0 & a \end{bmatrix} \begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} x & y \\ z & w \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}.$$

Hence

$$\begin{aligned} ax + \delta(a)z &= xa \\ ay + \delta(a)w &= ya \\ az &= za \\ aw &= wa \end{aligned}$$

for all $a \in A$. These relations imply $w, z \in F$ and as $\begin{bmatrix} x & y \\ z & w \end{bmatrix}$ is invertible, one of these scalars, say z, is nonzero. Putting $u = xz^{-1}$, we get $\delta(a) = ua - au$. As a result, δ is inner. ■

Corollary 1.25 *Let $\Bbbk$ be a field. Then any derivation of the matrix algebra $M_n(\Bbbk)$ is inner.*

Derivations of $NT_n(K)$ are given in [15] as follows:

- Let d be a diagonal matrix in $M_n(K)$. Then $i_d : x \to dx - xd$ is a derivation of $NT_n(K)$ which is called a diagonal derivation.

- If θ is a derivation of the ring K then $\bar{\theta} : [x_{i,j}] \to [\theta(x_{i,j})]$ is a derivation of $NT_n(K)$ called a ring derivation.

- Let s_t be a derivation of $NT_n(K)$. Then s_t is called a strongly nilpotent derivation of $NT_n(K)$ if $s_t(x) \in [NT_n(K)]^{k+1}$ for all $x \in [NT_n(K)]^k$.

In 2006, Chun and Park determined the derivations of the niltriangular matrix ring $NT_n(K)$.

Theorem 1.26 *[15] Any derivation δ of $NT_n(K)$ can be written as a sum of diagonal, ring and strongly nilpotent derivations.*

2 DERIVATIONS OF THE RING $R_n(K, J)$

Many authors have studied derivations of matrix rings and their subrings (see [16], [17], [18], [19], [20]). Let K be an associative ring with identity and J be an ideal of K. Recall that $R_n(K, J) = NT_n(K) + M_n(J)$ where $NT_n(K)$ is the set of all $n \times n$ matrices over K with zeros on and above the main diagonal and $M_n(J)$ is the set of all $n \times n$ matrices over J. In this section, we characterize all derivations of the ring $R_n(K, J)$.

The ideals of $R_n(K, J)$ are characterized in [21].

Definition 2.1 *An ideal J of a ring K is said to be a characteristic if it is invariant under any derivation of K (see [22]).*

It is obvious that K^m is a characteristic ideal of any associative ring K for every integer $m > 1$.

To compute the powers of $R = R_n(K, J)$, we use the infinite row of carpet ideals of K where each ideal is repeated n times. This terminology is originally introduced by Kargapolov and Merzljakov ([22, p107-108]). Regard the $n \times n$ matrix as a square array of n^2 points and cover it with carpet ideals $J^0 = K$, J, J^2, J^3, ... , J^m, ... as shown in the diagram for $n = 4$;

$$\begin{array}{ccccccccccccccccc}
K & K & K & K & J & J & J & J & J^2 & J^2 & J^2 & J^2 & J^3 & J^3 & \ldots & & \\
 & K & K & K & K & J & J & J & J & J^2 & J^2 & J^2 & J^2 & J^3 & \ldots & & \\
 & & K & K & K & K & J & J & J & J & J^2 & J^2 & J^2 & J^2 & \ldots & & \\
 & & & K & K & K & K & J & J & J & J & J^2 & J^2 & J^2 & J^2 \ldots & &
\end{array}$$

From the diagram, $R = \begin{bmatrix} J & J & J & J \\ K & J & J & J \\ K & K & J & J \\ K & K & K & J \end{bmatrix}$, $R^2 = \begin{bmatrix} J & J & J & J^2 \\ J & J & J & J \\ K & J & J & J \\ K & K & J & J \end{bmatrix}$,

$R^4 = \begin{bmatrix} J & J^2 & J^2 & J^2 \\ J & J & J^2 & J^2 \\ J & J & J & J^2 \\ J & J & J & J \end{bmatrix}$, $R^5 = \begin{bmatrix} J^2 & J^2 & J^2 & J^3 \\ J & J^2 & J^2 & J^2 \\ J & J & J^2 & J^2 \\ J & J & J & J^2 \end{bmatrix}$, $R^8 = \begin{bmatrix} J^2 & J^3 & J^3 & J^3 \\ J^2 & J^2 & J^3 & J^3 \\ J^2 & J^2 & J^2 & J^3 \\ J^2 & J^2 & J^2 & J^2 \end{bmatrix}$,

$$R^{4s} = \begin{bmatrix} J^s & J^{s+1} & J^{s+1} & J^{s+1} \\ J^s & J^s & J^{s+1} & J^{s+1} \\ J^s & J^s & J^s & J^{s+1} \\ J^s & J^s & J^s & J^s \end{bmatrix} \text{ and so on.}$$

Lemma 2.2 *Let J be an ideal of a ring K. If J is nilpotent, so is $R_n(K, J)$.*

The left annihilator of any ring K is denoted by $Ann_l(K)$ and

$$Ann_l(K) = \{x \in K : xK = 0\}.$$

The right annihilator of a ring can be defined in a similar way.
Let $R = R_n(K, J)$. By using carpet ideals, it can be seen that
$Ann_l(R^k) = \{[x_{i,j}] \in R : x_{u,v} \in Ann_l(J^{s+1}) \text{ for } v \leq t \text{ and } x_{u,v} \in Ann_l(J^s) \text{ for } v > t\}$
where $k = sn + t$ $(0 \leq t < n)$.

Proposition 2.3 *Let $Ann_K(J) = \{c \in K : cJ = Jc = 0\}$ and $Ann(R_n(K, J)) = \{A = [a_{i,j}] : AX = 0 = XA$ for all $X \in R_n(, J)\}$. Then $Ann(R_n(K, J)) = Ann_K(J)e_{n,1}$.*

Proof. Let $A = [a_{i,j}]$ be any element of $Ann(R_n(K, J))$. Then $Ax_{i,j}e_{i,j} = 0 = x_{i,j}e_{i,j}A$ for any $1 \leq i, j \leq n$ and $x_{i,j} \in I_{i,j}$. We know that $Ke_{i+1,i}$ and $Je_{1,n}$ generates all matrices in $R_n(K, J)$ since any elementary matrix can be written as

$$\prod_{k=0}^{i-j-1} e_{i-k,i-k-1} \text{ for } i > j \quad \textit{and}$$

$$\prod_{k=0}^{i-2} e_{i-k,i-k-1} y e_{1,n} \prod_{k=0}^{n-j-1} e_{n-k,n-k-1} \text{ for } i \leq j$$

where $x \in K$, $y \in J$. To determine the structure of a matrix $A \in Ann(R_n(K, J))$, it is sufficient to check $Axe_{i+1,i} = 0 = xe_{i+1,i}A$ and $Aye_{1,n} = 0 = ye_{1,n}A$ for all $x \in K$, $y \in J$.

$$0 = Axe_{i+1,i} = \sum_k A_{k,i+1} x e_{k,i} \Longrightarrow A_{k,i+1} = 0 \text{ for } 1 \leq k \leq n \text{ and } 1 \leq i < n,$$

$$0 = xe_{i+1,i}A = \sum_k x A_{i,k} e_{i+1,k} \Longrightarrow A_{i,k} = 0 \text{ for } 1 \leq k \leq n \text{ and } 1 \leq i < n.$$

and this means $A_{i,j} = 0$ except $(i,j) = (n,1)$. Furthermore, $A_{n,1}y = 0 = yA_{n,1}$ which means $A_{n,1} \in Ann_K(J)$ for all $y \in J$ since $0 = Aye_{1,n} = A_{n,1}ye_{n,n}$ implies $A_{n,1}y = 0$. and $0 = ye_{1,n}A = yA_{n,1}e_{1,1}$ implies $yA_{n,1} = 0$. As a result, it is clear that

$Ann(R_n(K,J)) = Ann_K(J)e_{n,1}$. ■

Now we define some derivations of $R_n(K,J)$.

Proposition 2.4 *Let* $\lambda_n : J \longrightarrow Ann_K(J)$, $\lambda_n(J^2) = 0$ *and* $\lambda_i : K \longrightarrow Ann_K(J)$, $\lambda_i(J) = 0$ *be additive maps for* $i = 1 < n$. *Then the map*

$$\Omega : R \longrightarrow Ann(R)$$
$$X = [x_{i,j}] \longrightarrow \left(\lambda_n(x_{1,n}) + \sum_{i=1}^{n-1} \lambda_i(x_{i+1,i})\right) e_{n,1}$$

determines a derivation of R. It is called an ***annihilator derivation****.*

Proof. Let X and Y be in R and $\lambda_n : J \longrightarrow Ann_K(J)$, $\lambda_i : K \longrightarrow Ann_K(J)$ $(i < n)$ be additive maps satisfying $\lambda_n(J^2) = 0$, $\lambda_i(J) = 0$. The map Ω is additive because λ_i is additive for all $i = 1, 2, ..., n$. On the other hand, $\Omega(XY)$ is equal to zero as

$$\Omega(XY) = \left(\sum_{k=1}^{n} \lambda_n(x_{1,k}y_{k,n}) + \sum_{i=1}^{n-1}\sum_{k=1}^{n} \lambda_i(x_{i+1,k}y_{k,i})\right) e_{n,1},$$

$\lambda_n(J^2) = 0$, $\lambda_i(J) = 0$ for $i = 1, 2, ..., n-1$. Finally, $\Omega(X)Y + X\Omega(Y) = 0$ considering that $\Omega : R \to Ann(R)$ where $Ann(R) = (Ann_K(J))\, e_{n,1}$. ■

Example 2.5 *Let* $K = \mathbb{Z}_9$ *and* J *be the ideal generated by* $\bar{3}$ *in* $\mathbb{Z}_9$. *Then*

$$\Omega : [x_{i,j}] \to \left(\sum_{i=1}^{n-1} \lambda(x_{i+1,i})\right) e_{n,1}$$

is an annihilator derivation of R where $\lambda : x \to 3x$ *is an additive map of K.*

Proposition 2.6 *If the additive group homomorphisms* $\sigma : J \longrightarrow Ann_K(J)$ *and* $\lambda, \mu : J \longrightarrow J$ *satisfy the following relations*

i) $\lambda(xy) = x\lambda(y)$

ii) $\mu(yx) = \mu(y)x$

iii) $\lambda(y)z + y\mu(z) = 0$

iv) $\sigma(J^2) = 0$

for $x \in K$ and $y, z \in J$, then the map

$$\begin{array}{rccc} \Delta : & R & \longrightarrow & R \\ & ye_{1,n} & \longrightarrow & \lambda(y)e_{1,1} + \mu(y)e_{n,n} + \sigma(y)e_{n,1} \\ & ye_{i,n} & \longrightarrow & \lambda(y)e_{i,1} \quad , \qquad 1 < i \leq n \\ & ye_{1,j} & \longrightarrow & \mu(y)e_{n,j} \quad , \qquad 1 \leq j < n \\ & x_{i,j}e_{i,j} & \longrightarrow & 0 \qquad , \qquad i > 1 \text{ and } j < n \end{array}$$

*determines a derivation of the ring R where $y \in J$ and $x_{i,j} \in I_{i,j}$. This derivation will be called an **almost annihilator** derivation.*

Proof. Let X and Y be in R and $\lambda : J \longrightarrow Ann_K(J)$, $\mu, \sigma : J \longrightarrow J$ be additive maps satisfying the conditions $i) - iv)$. It is obvious that Δ is additive because λ, μ and σ are additive maps. Besides, we have

$$\begin{aligned} \Delta(XY) = & \sum_{k=1}^{n} \lambda(x_{1,k}y_{k,n})e_{1,1} + \sum_{k=1}^{n} \mu(x_{1,k}y_{k,n})e_{n,n} \\ & + \sum_{i=2}^{n}\sum_{k=1}^{n} \lambda(x_{i,k}y_{k,n})e_{i,1} + \sum_{j=1}^{n-1}\sum_{k=1}^{n} \mu(x_{1,k}y_{k,j})e_{n,j} \end{aligned}$$

and

$$\begin{aligned} \Delta(X)Y + X\Delta(Y) = & \sum_{k=1}^{n} \lambda(x_{1,n})y_{1,k}e_{1,k} + \sum_{k=1}^{n} \mu(x_{1,n})y_{n,k}e_{n,k} \\ & + \sum_{i=2}^{n}\sum_{k=1}^{n} \lambda(x_{i,n})y_{1,k}e_{i,k} + \sum_{j=1}^{n-1}\sum_{k=1}^{n} \mu(x_{1,j})y_{j,k}e_{n,k} \\ & + \sum_{k=1}^{n} x_{k,1}\lambda(y_{1,n})e_{k,1} + \sum_{k=1}^{n} x_{k,n}\mu(y_{1,n})e_{k,n} \\ & + \sum_{i=2}^{n}\sum_{k=1}^{n} x_{k,i}\lambda(y_{i,n})e_{k,1} + \sum_{j=1}^{n-1}\sum_{k=1}^{n} x_{k,n}\mu(y_{1,j})e_{k,j} \end{aligned}$$

by $\sigma : J \longrightarrow Ann_K(J)$ and $iv)$. Now we need to show that these two are equal.

First of all, the $(1,1)$ coefficients of $\Delta(XY)$ and $\Delta(X)Y + X\Delta(Y)$ are $\sum_{k=1}^{n} \lambda(x_{1,k}y_{k,n})$ and $\left[\lambda(x_{1,n})y_{1,1} + x_{1,n}\mu(y_{1,1}) + \sum_{i=1}^{n} x_{1,i}\lambda(y_{i,n})\right]$, respectively and they are equal by the conditions $i)$ and $iii)$. Secondly, the (n,n) entries $\sum_{k=1}^{n} \mu(x_{1,k}y_{k,n})$ of $\Delta(XY)$ and $\left[\sum_{j=1}^{n} \mu(x_{1,j})y_{j,n} + \lambda(x_{n,n})y_{1,n} + x_{n,n}\mu(y_{1,n})\right]$ of $\Delta(X)Y + X\Delta(Y)$ are equal by $ii)$ and $iii)$. After that, the $(i,1)$ coefficients of $\Delta(XY)$ and $\Delta(X)Y + X\Delta(Y)$ for $i > 1$ are

$$\left[\sum_{k=1}^{n} \lambda(x_{1,k}y_{k,n})e_{1,1} + \sum_{i=2}^{n}\sum_{k=1}^{n} \lambda(x_{i,k}y_{k,n})e_{i,1} + \sum_{k=1}^{n} \mu(x_{1,k}y_{k,1})e_{n,1}\right]$$

and

$$\left[\sum_{i=1}^{n}\lambda(x_{i,n})y_{1,1}e_{i,1}+\sum_{j=1}^{n}\mu(x_{1,j})y_{j,1}e_{n,1}+\sum_{i=1}^{n}\sum_{k=1}^{n}x_{k,i}\lambda(y_{i,n})e_{k,1}+\sum_{k=1}^{n}x_{k,n}\mu(y_{1,1})e_{k,1}\right],$$

respectively. These two are equal to each other by $i)$, $ii)$ and $iii)$. Finally, the (n,j) entries of $\Delta(XY)$ and $\Delta(X)Y+X\Delta(Y)$ for $j<n$ are

$$\left[\sum_{k=1}^{n}\mu(x_{1,k}y_{k,n})e_{n,n}+\sum_{k=1}^{n}\lambda(x_{n,k}y_{k,n})e_{n,1}+\sum_{j=1}^{n-1}\sum_{k=1}^{n}\mu(x_{1,k}y_{k,j})e_{n,j}\right]$$

and

$$\sum_{k=1}^{n}\mu(x_{1,n})y_{n,k}e_{n,k}+\sum_{k=1}^{n}\lambda(x_{n,n})y_{1,k}e_{n,k}+\sum_{j=1}^{n-1}\sum_{k=1}^{n}\mu(x_{1,j})y_{j,k}e_{n,k}$$
$$+\sum_{i=1}^{n}x_{n,i}\lambda(y_{i,n})e_{n,1}+\sum_{j=1}^{n}x_{n,n}\mu(y_{1,j})e_{n,j},$$

respectively, and these two sums are equal as well by $i)$, $ii)$ and $iii)$.

Now, to complete the proof, we need to show that the entries except $(i,1)$ and (n,j) of $\Delta(X)Y+X\Delta(Y)$ are zero.

By excluding the $(i,1)$ and (n,j) entries of the matrix $\Delta(X)Y+X\Delta(Y)$, we get $\sum_{i=1}^{n-1}\sum_{k=2}^{n}\lambda(x_{i,n})y_{1,k}e_{i,k}+\sum_{j=2}^{n}\sum_{k=1}^{n-1}x_{k,n}\mu(y_{1,j})e_{k,j}$ which is equal to zero by $iii)$. ■

Example 2.7 *Let $K=Z_4$ and J be the ideal generated by $\bar{2}$ in K. Then the map*

$$\begin{array}{rccll}\Delta: & R_n(K,J) & \longrightarrow & R_n(K,J) & \\ & ye_{1,n} & \longrightarrow & \lambda(y)e_{1,1}+\mu(y)e_{n,n}+\sigma(y)e_{n,1} & \\ & ye_{i,n} & \longrightarrow & \lambda(y)e_{i,1}\quad, & 1<i\le n\\ & ye_{1,j} & \longrightarrow & \mu(y)e_{n,j}\quad, & 1\le j<n\\ & x_{i,j}e_{i,j} & \longrightarrow & 0\quad, & i>1 \text{ and } j<n\end{array}$$

is an almost annihilator derivation of R where $\lambda,\mu,\sigma: x\to x$. In particular, if n=3,

then $\Delta:[x_{i,j}]\to\begin{bmatrix}x_{1,3} & 0 & 0\\ x_{2,3} & 0 & 0\\ x_{1,3}+x_{3,3}+x_{1,1} & x_{1,2} & x_{1,3}\end{bmatrix}$ *is an almost annihilator derivation of $R_3(K,J)$ where $K=\mathbb{Z}_4$ and $J=(\bar{2})$.*

Proposition 2.8 *If θ is a derivation of the ring K and also of J, then* $\bar{\theta} : [x_{i,j}] \to \sum_{i,j=1}^{n} \theta(x_{i,j})e_{i,j}$ *is a derivation of R which is called a* ***ring derivation****.*

Proof. Let X and Y be arbitrary elements of R and θ be a derivation of K and also of J. Then $\bar{\theta}$ is additive as θ is an additive map of K. Besides, (i,j) entry of $\bar{\theta}(XY)$ is $\sum_{k=1}^{n} \theta(x_{i,k}y_{k,j})$ and (i,j) entry of $\bar{\theta}(X)Y + X\bar{\theta}(Y)$ is $\sum_{k=1}^{n} \theta(x_{i,k})y_{k,j} + \sum_{k=1}^{n} x_{i,k}\theta(y_{k,j})$. As a result of that $\theta(x_{i,k}y_{k,j}) = \theta(x_{i,k})y_{k,j} + x_{i,k}\theta(y_{k,j})$ for every $k = 1, 2, ..., n$, two sums $\sum_{k=1}^{n} \theta(x_{i,k}y_{k,j})$ and $\sum_{k=1}^{n} \theta(x_{i,k})y_{k,j} + \sum_{k=1}^{n} x_{i,k}\theta(y_{k,j})$ are equal. ∎

Let θ be an additive map of K and J. Then $\bar{\theta} : [x_{i,j}] \to \sum_{i,j=1}^{n} \theta(x_{i,j})e_{i,j}$ determines a derivation of the ring $R_2(K, J)$ if the relation $\theta(xy) = \theta(x)y + x\theta(y)$ holds and $\theta(1) = 0$ where $x \in K, y \in J$ or $x \in J, y \in K$. This derivation is called a (K^+, J)**-ring derivation**.

Example 2.9 *Let $K = \mathbb{R}[x]$ be the ring of all polynomials on $\mathbb{R}$ and $J = (x)$ be the ideal generated by x. Then $\bar{\theta} : [x_{i,j}] \to [\theta(x_{i,j})]$ is a ring derivation of $R_n(K, J)$ where θ is the ordinary derivation.*

Proposition 2.10 *For any ring R and any element a of this ring, the map $\Psi_a : x \longrightarrow ax - xa$ is a derivation of R which is called the* ***inner derivation*** *of R induced by the element a.*

Proof. Let $x, y \in R$ be arbitrary elements. It can easily be seen that Ψ_a is an additive map of R. We need to show $\Psi_a(xy) = \Psi_a(x)y + x\Psi_a(y)$;

$$\begin{aligned}
\Psi_a(xy) &= axy - xya \\
&= axy - (xay - xay) - xya \\
&= axy - xay + xay - xya \\
&= (ax - xa)y + x(ay - ya) \\
&= \Psi_a(x)y + x\Psi_a(y).
\end{aligned}$$

∎

Proposition 2.11 *Let $d = \sum_{i=1}^{n} d_i e_{i,i}$ $(d_i \in K)$. Then the map $\delta_d(x) = dx - xd$ is a derivation of $R_n(K, J)$ which is called the diagonal derivation induced by the diagonal matrix d.*

Proof. Same technique which is followed in the previous proposition applies. ∎

For the following theorem and lemmas, $R_n(K, J)$ will be denoted by R and n will be greater than 2 unless stated otherwise.

Theorem 2.12 *Every derivation φ of R is a sum of certain diagonal, inner, almost annihilator, annihilator and ring derivations.*

Lemma 2.13 *Let φ be a derivation of R for $n > 2$, $x \in I_{k,m}$ and*

$$\varphi(xe_{k,m}) = \sum_{s,t=1}^{n} \varphi_{s,t}^{k,m}(x) e_{s,t} = \begin{bmatrix} \varphi_{1,1}^{k,m}(x) & . \; . \; . & \varphi_{1,n}^{k,m}(x) \\ . & & . \\ . & & . \\ . & & . \\ \varphi_{n,1}^{k,m}(x) & . \; . \; . & \varphi_{n,n}^{k,m}(x) \end{bmatrix}$$

for any k,m. Then $\varphi(xe_{k,m})$ is exactly equal to the matrix

$$\begin{bmatrix}
0 & 0 & . \; . \; . & 0 & \varphi_{1,m}^{k,m}(x) & 0 & . \; . \; . & 0 \\
. & . & & . & . & . & & . \\
. & . & & . & . & . & & . \\
. & . & & . & . & . & & . \\
0 & 0 & . \; . \; . & 0 & \varphi_{k-1,m}^{k,m}(x) & 0 & . \; . \; . & 0 \\
\varphi_{k,1}^{k,m}(x) & \varphi_{k,2}^{k,m}(x) & . \; . \; . & \varphi_{k,m-1}^{k,m}(x) & \varphi_{k,m}^{k,m}(x) & \varphi_{k,m+1}^{k,m}(x) & . \; . \; . & \varphi_{k,n}^{k,m}(x) \\
0 & 0 & . \; . \; . & 0 & \varphi_{k+1,m}^{k,m}(x) & 0 & . \; . \; . & 0 \\
. & . & & . & . & . & & . \\
. & . & & . & . & . & & . \\
. & . & & . & . & . & & . \\
0 & 0 & . \; . \; . & 0 & \varphi_{n-1,m}^{k,m}(x) & 0 & . \; . \; . & 0 \\
\varphi_{n,1}^{k,m}(x) & 0 & . \; . \; . & 0 & \varphi_{n,m}^{k,m}(x) & 0 & . \; . \; . & 0
\end{bmatrix}.$$

Proof. If $m \neq i + 1$ we obtain

$$\begin{aligned}
0 &= \varphi(xe_{k,m}e_{i+1,i}) \\
&= \varphi(xe_{k,m})e_{i+1,i} + xe_{k,m}\varphi(e_{i+1,i}) \\
&= \left(\sum_{s,t=1}^{n} \varphi_{s,t}^{k,m}(x) e_{s,t} \right) e_{i+1,i} + xe_{k,m} \left(\sum_{s,t=1}^{n} x\varphi_{s,t}^{i+1,i}(1) e_{s,t} \right) \\
&= \sum_{s=1}^{n} \varphi_{s,i+1}^{k,m}(x) e_{s,i} + \sum_{t=1}^{n} x\varphi_{m,t}^{i+1,i}(1) e_{k,t} \\
&= \sum_{s \neq k} \varphi_{s,i+1}^{k,m}(x) e_{s,i} + \sum_{t \neq i} x\varphi_{m,t}^{i+1,i}(1) e_{k,t} + \left[\varphi_{k,i+1}^{k,m}(x) + x\varphi_{m,i}^{i+1,i}(1) \right] e_{k,i}
\end{aligned}$$

which can be written as

$$\begin{bmatrix} 0 & 0 & \cdots & 0 & \varphi^{k,m}_{1,i+1}(x) & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & \varphi^{k,m}_{2,i+1}(x) & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 & \varphi^{k,m}_{k-1,i+1}(x) & 0 & \cdots & 0 \\ x\varphi^{i+1,i}_{m,1}(1) & x\varphi^{i+1,i}_{m,2}(1) & \cdots & x\varphi^{i+1,i}_{m,i-1}(1) & \varphi^{k,m}_{k,i+1}(x) + x\varphi^{i+1,i}_{m,i}(1) & x\varphi^{i+1,i}_{m,i+1}(1) & \cdots & x\varphi^{i+1,i}_{m,n}(1) \\ 0 & 0 & \cdots & 0 & \varphi^{k,m}_{k+1,i+1}(x) & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 & \varphi^{k,m}_{n,i+1}(x) & 0 & \cdots & 0 \end{bmatrix}.$$

Thus

$$\varphi^{k,m}_{s,i+1}(x) = 0 \tag{i}$$

for $[s \neq k]$. It can be seen by (i) that $\varphi(xe_{k,m})$ has nonzero entries only in the k-th row, m-th column, and first column. If $i \neq k$ then we obtain

$$\begin{aligned} 0 &= \varphi(e_{i+1,i}xe_{k,m}) \\ &= \varphi(e_{i+1,i})xe_{k,m} + e_{i+1,i}\varphi(xe_{k,m}) \\ &= \left(\sum_{s,t=1}^{n} x\varphi^{i+1,i}_{s,t}(1)e_{s,t}\right) xe_{k,m} + e_{i+1,i}\left(\sum_{s,t=1}^{n} \varphi^{k,m}_{s,t}(x)e_{s,t}\right) \\ &= \sum_{s=1}^{n} \varphi^{i+1,i}_{s,k}(1)xe_{s,m} + \sum_{t=1}^{n} \varphi^{k,m}_{i,t}(x)e_{i+1,t} \\ &= \sum_{s\neq i+1}^{n} \varphi^{i+1,i}_{s,k}(1)xe_{s,m} + \sum_{t\neq m}^{n} \varphi^{k,m}_{i,t}(x)e_{i+1,t} + \left[\varphi^{i+1,i}_{i+1,k}(1)x + \varphi^{k,m}_{i,m}(x)\right] e_{i+1,m} \end{aligned}$$

which can be written as

$$\begin{bmatrix} 0 & 0 & \cdots & 0 & \varphi^{i+1,i}_{1,k}(1)x & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & \varphi^{i+1,i}_{2,k}(1)x & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 & \varphi^{i+1,i}_{i,k}(1)x & 0 & \cdots & 0 \\ \varphi^{k,m}_{i,1}(x) & \varphi^{k,m}_{i,2}(x) & \cdots & \varphi^{k,m}_{i,m-1}(x) & \varphi^{i+1,i}_{i+1,k}(1)x + \varphi^{k,m}_{i,m}(x) & \varphi^{k,m}_{i,m+1}(x) & \cdots & \varphi^{k,m}_{i,n}(x) \\ 0 & 0 & \cdots & 0 & \varphi^{i+1,i}_{i+2,k}(1)x & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 & \varphi^{i+1,i}_{n,k}(1)x & 0 & \cdots & 0 \end{bmatrix}.$$

Hence

$$\varphi^{k,m}_{i,t}(x) = 0 \quad [t \neq m]. \tag{ii}$$

It follows that the matrix $\varphi(xe_{k,m})$ has nonzero entries only on the k-th row, m-th column and n-th row for $x \in I_{k,m}$. Now if we combine (i) and (ii), then the image of $xe_{k,m}$ under φ is the matrix with zeros out of k-th row, m-th column and $(n,1)$ position. This completes the proof. ■

Lemma 2.14 *Let φ be a derivation of the ring R. Then there is a diagonal derivation δ_d of R such that (i+1, i)-coefficient of $[\varphi - \delta_d](e_{i+1,i})$ is zero for $1 \le i < n$.*

Proof. Let

$$d = \sum_{i=2}^{n} d_i e_{i,i}$$

be a matrix with $d_{i+1} = \sum_{k=1}^{i} c_k$ and $c_k = \varphi_{k+1,k}^{k+1,k}(1)$. Then

$$\begin{aligned} \delta_d(e_{i+1,i}) &= de_{i+1,i} - e_{i+1,i}d \\ &= d_{i+1}e_{i+1,i} - d_i e_{i+1,i} \\ &= c_i e_{i+1,i} \\ &= \varphi_{i+1,i}^{i+1,i}(1)e_{i+1,i}. \end{aligned}$$

Thus, the $(i+1,i)$-coefficient of the matrix $[\varphi - \delta_d](e_{i+1,i})$ is equal to zero for all i. ■

Lemma 2.15 *Let φ be a derivation of R such that (i+1,i) coefficient of the matrix $\varphi(e_{i+1,i})$ is zero for $i = 1, 2, ..., n-1$. Then there is an inner derivation Ψ satisfying that $[\varphi - \Psi](e_{i+1,i})$ has zero $i-th$ column and $(i+1,1)$ entries.*

Proof. Initially, we will see there is an inner derivation Ψ_A satisfying that $[\varphi - \Psi_A](e_{i+1,i})$ has nonzero entries only in $(i+1,1)$ and $(n,1)$ position for $1 < i < n-1$. Let $A_{k,k} = 0 = A_{j,1}$, $A_{u,i+1} = \varphi_{u,i}^{i+1,i}(1)$ for $u \neq i+1$, $1 < i < n$ and A be the matrix $[A_{i,j}]_{n\times n}$. Clearly A is equal to

$$\begin{bmatrix} 0 & \varphi_{1,1}^{2,1}(1) & \varphi_{1,2}^{3,2}(1) & . & . & . & \varphi_{1,n-1}^{n,n-1}(1) \\ 0 & 0 & \varphi_{2,2}^{3,2}(1) & . & . & . & \varphi_{2,n-1}^{n,n-1}(1) \\ 0 & \varphi_{3,1}^{2,1}(1) & 0 & . & . & . & \varphi_{3,n-1}^{n,n-1}(1) \\ 0 & \varphi_{4,1}^{2,1}(1) & \varphi_{4,2}^{3,2}(1) & . & . & . & \varphi_{4,n-1}^{n,n-1}(1) \\ . & . & . & & & & . \\ . & . & . & & & & . \\ . & . & . & & & & . \\ 0 & \varphi_{n-1,1}^{2,1}(1) & \varphi_{n-1,2}^{3,2}(1) & . & . & . & \varphi_{n-1,n-1}^{n,n-1}(1) \\ 0 & \varphi_{n,1}^{2,1}(1) & \varphi_{n,2}^{3,2}(1) & . & . & . & 0 \end{bmatrix}.$$

Consider the action of Ψ_A on the the matrices $e_{i+1,i}$. Then $\Psi_A(e_{i+1,i}) = Ae_{i+1,i} - e_{i+1,i}A$ and this is equal to

$$\begin{bmatrix} 0 & 0 & . & . & . & 0 & \varphi^{i+1,i}_{1,i}(1) & 0 & . & . & . & 0 \\ 0 & 0 & . & . & . & 0 & \varphi^{i+1,i}_{2,i}(1) & 0 & . & . & . & 0 \\ . & . & & & & . & . & . & & & & . \\ . & . & & & & . & . & . & & & & . \\ . & . & & & & . & . & . & & & & . \\ 0 & 0 & . & . & . & 0 & \varphi^{i+1,i}_{i,i}(1) & 0 & . & . & . & 0 \\ 0 & -\varphi^{2,1}_{i,1}(1) & . & . & . & -\varphi^{i-1,i-2}_{i,i-2}(1) & 0 & -\varphi^{i+1,i}_{i,i}(1) & . & . & . & -\varphi^{n,n-1}_{i,n-1}(1) \\ 0 & 0 & . & . & . & 0 & \varphi^{i+1,i}_{i+2,i}(1) & 0 & . & . & . & 0 \\ . & . & & & & . & . & . & & & & . \\ . & . & & & & . & . & . & & & & . \\ . & . & & & & . & . & . & & & & . \\ 0 & 0 & . & . & . & 0 & \varphi^{i+1,i}_{n,i}(1) & 0 & . & . & . & 0 \end{bmatrix}.$$

On the other hand, we find

$$\begin{aligned} 0 = \varphi(e_{i,j}e_{k,m}) &= \varphi(e_{i,j})e_{k,m} + e_{i,j}\varphi(e_{k,m}) \\ &= \left[\sum_{s,t=1}^{n} \varphi^{i,j}_{s,t}(1)e_{s,t}\right] e_{k,m} + e_{i,j}\left[\sum_{s,t=1}^{n} \varphi^{k,m}_{s,t}(1)e_{s,t}\right] \\ &= \sum_{s=1}^{n} \varphi^{i,j}_{s,k}(1)e_{s,m} + \sum_{t=1}^{n} \varphi^{k,m}_{j,t}(1)e_{i,t} \end{aligned}$$

for $k > m,\ i > j$ and $j \neq k$. Thus

$$\varphi^{i,j}_{i,k}(1) + \varphi^{k,m}_{j,m}(1) = 0. \qquad \text{(iii)}$$

By (iii), it is clear that

$$[\varphi - \Psi_A](e_{i+1,i}) = \varphi^{i+1,i}_{i+1,1}(1)e_{i+1,1} + \varphi^{i+1,i}_{n,1}(1)e_{n,1}.$$

Now let $\Gamma = \varphi - \Psi_A$ for brevity. Then

$$\Gamma(e_{i+1,i}) = \Gamma^{i+1,i}_{i+1,1}(1)e_{i+1,1} + \Gamma^{i+1,i}_{n,1}(1)e_{n,1} \quad \text{for} \quad 1 < i < n-1.$$

In particular, $\Gamma(e_{2,1}) = 0$. Let

$$B = \begin{bmatrix} 0 & 0 & . & . & . & 0 & 0 \\ -b_3 & 0 & . & . & . & 0 & 0 \\ . & . & & & & . & . \\ . & . & & & & . & . \\ . & . & & & & . & . \\ -b_n & 0 & . & . & . & 0 & 0 \\ 0 & 0 & . & . & . & 0 & 0 \end{bmatrix}$$

where $b_{j+1} = \Gamma^{j+1,j}_{j+1,1}(1)$ for $2 < j < n$ and denote Ψ_B the inner derivation induced by the matrix B. In this case it can be easily seen that

$$\Psi_B(e_{i+1,i}) = Be_{i+1,i} - e_{i+1,i}B = \Gamma^{i+1,i}_{i+1,1}(1)e_{i+1,1} \quad for \quad i = 2, ..., n-2$$

and this completes the proof. ∎

Let $\Pi = \varphi - \Psi_A - \Psi_B$ for brevity. Obviously $\Pi(e_{2,1}) = 0 = \Pi(e_{n,n-1})$.

Lemma 2.16 *There exists an annihilator derivation Ω such that $(n,1)$ entries of $[\Pi - \Omega](ye_{1,n})$ and $[\Pi - \Omega](xe_{i+1,i})$ are zeros where $y \in J$ and $x \in K$.*

Proof. Firstly, we want to show that the additive mappings $\Pi^{i+1,i}_{n,1}$ satisfy the conditions of an annihilator derivation. $(n,1)$ entry of the relation

$$\begin{aligned}\Pi(xye_{2,1}) &= \Pi(xe_{2,1}ye_{1,1}) \\ &= \Pi(xe_{2,1})ye_{1,1} + xe_{2,1}\Pi(ye_{1,1})\end{aligned}$$

gives $\Pi^{2,1}_{n,1}(xy) = \Pi^{2,1}_{n,1}(x)y$ and $\Pi^{2,1}_{n,1}(y) = \Pi^{2,1}_{n,1}(1)y = 0$ for $x = 1$ considering that $\Pi(e_{2,1}) = 0$. Hence $\Pi^{2,1}_{n,1}(J) = 0$ and $\Pi^{2,1}_{n,1}(xy) = \Pi^{2,1}_{n,1}(x)y = 0$. Moreover, we have $0 = \Pi^{n,n}_{n,2}(y)x + y\Pi^{2,1}_{n,1}(x)$ by $(n,1)$ coefficient of the relation

$$\begin{aligned}0 &= \Pi(ye_{n,n}xe_{2,1}) \\ &= \Pi(ye_{n,n})xe_{2,1} + ye_{n,n}\Pi(xe_{2,1})\end{aligned}$$

and since $\Pi(e_{2,1}) = 0$, we get $\Pi^{n,n}_{n,2}(y) = 0$ while $x = 1$. This implies $y\Pi^{2,1}_{n,1}(x) = 0$. Consequently, we have $\Pi^{2,1}_{n,1} : K \longrightarrow Ann_K(J)$. Besides, we get $\Pi^{n,n-1}_{n,1}(J) = 0$ by $\Pi(e_{n,n-1}) = 0$ and $y\Pi^{n,n-1}_{n,1}(x) = 0$ by $(n,1) - th$ coefficient of the relation

$$\begin{aligned}\Pi(yxe_{n,n-1}) &= \Pi(ye_{n,n}xe_{n,n-1}) \\ &= \Pi(ye_{n,n})xe_{n,n-1} + ye_{n,n}\Pi(xe_{n,n-1}).\end{aligned}$$

Forasmuch as $\Pi(e_{n,n-1}) = 0$, we have $\Pi^{1,1}_{n-1,1}(y) = 0$ and it follows $\Pi^{n,n-1}_{n,1}(x)y = 0$ by $(n,1) - th$ coefficient of the relation

$$\begin{aligned}0 &= \Pi(xe_{n,n-1}ye_{1,1}) \\ &= \Pi(xe_{n,n-1})ye_{1,1} + xe_{n,n-1}\Pi(ye_{1,1}).\end{aligned}$$

In addition, for $1 < i < n-1$, it is obtained $\Pi_{n,1}^{i+1,i}(x)y = 0$ and $y\Pi_{n,1}^{i+1,i}(x) = 0$ by (n,n) and $(1,1)-th$ coefficients of the relations

$$\begin{aligned} 0 &= \Pi(xe_{i+1,i}ye_{1,n}) \\ &= \Pi(xe_{i+1,i})ye_{1,n} + xe_{i+1,i}\Pi(ye_{1,n}) \end{aligned}$$

and

$$\begin{aligned} 0 &= \Pi(ye_{1,n}xe_{i+1,i}) \\ &= \Pi(ye_{1,n})xe_{i+1,,i} + ye_{1,n}\Pi(xe_{i+1,i}), \end{aligned}$$

respectively. In that $(n,1)-th$ coefficient of the relation

$$\begin{aligned} \Pi(xye_{i+1,i}) &= \Pi(xe_{i+1,i}ye_{i,i}) \\ &= \Pi(xe_{i+1,i})ye_{i,i} + xe_{i+1,i}\Pi(ye_{i,i}) \end{aligned}$$

is zero, we have $\Pi_{n,1}^{i+1,i}(J) = 0$.

Secondly, $(n,1)-th$ coefficient of the relation

$$\begin{aligned} \Pi(yze_{1,n}) &= \Pi(ye_{1,2}ze_{2,n}) \\ &= \Pi(ye_{1,2})ze_{2,n} + ye_{1,2}\Pi(ze_{2,n}) \end{aligned}$$

gives $\Pi_{n,1}^{1,n}(J^2) = 0$ and $(n,1)-th$ coefficients of the relations

$$\begin{aligned} 0 &= \Pi(ye_{1,n}ze_{1,1}) \\ &= \Pi(ye_{1,n})ze_{1,1} + ye_{1,n}\Pi(ze_{1,1}), \end{aligned}$$

$$\begin{aligned} 0 &= \Pi(ye_{n,n}ze_{1,n}) \\ &= \Pi(ye_{n,n})ze_{1,n} + ye_{n,n}\Pi(ze_{1,n}) \end{aligned}$$

give $\Pi_{n,1}^{1,n} : J \to Ann_K(J)$.

Finally,

$$\begin{array}{rccc} \Omega : & R & \to & Ann(R) \\ & [x_{i,j}] & \to & \left(\sum_{i=1}^{n-1} \Pi_{n,1}^{i+1,i}(x_{i+1,i}) + \Pi_{n,1}^{1,n}(x_{1,n}) \right) e_{n,1} \end{array}$$

is the desired annihilator derivation of R and $[\Pi - \Omega](e_{i+1,i}) = 0$. ■

Let $\Xi = \Pi - \Omega$ for brevity. Now consider the relations

$$\begin{aligned}0 &= \Xi(xe_{i+1,i}e_{j+1,j}) \qquad\qquad (i \neq j+1)\\ &= \Xi(xe_{i+1,i})e_{j+1,j} + xe_{i+1,i}\Xi(e_{j+1,j})\\ &= \Xi(xe_{i+1,i})e_{j+1,j}\\ &= \sum_{s=1}^{n} \Xi_{s,j+1}^{i+1,i}(x)e_{s,j},\end{aligned}$$

$$\begin{aligned}0 &= \Xi(e_{u+1,u}xe_{i+1,i}) \qquad\qquad (i+1 \neq u)\\ &= \Xi(e_{u+1,u})xe_{i+1,i} + e_{u+1,u}\Xi(xe_{i+1,i})\\ &= e_{u+1,u}\Xi(xe_{i+1,i})\\ &= \sum_{t=1}^{n} \Xi_{u,t}^{i+1,i}(x)e_{u+1,t},\end{aligned}$$

$$\begin{aligned}0 &= \Xi(ye_{1,n}e_{v+1,v}) \qquad\qquad (v < n-1)\\ &= \Xi(ye_{1,n})e_{v+1,v} + ye_{1,n}\Xi(e_{v+1,v})\\ &= \Xi(ye_{1,n})e_{v+1,v}\\ &= \sum_{s=1}^{n} \Xi_{s,v+1}^{1,n}(y)e_{s,v}\end{aligned}$$

and

$$\begin{aligned}0 &= \Xi(e_{a+1,a}ye_{1,n}) \qquad\qquad (a \neq 1)\\ &= \Xi(e_{a+1,a})ye_{1,n} + e_{a+1,a}\Xi(ye_{1,n})\\ &= e_{a+1,a}\Xi(ye_{1,n})\\ &= \sum_{t=1}^{n} \Xi_{a,t}^{1,n}(y)e_{a+1,t}.\end{aligned}$$

It follows that $(j+1)-th$ column of $\Xi(xe_{i+1,i})$ is zero for $i \neq j+1$ where $j < n$, $u-th$ row of $\Xi(xe_{i+1,i})$ is zero for $u \neq i+1$ where $u < n$, $(v+1)-th$ column of $\Xi(ye_{1,n})$ is zero for $v < n-1$ and $a-th$ row of $\Xi(ye_{1,n})$ is zero for $1 < a < n$. That means $\Xi(xe_{i+1,i})$ is zero except its $(i+1,1)$, $(i+1,i)$, (n,i) coefficients and $\Xi(ye_{1,n})$ is zero except its $(1,1)$, $(1,n)$, (n,n) coefficients. In other words, $\Xi(xe_{i+1,i}) = \Xi_{i+1,1}^{i+1,i}(x)e_{i+1,1} + \Xi_{i+1,i}^{i+1,i}(x)e_{i+1,i} + \Xi_{n,i}^{i+1,i}(x)e_{n,i}$ for $1 < i < n-1$ and $\Xi(ye_{1,n}) = \Xi_{1,1}^{1,n}(y)e_{1,1} + \Xi_{1,n}^{1,n}(y)e_{1,n} + \Xi_{n,n}^{1,n}(y)e_{n,n}$. In particular, $\Xi(xe_{2,1}) = \Xi_{2,1}^{2,1}(x)e_{2,1}$ and $\Xi(xe_{n,n-1}) = \Xi_{n,n-1}^{n,n-1}(x)e_{n,n-1}$.

Lemma 2.17 *There exists an almost annihilator derivation* Δ *such that* $(1,1)$, (n,n) *coefficients of* $[\Xi - \Delta](ye_{1,n})$, (n,j) *coefficient of* $[\Xi - \Delta](ye_{1,j})$ *and* $(i,1)$ *coefficient of* $[\Xi - \Delta](ye_{i,n})$ *are zeros where* $y \in J$, $1 < i$ *and* $j < n$.

Proof. We need to show that the conditions of an almost annihilator derivation are satisfied. If $1 < i \leq n$, $x \in K$ and $y \in J$, then we get $\Xi_{1,1}^{1,n}(xy) = x\Xi_{1,1}^{1,n}(y)$ and $\Xi_{i,1}^{i,n} = \Xi_{1,1}^{1,n}$ for $x = 1$ by $(i,1) - th$ coefficient of the relation

$$\begin{aligned}\Xi(xye_{i,n}) &= \Xi(xe_{i,1}ye_{1,n}) \\ &= \Xi(xe_{i,1})ye_{1,n} + xe_{i,1}\Xi(ye_{1,n}).\end{aligned}$$

In addition, if $1 \leq i < n$, $x \in K$ and $y \in J$, then it is obtained $\Xi_{n,n}^{1,n}(yx) = \Xi_{n,n}^{1,n}(y)x$ and $\Xi_{n,i}^{1,i} = \Xi_{n,n}^{1,n}$ for $x = 1$ by the $(n,i) - th$ coefficient of the relation

$$\begin{aligned}\Xi(yxe_{1,i}) &= \Xi(ye_{1,n}xe_{n,i}) \\ &= \Xi(ye_{1,n})xe_{n,i} + ye_{1,n}\Xi(xe_{n,i}).\end{aligned}$$

Say $\lambda := \Xi_{i,1}^{i,n} = \Xi_{1,1}^{1,n}$ and $\mu := \Xi_{n,n}^{1,n} = \Xi_{n,j}^{1,j}$. Then the following map is an almost annihilator derivation of R;

$$\begin{aligned}\Delta: \quad R &\rightarrow R \\ ye_{1,n} &\rightarrow \lambda(y)e_{1,1} + \mu(y)e_{n,n} \\ ye_{1,j} &\rightarrow \mu(y)e_{n,j} \qquad (j < n) \\ ye_{i,n} &\rightarrow \lambda(y)e_{i,1} \qquad (i > 1) \\ x_{i,j}e_{i,j} &\rightarrow 0 \ (i > 1 \text{ and } j < n).\end{aligned}$$

■

Let $\xi := \Xi - \Delta$ for brevity. By comparing the relations

$$\begin{aligned}\xi(xe_{i+1,i-1}) &= \xi(xe_{i+1,i}e_{i,i-1}) \\ &= \xi(xe_{i+1,i})e_{i,i-1} + xe_{i+1,i}\xi(xe_{i,i-1}) \\ &= \xi(xe_{i+1,i})e_{i,i-1} \\ &= \xi_{i+1,i}^{i+1,i}(x)e_{i+1,i-1} + \xi_{n,i}^{i+1,i}(x)e_{n,i-1}\end{aligned}$$

and

$$
\begin{aligned}
\xi(xe_{i+1,i-1}) &= \xi(e_{i+1,i}xe_{i,i-1}) \\
&= \xi(e_{i+1,i})xe_{i,i-1} + e_{i+1,i}\xi(xe_{i,i-1}) \\
&= e_{i+1,i}\xi(xe_{i,i-1}) \\
&= \xi_{i,1}^{i,i-1}(x)e_{i+1,1} + \xi_{i,i-1}^{i,i-1}(x)e_{i+1,i-1} \ ,
\end{aligned}
$$

it can be obtained $\xi_{i+1,1}^{i+1,i} = 0,\ \xi_{n,i}^{i+1,i} = 0$ and one can see that $\xi(xe_{i+1,i}) = \xi_{i+1,i}^{i+1,i}(x)e_{i+1,i}$.

Lemma 2.18 *There exists a ring derivation $\bar{\theta}$ such that $\bar{\theta} = \xi$.*

Proof. We first need to show $\xi_{i,j}^{i,j}$ is a derivation of the coefficient ring. Since $\xi(xe_{i+1,i}) = \xi_{i+1,i}^{i+1,i}(x)e_{i+1,i}$, clearly $\xi(e_{i+1,i}) = 0$. So $\xi(e_{i,j})$ is equal to zero for $i > j$. By (i,k) coefficient of the relation

$$
\begin{aligned}
\xi(xe_{i,k}) &= \xi(x_1e_{i,j}x_2e_{j,k}) \\
&= \xi(x_1e_{i,j})x_2e_{j,k} + x_1e_{i,j}\xi(x_2e_{j,k}) \ ,
\end{aligned}
$$

we have $\xi_{i,k}^{i,k}(x_1x_2) = \xi_{i,j}^{i,j}(x_1)x_2 + x_1\xi_{j,k}^{j,k}(x_2)$ for $i > j > k$ and $\xi_{i,k}^{i,k} = \xi_{i,j}^{i,j} = \xi_{j,k}^{j,k}$ as $\xi(e_{i,k}),\ \xi(e_{i,j}),\ \xi(e_{j,k})$ are all zeros. This means $\xi_{u,v}^{u,v} = \xi_{s,t}^{s,t}$ are derivations for all $u > v$ and $s > t$. On the other hand, we have $\xi_{i,1}^{i,1} = \xi_{i,j}^{i,j}$ for $y \in J$ and $i \leq j$ by $(i,1)$ coefficient of the relation

$$
\begin{aligned}
\xi(ye_{i,1}) &= \xi(ye_{i,j}e_{j,1}) \\
&= \xi(ye_{i,j})e_{j,1} + ye_{i,j}\xi(e_{i,1}) \\
&= \xi(ye_{i,j})e_{j,1}.
\end{aligned}
$$

This means $\xi_{u,v}^{u,v} = \xi_{s,t}^{s,t}$ for all $u \leq v$ and $s \leq t$ because $\xi_{i,1}^{i,1} = \xi_{j,1}^{j,1}$ for all i,j. Now it is easy to see that $\xi_{i,j}^{i,j} = \xi_{k,m}^{k,m}$ are all derivations for any i,j,k,m. Say $\theta := \xi_{i,j}^{i,j}$ for all $1 \leq i,j \leq n$. Now $\left(\xi - \overline{\theta}\right)(xe_{i+1,i}) = 0$ and $\left(\xi - \overline{\theta}\right)(ye_{1,n}) = 0$ where $\overline{\theta}$ is a ring derivation of R such that $\overline{\theta}(A) = \sum_{i,j=1}^{n} \theta(a_{i,j})e_{i,j},\ A = [a_{i,j}] \in R$. Considering that any element $xe_{i,j}$ can be written as $xe_{i,j} = x\prod_{k=0}^{i-j-1} e_{i-k,i-k-1}$ for $i > j$ and $ye_{i,j} = \prod_{k=0}^{i-2} e_{i-k,i-k-1}ye_{1,n}\prod_{k=0}^{n-j-1} e_{n-k,n-k-1}$ for $i \leq j$ where $x \in K$, $y \in J$, the map $\xi - \overline{\theta}$ is equal to zero. This completes the proof. ■

We showed that any derivation φ of $R_n(K,J)$ can be written as a sum of a certain diagonal derivation δ_d, an inner derivation Ψ, an almost annihilator derivation Ω, an annihilator derivation Δ and a ring derivation $\bar{\theta}$, i.e.

$$\varphi = \delta_d + \Psi + \Omega + \Delta + \bar{\theta}.$$

If **n=2**, we first consider the action of φ on the relations

$$ye_{1,1}e_{2,1} = 0\ ,\quad xe_{2,1}e_{2,1} = 0,\quad e_{2,1}ye_{2,2} = 0\quad (x \in K,\ y \in J)$$

and we obtain $\varphi^{1,1}_{2,2} = 0$ by (2,1) coefficient of the relation $0 = \varphi(ye_{1,1}e_{2,1}) = \varphi(ye_{1,1})e_{2,1} + ye_{1,1}\varphi(e_{2,1})$, $\varphi^{2,1}_{1,2} = 0$, by (1,1) coefficient of $0 = \varphi(xe_{2,1}e_{2,1}) = \varphi(xe_{2,1})e_{2,1} + xe_{2,1}\varphi(e_{2,1})$ and $\varphi^{2,2}_{1,1} = 0$ by (2,1) coefficient of $0 = \varphi(e_{2,1}ye_{2,2}) = \varphi(e_{2,1})ye_{2,2} + e_{2,1}\varphi(ye_{2,2})$. Then we have

$$\varphi(ye_{1,1}) = \varphi^{1,1}_{1,1}(y)e_{1,1} + \varphi^{1,1}_{1,2}(y)e_{1,2} + \varphi^{1,1}_{2,1}(y)e_{2,1}\ ,$$
$$\varphi(xe_{2,1}) = \varphi^{2,1}_{1,1}(x)e_{1,1} + \varphi^{2,1}_{2,1}(x)e_{2,1} + \varphi^{2,1}_{2,2}(x)e_{2,2}\ ,$$
$$\varphi(ye_{2,2}) = \varphi^{2,2}_{1,2}(y)e_{1,2} + \varphi^{2,2}_{2,1}(y)e_{2,1} + \varphi^{2,2}_{2,2}(y)e_{2,2}.$$

Now let $d = \varphi^{2,1}_{2,1}(1)e_{2,2}$ be a diagonal matrix and $A = \varphi^{2,1}_{1,1}(1)e_{1,2} \in R_2(K,J)$. Then we find

$$(\varphi - \delta_d - \Psi_A)\,(e_{2,1}) = 0$$

where δ_d is the diagonal derivation induced by the diagonal matrix d and Ψ_A is the inner derivation induced by the matrix A since $\delta_d(e_{2,1}) = \varphi^{2,1}_{2,1}(1)e_{2,2}e_{2,1} - e_{2,1}\varphi^{2,1}_{2,1}(1)e_{2,2} = \varphi^{2,1}_{2,1}(1)e_{2,1}$, $\Psi_A(e_{2,1}) = \varphi^{2,1}_{1,1}(1)e_{1,2}e_{2,1} - e_{2,1}\varphi^{2,1}_{1,1}(1)e_{1,2} = \varphi^{2,1}_{1,1}(1)e_{1,1} - \varphi^{2,1}_{1,1}(1)e_{2,2}$ and $\varphi^{2,1}_{2,2}(1) + \varphi^{2,1}_{1,1}(1) = 0$ by (2,1) coefficient of the relation $\varphi(e_{2,1}e_{2,1}) = 0$.
Let $\varphi - \delta_d - \Psi_A = \Pi$. For $x, x_1, x_2 \in K$ and $y \in J$, the relations $\Pi(ye_{1,1}e_{2,1}) = 0$, $\Pi(x_1e_{2,1}x_2e_{2,1}) = 0$, $\Pi(e_{2,1}ye_{2,2}) = 0$ give

$$\Pi(ye_{1,1}) = \Pi^{1,1}_{1,1}(y)e_{1,1} + \Pi^{1,1}_{2,1}(y)e_{2,1}\ ,$$
$$\Pi(ye_{1,2}) = \Pi^{1,2}_{1,1}(y)e_{1,1} + \Pi^{1,2}_{1,2}(y)e_{1,2} + \Pi^{1,2}_{2,1}(y)e_{2,1} + \Pi^{1,2}_{2,2}(y)e_{2,2}\ ,$$
$$\Pi(xe_{2,1}) = \Pi^{2,1}_{2,1}(x)e_{2,1}\ ,$$
$$\Pi(ye_{2,2}) = \Pi^{2,2}_{2,1}(y)e_{2,1} + \Pi^{2,2}_{2,2}(y)e_{2,2}\ .$$

On the other hand, we get $\Pi_{2,1}^{1,1}(yx) = \Pi_{2,2}^{1,2}(y)x$, $\Pi_{2,2}^{1,2} = \Pi_{2,1}^{1,1}$, $\Pi_{2,1}^{2,2}(xy) = x\Pi_{1,1}^{1,2}(y)$ and $\Pi_{1,1}^{1,2} = \Pi_{2,1}^{2,2}$ by (2,1) coefficients of the relations $\Pi(ye_{1,2}xe_{2,1}) = \Pi(yxe_{1,1})$ and $\Pi(xe_{2,1}ye_{1,2}) = \Pi(xye_{2,2})$. In addition, (1,1) and (2,2) coefficients of the relation $0 = \Pi(ye_{1,2}ze_{1,2}) = \Pi(ye_{1,2})ze_{1,2} + ye_{1,2}\Pi(ze_{1,2})$ gives $\Pi_{2,1}^{1,2} : J \to Ann_K(J)$. Besides, we have $\Pi_{2,1}^{1,2}(J^2) = 0$ by (2,1) coefficient of the relation $\Pi(yze_{1,2}) = \Pi(ye_{1,1}ze_{1,2}) = \Pi(ye_{1,1})ze_{1,2} + ye_{1,1}\Pi(ze_{1,2})$. Furthermore, we obtain $\Pi_{1,1}^{1,2}(y)z + y\Pi_{2,2}^{1,2}(z) = 0$ by (1,2) coefficient of $0 = \Pi(ye_{1,2}ze_{1,2}) = \Pi(ye_{1,2})ze_{1,2}+ye_{1,2}\Pi(ze_{1,2})$. Put $\lambda = \Pi_{1,1}^{1,2} = \Pi_{2,1}^{2,2}$, $\mu = \Pi_{2,2}^{1,2} = \Pi_{2,1}^{1,1}$, $\sigma = \Pi_{2,1}^{1,2}$. Then all conditions of almost annihilator derivation are satisfied and the map

$$\begin{array}{rlcl} \Delta: & R_2(K,J) & \to & R_2(K,J) \\ & ye_{1,2} & \to & \lambda(y)e_{1,1} + \mu(y)e_{2,2} + \sigma(y)e_{2,1} \\ & ye_{1,1} & \to & \mu(y)e_{2,1} \\ & ye_{2,2} & \to & \lambda(y)e_{2,1} \\ & xe_{2,1} & \to & 0 \end{array}$$

becomes an almost annihilator derivation of $R_2(K,J)$. Let $\xi = \Pi - \Delta$. Then $\xi(x_{i,j}e_{i,j}) = \xi_{i,j}^{i,j}(x_{i,j})e_{i,j}$. For $x \in K$ and $y \in J$, the relations $\xi(xe_{2,1}ye_{1,1}) = \xi(xye_{2,1})$, $\xi(ye_{2,2}xe_{2,1}) = \xi(yxe_{2,1})$, $\xi(xe_{2,1}ye_{1,2}) = \xi(xye_{2,2})$ give $\theta = \xi_{1,1}^{1,1} = \xi_{2,1}^{2,1} = \xi_{1,2}^{1,2} = \xi_{2,2}^{2,2}$, $\xi_{2,1}^{2,1}(xy) = \xi_{2,1}^{2,1}(x)y + x\xi_{2,1}^{2,1}(y)$ and $\xi_{2,1}^{2,1}(yx) = \xi_{2,1}^{2,1}(y)x + y\xi_{2,1}^{2,1}(x)$. Then $\bar{\theta} : [x_{i,j}] \to [\theta(x_{i,j})]$ is a (K^+, J)- ring derivation since $\theta(1) = \xi_{2,1}^{2,1}(1) = 0$ and finally $\xi - \bar{\theta} = 0$. This means any derivation φ of $R_2(K,J)$ can be written as a sum of a certain diagonal derivation δ_d, an inner derivation Ψ_A, an almost annihilator derivation Δ and a ring derivation $\bar{\theta}$ of $R_2(K,J)$, i.e.

$$\varphi = \delta_d + \Psi_A + \Delta + \bar{\theta}.$$

Some part of this section of this thesis is published in 2017 (see [23]).

3 JORDAN DERIVATIONS OF THE RING $R_n(K, J)$

In this section, we describe all Jordan derivations of the ring $R_n(K, J)$. An additive map $d : K \longrightarrow K$ is a Jordan derivation if it satisfies

$$d(r \circ s) = d(rs + sr) = d(r)s + sd(r) + rd(s) + d(s)r$$

for arbitrary elements $r, s \in K$ where $r \circ s = rs + sr$. Jordan derivations of some rings and algebras have been studied by some researchers ([24],[25],[26],[27],[28],[29]).

Derivations are examples of Jordan derivations and often it turns out that they are actually the only possible examples. However, there exist Jordan derivations which are not derivations. An example of non-trivial Jordan derivation is given as follows:

Example 3.1 *Let $S = \mathbb{C}[x]$ with the relation $x^2 = 0$ and let $I = \mathbb{C}x$. Obviously I is an ideal of S as $x^2 = 0$. Now let $\bar{R} = \left\{ \begin{bmatrix} x & y \\ z & w \end{bmatrix} : x, y, w \in S,\ z \in I \right\}$. Obviously $\bar{R}$ is a ring with matrix addition and multiplication. For any $t = \begin{bmatrix} x & y \\ z & w \end{bmatrix} \in \bar{R}$, define $\delta(t) = \begin{bmatrix} 0 & z \\ 0 & 0 \end{bmatrix}$. Then δ is not a derivation but a Jordan derivation of $\bar{R}$. In order to see that, we need to show δ is additive,*

$$\begin{aligned} \delta\left(q \circ q'\right) &= \delta\left(q\right) \circ q' + q \circ \delta(q') \\ &= \delta(q)q' + q'\delta(q) + q\delta(q') + \delta(q')q \end{aligned}$$

holds and $\delta\left(qq'\right) \neq \delta(q)q' + q\delta(q')$ for any $q = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$, $q' = \begin{bmatrix} x' & y' \\ z' & w' \end{bmatrix} \in \bar{R}$.

First of all, δ is clearly additive, S is commutative and $I^2 = 0$ by definition. Two *relations*

$$\delta\left(\begin{bmatrix} x & y \\ z & w \end{bmatrix} \circ \begin{bmatrix} x' & y' \\ z' & w' \end{bmatrix}\right) = \delta\left(\begin{bmatrix} x & y \\ z & w \end{bmatrix}\begin{bmatrix} x' & y' \\ z' & w' \end{bmatrix} + \begin{bmatrix} x' & y' \\ z' & w' \end{bmatrix}\begin{bmatrix} x & y \\ z & w \end{bmatrix}\right)$$
$$= \delta\left(\begin{bmatrix} x & y \\ z & w \end{bmatrix}\begin{bmatrix} x' & y' \\ z' & w' \end{bmatrix}\right) + \delta\left(\begin{bmatrix} x' & y' \\ z' & w' \end{bmatrix}\begin{bmatrix} x & y \\ z & w \end{bmatrix}\right)$$
$$= \delta\left(\begin{bmatrix} xx' + yz' & xy' + yw' \\ zx^{'} + wz^{'} & zy' + ww' \end{bmatrix}\right)$$
$$+ \delta\left(\begin{bmatrix} x'x + y'z & x'y + y'w \\ z'x + w'z & z'y + w'w \end{bmatrix}\right)$$
$$= \begin{bmatrix} 0 & zx^{'} + wz^{'} \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & z'x + w'z \\ 0 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & zx^{'} + wz^{'} + z'x + w'z \\ 0 & 0 \end{bmatrix}$$

and

$$\delta\left(\begin{bmatrix} x & y \\ z & w \end{bmatrix}\right)\begin{bmatrix} x' & y' \\ z' & w' \end{bmatrix} + \begin{bmatrix} x' & y' \\ z' & w' \end{bmatrix}\delta\left(\begin{bmatrix} x & y \\ z & w \end{bmatrix}\right)$$
$$+ \begin{bmatrix} x & y \\ z & w \end{bmatrix}\delta\left(\begin{bmatrix} x' & y' \\ z' & w' \end{bmatrix}\right) + \delta\left(\begin{bmatrix} x' & y' \\ z' & w' \end{bmatrix}\right)\begin{bmatrix} x & y \\ z & w \end{bmatrix}$$
$$= \begin{bmatrix} 0 & z \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x' & y' \\ z' & w' \end{bmatrix} + \begin{bmatrix} x' & y' \\ z' & w' \end{bmatrix}\begin{bmatrix} 0 & z \\ 0 & 0 \end{bmatrix}$$
$$+ \begin{bmatrix} x & y \\ z & w \end{bmatrix}\begin{bmatrix} 0 & z' \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & z' \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x & y \\ z & w \end{bmatrix}$$
$$= \begin{bmatrix} zz' & zw' \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & x'z \\ 0 & z'z \end{bmatrix} + \begin{bmatrix} 0 & xz' \\ 0 & zz' \end{bmatrix} + \begin{bmatrix} z'z & z'w \\ 0 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & zw' + x'z + xz' + z'w \\ 0 & 0 \end{bmatrix} \qquad (zz', z'z \in I^2 = 0)$$

are clearly equal and we get $\delta(q \circ q^{'}) = \delta\,(q)\,q' + q'\delta(q) + q\delta(q') + \delta(q')q$. *It means* δ *is* *a Jordan derivation. On the other hand,*

$$\delta\left(\begin{bmatrix} x & y \\ z & w \end{bmatrix}\begin{bmatrix} x' & y' \\ z' & w' \end{bmatrix}\right) = \delta\left(\begin{bmatrix} xx'+yz' & xy'+yw' \\ zx'+wz' & zy'+ww' \end{bmatrix}\right)$$
$$= \begin{bmatrix} 0 & zx'+wz' \\ 0 & 0 \end{bmatrix}$$

and

$$\delta\left(\begin{bmatrix} x & y \\ z & w \end{bmatrix}\right)\begin{bmatrix} x' & y' \\ z' & w' \end{bmatrix} + \begin{bmatrix} x & y \\ z & w \end{bmatrix}\delta\left(\begin{bmatrix} x' & y' \\ z' & w' \end{bmatrix}\right) = \begin{bmatrix} 0 & z \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x' & y' \\ z' & w' \end{bmatrix}$$
$$+ \begin{bmatrix} x & y \\ z & w \end{bmatrix}\begin{bmatrix} 0 & z' \\ 0 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} zz' & zw' \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & xz' \\ 0 & zz' \end{bmatrix}$$
$$= \begin{bmatrix} 0 & zw'+xz' \\ 0 & 0 \end{bmatrix}$$

as zz' and $z'z$ are elements in $I^2 = 0$. Then $\delta(qq')$ and $\delta(q)q' + q\delta(q')$ are obviously different from each other and this means δ is not a derivation.

Let $NT_n(K)$ be the ring of all (lower) niltriangular $n \times n$ matrices over any associative ring with identity whose entries are all zeros on and above the main diagonal. The following is an example of a Jordan derivation of $NT_n(K)$ that is not a derivation of the same ring.

Example 3.2 *[28] Let $a, b, x \in K$ such that $aK^2 = 0 = b(K \circ K)$. Then the map $\tau : xe_{2,1} \to bxe_{n,2} + axe_{n,3},\ xe_{3,1} \to axe_{n,2}$ is a Jordan derivation but not a derivation of $NT_n(K)$ for $n > 3$. To see that τ is a Jordan derivation, we need to show τ is additive and $\tau(S \circ T) = \tau(S) \circ T + S \circ \tau(T)$ for arbitrary matrices $[s_{i,j}] = S$, $[t_{i,j}] = T \in NT_n(K)$.*

τ is additive because

$$
\begin{aligned}
\tau(S+T) &= \tau([s_{i,j}]+[t_{i,j}]) \\
&= \tau([s_{i,j}+t_{i,j}]) \\
&= b(s_{2,1}+t_{2,1})e_{n,2} + a(s_{2,1}+t_{2,1})e_{n,3} + a(s_{3,1}+t_{3,1})e_{n,2} \\
&= bs_{2,1}e_{n,2} + bt_{2,1}e_{n,2} + as_{2,1}e_{n,3} + at_{2,1}e_{n,3} + as_{3,1}e_{n,2} + at_{3,1}e_{n,2} \\
&= bs_{2,1}e_{n,2} + as_{2,1}e_{n,3} + as_{3,1}e_{n,2} + bt_{2,1}e_{n,2} + as_{2,1}e_{n,3} + at_{3,1}e_{n,2} \\
&= \tau(S)+\tau(T).
\end{aligned}
$$

In addition,

$$
\begin{aligned}
\tau(S\circ T) &= \tau(ST+TS) \\
&= b\sum_k s_{2,k}t_{k,1}e_{n,2} + a\sum_k s_{2,k}t_{k,1}e_{n,3} + a\sum_k s_{3,k}t_{k,1}e_{n,2} \\
&\quad + b\sum_k t_{2,k}s_{k,1}e_{n,2} + a\sum_k t_{2,k}s_{k,1}e_{n,3} + a\sum_k t_{3,k}s_{k,1}e_{n,2} \\
&= a(s_{3,2}t_{2,1}+t_{3,2}s_{2,1})e_{n,2} \qquad (s_{i,j}=t_{i,j}=0 \textit{ if } i\le j) \\
&= 0
\end{aligned}
$$

and

$$
\begin{aligned}
\tau(S)\circ T + S\circ\tau(T) &= \tau(S)T + T\tau(S) + S\tau(T) + \tau(T)S \\
&= (bs_{2,1}e_{n,2} + as_{2,1}e_{n,3} + as_{3,1}e_{n,2})\,T \\
&\quad + T\,(bs_{2,1}e_{n,2} + as_{2,1}e_{n,3} + as_{3,1}e_{n,2}) \\
&\quad + S\,(bt_{2,1}e_{n,2} + at_{2,1}e_{n,3} + at_{3,1}e_{n,2}) \\
&\quad + (bt_{2,1}e_{n,2} + at_{2,1}e_{n,3} + at_{3,1}e_{n,2})\,S \\
&= \sum_k bs_{2,1}t_{2,k}e_{n,k} + \sum_k as_{2,1}t_{3,k}e_{n,k} + \sum_k as_{3,1}t_{2,k}e_{n,k} \\
&\quad + \sum_k t_{k,n}bs_{2,1}e_{k,2} + \sum_k t_{k,n}as_{2,1}e_{k,3} + \sum_k t_{k,n}as_{3,1}e_{k,2} \\
&\quad + \sum_k s_{k,n}bt_{2,1}e_{k,2} + \sum_k s_{k,n}at_{2,1}e_{k,3} + \sum_k s_{k,n}at_{3,1}e_{k,2} \\
&\quad + \sum_k bt_{2,1}s_{2,k}e_{n,k} + \sum_k at_{2,1}s_{3,k}e_{n,k} + \sum_k at_{3,1}s_{2,k}e_{n,k} \\
&= bs_{2,1}t_{2,1}e_{n,1} + as_{2,1}t_{3,1}e_{n,1} + as_{2,1}t_{3,2}e_{n,2} + as_{3,1}t_{2,1}e_{n,1} \\
&\quad + bt_{2,1}s_{2,1}e_{n,1} + at_{2,1}s_{3,1}e_{n,1} + at_{2,1}s_{3,2}e_{n,2} + at_{3,1}s_{2,1}e_{n,1} \\
&= b(s_{2,1}t_{2,1} + t_{2,1}s_{2,1})e_{n,1} \qquad (aK^2 = 0) \\
&= 0.
\end{aligned}
$$

This means τ *is a Jordan derivataion forasmuch as* $\tau(S\circ T) = \tau(S)\circ T + S\circ\tau(T)$. *However,* τ *is not a derivation as the right sides of the equalities*

$$
\begin{aligned}
\tau(ST) &= b\sum_k s_{2,k}t_{k,1}e_{n,2} + a\sum_k s_{2,k}t_{k,1}e_{n,3} + a\sum_k s_{3,k}t_{k,1}e_{n,2} \\
&= 0 \qquad (aK^2 = 0 \text{ and } s_{i,j} = y_{i,j} = 0 \text{ for } i \le j)
\end{aligned}
$$

and

$$
\begin{aligned}
\tau(S)T + S\tau(T) &= (bs_{2,1}e_{n,2} + as_{2,1}e_{n,3} + as_{3,1}e_{n,2})\,T \\
&\quad + S\,(bt_{2,1}e_{n,2} + at_{2,1}e_{n,3} + at_{3,1}e_{n,2}) \\
&= \sum_k bs_{2,1}t_{2,k}e_{n,k} + \sum_k as_{2,1}t_{3,k}e_{n,k} + \sum_k as_{3,1}t_{2,k}e_{n,k} \\
&\quad + \sum_k s_{k,n}bt_{2,1}e_{k,2} + \sum_k s_{k,n}at_{2,1}e_{k,3} + \sum_k s_{k,n}at_{3,1}e_{k,2} \\
&= bs_{2,1}t_{2,1}e_{n,1} \qquad (aK^2 = 0 \text{ and } s_{i,j} = y_{i,j} = 0 \text{ for } i \le j)
\end{aligned}
$$

are different from each other.

The following is another example of a Jordan derivation of $NT_n(K)$ that is not a derivation of the same ring.

Example 3.3 *[28] Let K be an associative ring and $c, d, x \in K$ such that $(K \circ K)c = 0 = K^2 d$. Then the map $\omega : xe_{n,n-1} \to xce_{n-1,1} + xde_{n-2,1}$, $xe_{n,n-2} \to xde_{n-1,1}$ is a Jordan derivation but not a derivation of $NT_n(K)$ for $n > 3$. So as to see that ω is a Jordan derivation, we need to show ω is additive and $\omega(S \circ T) = \omega(S) \circ T + S \circ \omega(T)$ for arbitrary matrices $[s_{i,j}] = S$, $[t_{i,j}] = T \in NT_n(K)$.*

$\omega : NT_n(K) \to NT_n(K)$ is additive:

$$
\begin{aligned}
\omega(S+T) &= \omega([s_{i,j}] + [t_{i,j}]) \\
&= \omega([s_{i,j} + t_{i,j}]) \\
&= (s_{n,n-1} + t_{n,n-1})ce_{n-1,1} + (s_{n,n-1} + t_{n,n-1})de_{n-2,1} \\
&+ (s_{n,n-2} + t_{n,n-2})de_{n-1,1} \\
&= s_{n,n-1}ce_{n-1,1} + t_{n,n-1}ce_{n-1,1} + s_{n,n-1}de_{n-2,1} \\
&+ t_{n,n-1}de_{n-2,1} + s_{n,n-2}de_{n-1,1} + t_{n,n-2}de_{n-1,1} \\
&= s_{n,n-1}ce_{n-1,1} + s_{n,n-1}de_{n-2,1} + s_{n,n-2}de_{n-1,1} \\
&+ t_{n,n-1}ce_{n-1,1} + t_{n,n-1}de_{n-2,1} + t_{n,n-2}de_{n-1,1} \\
&= \omega(S) + \omega(T).
\end{aligned}
$$

ω is a Jordan derivation since two relations

$$
\begin{aligned}
\omega(S \circ T) &= \omega(ST + TS) \\
&= \omega(ST) + \omega(TS) \\
&= \sum_k s_{n,k}t_{k,n-1}ce_{n-1,1} + \sum_k s_{n,k}t_{k,n-1}de_{n-2,1} \\
&+ \sum_k s_{n,k}t_{k,n-2}de_{n-1,1} + \sum_k t_{n,k}s_{k,n-1}ce_{n-1,1} \\
&+ \sum_k t_{n,k}s_{k,n-1}de_{n-2,1} + \sum_k t_{n,k}s_{k,n-2}de_{n-1,1} \\
&= 0 \qquad (K^2 d = 0,\ s_{i,j} = t_{i,j} = 0 \text{ for } i \le j)
\end{aligned}
$$

and

$$\begin{aligned}
\omega(S)\circ T + S\circ\omega(T) &= \omega(S)T + T\omega(S) + S\omega(T) + \omega(T)S \\
&= (s_{n,n-1}ce_{n-1,1} + s_{n,n-1}de_{n-2,1} + s_{n,n-2}de_{n-1,1})T \\
&+ T(s_{n,n-1}ce_{n-1,1} + s_{n,n-1}de_{n-2,1} + s_{n,n-2}de_{n-1,1}) \\
&+ S(t_{n,n-1}ce_{n-1,1} + t_{n,n-1}de_{n-2,1} + t_{n,n-2}de_{n-1,1}) \\
&+ (t_{n,n-1}ce_{n-1,1} + t_{n,n-1}de_{n-2,1} + t_{n,n-2}de_{n-1,1})S \\
&= \sum_k s_{n,n-1}ct_{1,k}e_{n-1,k} + \sum_k s_{n,n-1}dt_{1,k}e_{n-2,k} \\
&+ \sum_k s_{n,n-2}dt_{1,k}e_{n-1,k} + \sum_k t_{k,n-1}s_{n,n-1}ce_{k,1} \\
&+ \sum_k t_{k,n-2}s_{n,n-1}de_{k,1} + \sum_k t_{k,n-1}s_{n,n-2}de_{k,1} \\
&+ \sum_k s_{k,n-1}t_{n,n-1}ce_{k,1} + \sum_k s_{k,n-2}t_{n,n-1}de_{k,1} \\
&+ \sum_k s_{k,n-1}t_{n,n-2}de_{k,1} + \sum_k t_{n,n-1}cs_{1,k}e_{n-1,k} \\
&+ \sum_k t_{n,n-1}ds_{1,k}e_{n-2,k} + \sum_k t_{n,n-2}ds_{1,k}e_{n-1,k} \\
&= (t_{n,n-1}s_{n,n-1} + s_{n,n-1}t_{n,n-1})ce_{n,1} \qquad (K^2d = 0) \\
&= (t_{n,n-1}\circ s_{n,n-1})ce_{n,1} \\
&= 0 \qquad ((K\circ K)c = 0).
\end{aligned}$$

are equal to each other. But it is not a derivation since two equalities are distinct from each other:

$$\begin{aligned}
\omega(ST) &= \sum_k s_{n,k}t_{k,n-1}ce_{n-1,1} + \sum_k s_{n,k}t_{k,n-1}de_{n-2,1} \\
&+ \sum_k s_{n,k}t_{k,n-2}de_{n-1,1} \\
&= 0 \qquad (K^2d = 0 \text{ and } s_{i,j} = t_{i,j} = 0 \text{ for } i \le j),
\end{aligned}$$

$$\begin{aligned}
\omega(S)T + S\omega(T) &= (s_{n,n-1}ce_{n-1,1} + s_{n,n-1}de_{n-2,1} + s_{n,n-2}de_{n-1,1})S \\
&+ S(t_{n,n-1}ce_{n-1,1} + t_{n,n-1}de_{n-2,1} + t_{n,n-2}de_{n-1,1}) \\
&= \sum_k s_{n,n-1}ct_{1,k}e_{n-1,k} + \sum_k s_{n,n-1}dt_{1,k}e_{n-2,k} \\
&+ \sum_k s_{n,n-2}dt_{1,k}e_{n-1,k} + \sum_k s_{k,n-1}t_{n,n-1}ce_{k,1} \\
&+ \sum_k s_{k,n-2}t_{n,n-1}de_{k,1} + \sum_k s_{k,n-1}t_{n,n-2}de_{k,1} \\
&= s_{n,n-1}t_{n,n-1}ce_{n,1} \qquad (K^2d = 0 \textit{ and } s_{i,j} = t_{i,j} = 0 \textit{ for } i \leq j).
\end{aligned}$$

The following is an example of a Jordan derivation of $NT_3(K)$ that is not a derivation of the same ring.

Example 3.4 *[28] Let ρ be an additive map of a ring K with $\rho : xe_{2,1} \rightarrow axe_{3,2}$ where $a(K \circ K) = 0$. Then ρ is a Jordan derivation but not a derivation of $NT_3(K)$.*
To see that, choose arbitrary matrices $S, T \in NT_3(K)$. Then ρ is a Jordan derivation of $NT_3(K)$ because the right sides of the equalities

$$\begin{aligned}
\rho(S \circ T) &= \rho(ST + TS) \\
&= \rho(ST) + \rho(TS) \\
&= a\sum_k s_{2,k}t_{k,1}e_{3,2} + a\sum_k t_{2,k}s_{k,1}e_{3,2} \\
&= 0 \quad (s_{i,j} = t_{i,j} = 0 \textit{ for } i \leq j)
\end{aligned}$$

and

$$\begin{aligned}
\rho(S) \circ T + S \circ \rho(T) &= \rho(S)T + T\rho(S) + S\rho(T) + \rho(T)S \\
&= as_{2,1}e_{3,2}T + Tas_{2,1}e_{3,2} + Sat_{2,1}e_{3,2} + at_{2,1}e_{3,2}S \\
&= \sum_k as_{2,1}t_{2,k}e_{3,k} + \sum_k t_{k,3}as_{2,1}e_{k,2} \\
&+ \sum_k s_{k,3}at_{2,1}e_{k,2} + \sum_k at_{2,1}s_{2,k}e_{3,k} \\
&= as_{2,1}t_{2,1}e_{3,1} + at_{2,1}s_{2,1}e_{3,1} \quad (s_{i,j} = t_{i,j} = 0 \textit{ for } i \leq j) \\
&= a(s_{2,1} \circ t_{2,1})e_{3,1} \\
&= 0 \quad (a(K \circ K) = 0),
\end{aligned}$$

are equal. However, ρ is not a derivation of $NT_3(K)=0$ as a result of that the right sides of the equalities

$$
\begin{aligned}
\rho(ST) &= a\sum_k s_{2,k}t_{k,1}e_{3,2}\\
&= 0 \quad (s_{i,j}=t_{i,j}=0 \textit{ for } i\le j)
\end{aligned}
$$

and

$$
\begin{aligned}
\rho(S)T+S\rho(T) &= as_{2,1}e_{3,2}T + Sat_{2,1}e_{3,2}\\
&= \sum_k as_{2,1}t_{2,k}e_{3,k} + \sum_k s_{k,3}at_{2,1}e_{k,2}\\
&= as_{2,1}t_{2,1}e_{3,1} \quad (s_{i,j}=t_{i,j}=0 \textit{ for } i\le j)
\end{aligned}
$$

are different from each other.

The following is another example of a Jordan derivation of $NT_3(K)$ that is not a derivation of the same ring.

Example 3.5 *[28] The additive map $\bar{\rho}: xe_{3,2}\to axe_{2,1}$ of K is a Jordan derivation but not a derivation of $NT_3(K)$ if $ras+sar=0$ for all $r,s\in K$:*
Let S and T be arbitrary elements of $NT_3(K)$. Then $\bar{\rho}$ is a Jordan derivation because the right sides of the equalities

$$
\begin{aligned}
\bar{\rho}(S\circ T) &= \bar{\rho}(ST+TS)\\
&= \bar{\rho}(ST)+\bar{\rho}(TS)\\
&= a\sum_k s_{3,k}t_{k,2}e_{2,1} + a\sum_k t_{3,k}s_{k,2}e_{2,1}\\
&= 0 \quad (s_{i,j}=t_{i,j}=0 \textit{ for } i\le j)
\end{aligned}
$$

and

$$
\begin{aligned}
\bar{\rho}(S)\circ T + S\circ\bar{\rho}(T) &= \bar{\rho}(S)T + T\bar{\rho}(S) + S\bar{\rho}(T) + \bar{\rho}(T)S\\
&= as_{3,2}e_{2,1}T + Tas_{3,2}e_{2,1} + Sat_{3,2}e_{2,1} + at_{3,2}e_{2,1}S\\
&= \sum_k as_{3,2}t_{1,k}e_{2,k} + \sum_k t_{k,2}as_{3,2}e_{k,1}\\
&+ \sum_k s_{k,2}at_{3,2}e_{k,1} + \sum_k at_{3,2}s_{1,k}e_{2,k}\\
&= (t_{3,2}as_{3,2}+s_{3,2}at_{3,2})e_{3,1} \quad (s_{i,j}=t_{i,j}=0 \textit{ for } i\le j)\\
&= 0 \quad (ras+sar=0)
\end{aligned}
$$

are equal to each other. But $\bar{\rho}$ *is not a derivation of* $NT_3(K)$ *since* $\bar{\rho}(ST) \neq \bar{\rho}(S)T + S\bar{\rho}(T)$ *by*

$$\begin{aligned}\bar{\rho}(ST) &= a\sum_k s_{3,k}t_{k,2}e_{2,1} \\ &= 0 \quad (s_{i,j} = t_{i,j} = 0 \text{ for } i \leq j)\end{aligned}$$

and

$$\begin{aligned}\bar{\rho}(S)T + S\bar{\rho}(T) &= as_{3,2}e_{2,1}T + Sat_{3,2}e_{2,1} \\ &= \sum_k as_{3,2}t_{1,k}e_{2,k} + \sum_k s_{k,2}at_{3,2}e_{k,1} \\ &= s_{3,2}at_{3,2}e_{3,1} \qquad (s_{i,j} = t_{i,j} = 0 \text{ for } i \leq j).\end{aligned}$$

Definition 3.6 *A 2-torsion free ring* K *is a ring with the property that* $2x = 0$ *implies* $x = 0$ *for* $x \in K$.

It is clear that any field of characteristic not 2 is 2-torsion free, moreover, $M_2(\mathbb{Z})$ is 2-torsion free as well since $\mathbb{Z}$ is 2-torsion free.

If d is a Jordan derivation of a 2-torsion free ring K then the following equalities hold where r and s are arbitrary elements of K.

$$\begin{aligned} i)\ & d(r^2) = d(r)r + rd(r) \\ ii)\ & d(rsr) = d(r)sr + rd(s)r + rsd(r).\end{aligned}$$

A ring K is called prime if and only if $aKb = 0$ implies $a = 0$ or $b = 0$ where $a, b \in K$.

In 1957, Herstein proved the theorem given below:

Theorem 3.7 *[27, Theorem 3.1] Every Jordan derivation of a prime ring of characteristic different from 2 is an ordinary derivation.*

Taking into account that K is 2-torsion free in this section, a useful proposition can be given:

Proposition 3.8 *Let K be a 2-torsion free ring, $d : K \to K$ be an additive map and k be an arbitrary element of K. Then d is a Jordan derivation of K if and only if*

$$d(k^2) = d(k)k + kd(k).$$

Proof. Let K be a 2-torsion free ring, d be a Jordan derivation of K and $k \in K$. Forasmuch as

$$\begin{aligned} 2d(k^2) &= d(2k^2) \\ &= d(kk + kk) \\ &= d(k \circ k) \\ &= d(k)k + kd(k) + kd(k) + d(k)k \\ &= 2(d(k)k + kd(k)) \end{aligned}$$

one can deduce that $2\{d(k^2) - [d(k)k + kd(k)]\} = 0$. Then, as K is a 2-torsion free ring, it is obtained $d(k^2) = d(k)k + kd(k)$.

Now let d be an additive map of K satisfying $d(k^2) = d(k)k + kd(k)$ for all $k \in K$ and $s,\ v$ be arbitrary elements of K. In that right sides of the equations

$$\begin{aligned} d\left[(s+v)^2\right] &= d(s^2 + sv + vs + v^2) \\ &= d(s^2) + d(sv) + d(vs) + d(v^2) \\ &= d(s)s + sd(s) + d(sv) + d(vs) + d(v)v + vd(v) \end{aligned}$$

and

$$\begin{aligned} d\left[(s+v)^2\right] &= d(s+v).(s+v) + (s+v).d(s+v) \\ &= [d(s) + d(v)]\,(s+v) + (s+v)\,[d(s) + d(v)] \\ &= d(s)s + d(s)v + d(v)s + d(v)v \\ &\quad + vd(s) + sd(v) + vd(s) + vd(v) \end{aligned}$$

are equal, we get $d(sv + vs) = d(s)v + d(v)s + sd(v) + vd(s)$ which means d is a Jordan derivation of K. ■

Besides, we have the following property for Jordan derivations of 2-torsion free rings.

Proposition 3.9 *If K is a 2-torsion free ring and d is a Jordan derivation of K, then $d(wtw) = d(w)tw + wd(t)w + wtd(w)$ for all $w, t \in K$.*

Proof. Let $d : K \to K$ be a Jordan derivation and w, t be arbitrary elements of K. By the relations

$$\begin{aligned}
d(w \circ (w \circ t)) &= d(w) \circ (w \circ t) + w \circ d(w \circ t) \\
&= d(w) \circ (wt + tw) + w \circ (d(w) \circ t + w \circ d(t)) \\
&= d(w)(wt + tw) + (wt + tw)d(w) \\
&\quad + w \circ (d(w)t + td(w) + wd(t) + d(t)w) \\
&= d(w)wt + d(w)tw + wtd(w) + twd(w) \\
&\quad + wd(w)t + wtd(w) + w^2d(t) + wd(t)w \\
&\quad + d(w)tw + td(w)w + wd(t)w + d(t)w^2 \\
&= 2(d(w)tw + wd(t)w + wtd(w)) \\
&\quad + d(w)wt + twd(w) + wd(w)t \\
&\quad + w^2d(t) + td(w)w + d(t)w^2
\end{aligned}$$

and

$$\begin{aligned}
d(w \circ (w \circ t)) &= d(w \circ (wt + tw)) \\
&= d(w^2t + wtw + wtw + tw^2) \\
&= d(w^2 \circ t + 2wtw) \\
&= d(w^2 \circ t) + 2d(wtw) \\
&= d(w^2) \circ t + w^2 \circ d(t) + 2d(wtw) \\
&= d(w^2)t + td(w^2) + w^2d(t) + d(t)w^2 + 2d(wtw) \\
&= (d(w)w + wd(w))t + t(d(w)w + wd(w)) \text{ (Proposition 3.8)} \\
&\quad + w^2d(t) + d(t)w^2 + 2d(wtw) \\
&= d(w)wt + wd(w)t + td(w)w + twd(w) \\
&\quad + w^2d(t) + d(t)w^2 + 2d(wtw)
\end{aligned}$$

we get

$$2d(wtw) = 2(d(w)tw + wd(t)w + wtd(w))$$

and

$$d(wtw) = d(w)tw + wd(t)w + wtd(w)$$

as K is a 2-torsion free ring. ■

For many researchers, the usual goal has been to describe the nontrivial Jordan derivations.

The problem was studied for semi-prime rings (see [30],[25]) and certain algebras of triangular $n \times n$ matrices over a 2-torsion free commutative ring (see [24]). All Jordan derivations are trivial for semi-prime rings and any algebra of triangular $n \times n$ matrices over a 2-torsion free ring.

In 2011, Kuzucuğlu described the Jordan derivations of $NT_n(K)$.

Theorem 3.10 *[28] Every Jordan derivation of $NT_n(K)$ can be written as a sum of a derivation and an extremal Jordan derivation.*

From now on, we denote a 2-torsion free and associative ring with identity by K and an ideal of K by J. As it is stated before, we will show that any Jordan derivation of $R_n(K, J)$ can be written as a sum of a derivation and an extremal Jordan derivation. Now we define some extremal Jordan derivations as follows:

Proposition 3.11 *If the additive maps $\alpha, \beta, \gamma : J \to Ann_K(J)$ satisfy the conditions*

i) $\alpha(yx) = x\alpha(y)$

ii) $\beta(yx) = x\beta(y)$

iii) $\beta(xy) = \beta(y)x$

iv) $\gamma(xy) = \gamma(y)x$

v) $\alpha(J^2) = \beta(J^2) = \gamma(J^2) = 0$

for $x \in K$ and $y \in J$, then the map

$$\begin{array}{rccl} \Omega : & R_n(K, J) & \longrightarrow & R_n(K, J) \\ & ye_{1,n} & \longrightarrow & \alpha(y)e_{n-1,1} + \beta(y)e_{n-1,2} + \gamma(y)e_{n,2} \\ & ye_{1,n-1} & \longrightarrow & \alpha(y)e_{n,1} + \beta(y)e_{n,2} \\ & ye_{2,n-1} & \longrightarrow & \beta(y)e_{n,1} \\ & ye_{2,n} & \longrightarrow & \beta(y)e_{n-1,1} + \gamma(y)e_{n,1} \\ & x_{i,j}e_{i,j} & \longrightarrow & 0 \quad ((i,j) \neq (1,n), (1,n-1), (2,n-1), (2,n)) \end{array}$$

determines a Jordan derivation of the ring R which will be called an extremal Jordan derivation.

Proof. Let X and Y be arbitrary elements of R and $\alpha, \beta, \gamma : J \to Ann_K(J)$ be additive maps satisfying $\alpha(yx) = x\alpha(y)$, $\beta(yx) = x\beta(y)$, $\beta(xy) = \beta(y)x$, $\gamma(xy) = \gamma(y)x$, $\alpha(J^2) = 0$, $\beta(J^2) = 0$ and $\gamma(J^2) = 0$. Then

$$\begin{aligned}
\Omega(X \circ Y) &= \Omega(XY + YX) \\
&= \left[\sum_{k=1}^{n} \alpha(x_{1,k}y_{k,n} + y_{1,k}x_{k,n})\right] e_{n-1,1} \\
&+ \left[\sum_{k=1}^{n} \beta(x_{1,k}y_{k,n} + y_{1,k}x_{k,n})\right] e_{n-1,2} \\
&+ \left[\sum_{k=1}^{n} \gamma(x_{1,k}y_{k,n} + y_{1,k}x_{k,n})\right] e_{n,2} \\
&+ \left[\sum_{k=1}^{n} \alpha(x_{1,k}y_{k,n-1} + y_{1,k}x_{k,n-1})\right] e_{n,1} \\
&+ \left[\sum_{k=1}^{n} \beta(x_{1,k}y_{k,n-1} + y_{1,k}x_{k,n-1})\right] e_{n,2} \\
&+ \left[\sum_{k=1}^{n} \beta(x_{2,k}y_{k,n-1} + y_{2,k}x_{k,n-1})\right] e_{n,1} \\
&+ \left[\sum_{k=1}^{n} \beta(x_{2,k}y_{k,n} + y_{2,k}x_{k,n})\right] e_{n-1,1} \\
&+ \left[\sum_{k=1}^{n} \gamma(x_{2,k}y_{k,n} + y_{2,k}x_{k,n})\right] e_{n,1} \\
&= [\alpha(x_{1,n}y_{n,n-1} + y_{1,n}x_{n,n-1}) + \beta(x_{2,1}y_{1,n-1} + y_{2,1}x_{1,n-1})]e_{n,1} \\
&+ [\beta(x_{2,n}y_{n,n-1} + y_{2,n}x_{n,n-1}) + \gamma(x_{2,1}y_{1,n} + y_{2,1}x_{1,n})]e_{n,1} \\
&+ [\beta(x_{2,1}y_{1,n} + y_{2,1}x_{1,n})]\, e_{n-1,1} \\
&+ [\beta(x_{1,n}y_{n,n-1} + y_{1,n}x_{n,n-1})]\, e_{n,2}
\end{aligned}$$

considering that $\alpha(J^2) = 0$, $\beta(J^2) = 0$, $\gamma(J^2) = 0$. On the other hand

$$
\begin{aligned}
\Omega(X) \circ Y + X \circ \Omega(Y) &= \Omega(X)Y + Y\Omega(X) + X\Omega(Y) + \Omega(Y)X \\
&= \left[\sum_{k=1}^{n} \alpha(x_{1,n})y_{1,k} + \alpha(y_{1,n})x_{1,k}\right] e_{n-1,k} \\
&+ \left[\sum_{k=1}^{n} \beta(x_{1,n})y_{2,k} + \beta(y_{1,n})x_{2,k}\right] e_{n-1,k} \\
&+ \left[\sum_{k=1}^{n} \gamma(x_{1,n})y_{2,k} + \gamma(y_{1,n})x_{2,k}\right] e_{n,k} \\
&+ \left[\sum_{k=1}^{n} \alpha(x_{1,n-1})y_{1,k} + \alpha(y_{1,n-1})x_{1,k}\right] e_{n,k} \\
&+ \left[\sum_{k=1}^{n} \beta(x_{1,n-1})y_{2,k} + \beta(y_{1,n-1})x_{2,k}\right] e_{n,k} \\
&+ \left[\sum_{k=1}^{n} \beta(x_{2,n-1})y_{1,k} + \beta(y_{2,n-1})x_{1,k}\right] e_{n,k} \\
&+ \left[\sum_{k=1}^{n} \beta(x_{2,n})y_{1,k} + \beta(y_{2,n})x_{1,k}\right] e_{n-1,k} \\
&+ \left[\sum_{k=1}^{n} \gamma(x_{2,n})y_{1,k} + \gamma(y_{2,n})x_{1,k}\right] e_{n,k} \\
&= [y_{n,n-1}\alpha(x_{1,n}) + x_{n,n-1}\alpha(y_{1,n}) + \beta(y_{1,n-1})x_{2,1}]e_{n,1} \\
&+ [\beta(x_{1,n-1})y_{2,1} + y_{n,n-1}\beta(x_{2,n}) + x_{n,n-1}\beta(y_{2,n})]e_{n,1} \\
&+ [\gamma(y_{1,n})x_{2,1} + \gamma(x_{1,n})y_{2,1}]e_{n,1} \\
&+ [\beta(y_{1,n})x_{2,1} + \beta(x_{1,n})y_{2,1}]\, e_{n-1,1} \\
&+ [y_{n,n-1}\beta(x_{1,n}) + x_{n,n-1}\beta(y_{1,n})]\, e_{n,2}
\end{aligned}
$$

since $\alpha(J), \beta(J), \gamma(J)$ are all contained in $Ann_K(J)$. Now, by considering the conditions $i) - iv)$, one can easily see that

$$\Omega(X) \circ Y + X \circ \Omega(Y) = \Omega(X \circ Y)$$

which completes the proof. ■

Example 3.12 *Let K_1 be a commutative ring with identity and let J_1 be an ideal of K_1 which is nilpotent of class two. If $K = K_1 \times K_1$ and $J = J_1 \times J_1$ is an ideal of K,* *then the maps*

$$\alpha : J \to Ann_K(J)$$
$$: (a,b) \to (a,0)$$

$$\beta : J \to Ann_K(J)$$
$$: (a,b) \to (0,b)$$

$$\gamma : J \to Ann_K(J)$$
$$: (a,b) \to (a,b)$$

are all additive and satisfy all the conditions i)-v). This means

$$\begin{array}{rccl} \Omega: & R_n(K,J) & \longrightarrow & R_n(K,J) \\ & ye_{1,n} & \longrightarrow & \alpha(y)e_{n-1,1} + \beta(y)e_{n-1,2} + \gamma(y)e_{n,2} \\ & ye_{1,n-1} & \longrightarrow & \alpha(y)e_{n,1} + \beta(y)e_{n,2} \\ & ye_{2,n-1} & \longrightarrow & \beta(y)e_{n,1} \\ & ye_{2,n} & \longrightarrow & \beta(y)e_{n-1,1} + \gamma(y)e_{n,1} \\ & x_{i,j}e_{i,j} & \longrightarrow & 0 \quad ((i,j) \neq (1,n),(1,n-1),(2,n-1),(2,n)) \end{array}$$

is a Jordan derivation of $R_n(K,J)$.

Proposition 3.13 *If additive maps* $\alpha_1, \alpha_2 : J \to Ann_K(J)$ *satisfy the conditions*

i) $\alpha_1(xy) = \alpha_1(y)x$

ii) $\alpha_2(yx) = x\alpha_2(y)$

iii) $\alpha_1(J^2) = 0 = \alpha_2(J^2)$

for $x \in K$ *and* $y \in J$, *then the map*

$$\begin{array}{rccl} F: & R_3(K,J) & \to & R_3(K,J) \\ & ye_{1,3} & \to & \alpha_1(y)e_{3,2} + \alpha_2(y)e_{2,1} \\ & ye_{1,2} & \to & \alpha_2(y)e_{3,1} \\ & ye_{2,3} & \to & \alpha_1(y)e_{3,1} \\ & x_{i,j}e_{i,j} & \to & 0 \quad ((i,j) \neq (1,2),(1,3),(2,3)) \end{array}$$

determines a Jordan derivation which will be called an extremal Jordan derivation as well.

Proof. Let $P = [p_{i,j}]$ and $Q = [q_{i,j}]$ be arbitrary matrices in $R_3(K, J)$ and $\alpha_1, \alpha_2 : J \to Ann_K(J)$ be additive maps with the properties $i)$, $ii)$ and $iii)$. $\digamma$ is clearly an additive map by definition. In addition, right sides of the two equalities

$$\begin{aligned}
\digamma(P) \circ Q + P \circ \digamma(Q) &= \digamma(P)Q + Q\digamma(P) + P\digamma(Q) + \digamma(Q)P \\
&= [\alpha_1(p_{1,3})e_{3,2} + \alpha_2(p_{1,3})e_{2,1} + \alpha_2(p_{1,2})e_{3,1} + \alpha_1(p_{2,3})e_{3,1}]Q \\
&\quad + Q[\alpha_1(p_{1,3})e_{3,2} + \alpha_2(p_{1,3})e_{2,1} + \alpha_2(p_{1,2})e_{3,1} + \alpha_1(p_{2,3})e_{3,1}] \\
&\quad + P[\alpha_1(q_{1,3})e_{3,2} + \alpha_2(q_{1,3})e_{2,1} + \alpha_2(q_{1,2})e_{3,1} + \alpha_1(q_{2,3})e_{3,1}] \\
&\quad + [\alpha_1(q_{1,3})e_{3,2} + \alpha_2(q_{1,3})e_{2,1} + \alpha_2(q_{1,2})e_{3,1} + \alpha_1(q_{2,3})e_{3,1}]P \\
&= \sum_k \alpha_1(p_{1,3})q_{2,k}e_{3,k} + \sum_k \alpha_2(p_{1,3})q_{1,k}e_{2,k} \\
&\quad + \sum_k \alpha_2(p_{1,2})q_{1,k}e_{3,k} + \sum_k \alpha_1(p_{2,3})q_{1,k}e_{3,k} \\
&\quad + \sum_k q_{k,3}\alpha_1(q_{1,3})e_{k,2} + \sum_k q_{k,2}\alpha_2(p_{1,3})e_{k,1} \\
&\quad + \sum_k q_{k,3}\alpha_2(p_{1,2})e_{k,1} + \sum_k q_{k,3}\alpha_1(p_{2,3})e_{k,1} \\
&\quad + \sum_k p_{k,3}\alpha_1(q_{1,3})e_{k,2} + \sum_k p_{k,2}\alpha_2(q_{1,3})e_{k,1} \\
&\quad + \sum_k p_{k,3}\alpha_2(q_{1,2})e_{k,1} + \sum_k p_{k,3}\alpha_1(q_{2,3})e_{k,1} \\
&\quad + \sum_k \alpha_1(q_{1,3})p_{2,k}e_{3,k} + \sum_k \alpha_2(q_{1,3})p_{1,k}e_{2,k} \\
&\quad + \sum_k \alpha_2(q_{1,2})p_{1,k}e_{3,k} + \sum_k \alpha_1(q_{2,3})p_{1,k}e_{3,k} \\
&= \alpha_1(p_{1,3})y_{2,1}e_{3,1} + q_{3,2}\alpha_2(p_{1,3})e_{3,1} \\
&+ p_{3,2}\alpha_2(q_{1,3})e_{3,1} + \alpha_1(q_{1,3})x_{2,1}e_{3,1}
\end{aligned}$$

and

$$\begin{aligned}
\digamma(P \circ Q) &= \digamma(PQ + QP) \\
&= \digamma(PQ) + \digamma(QP) \\
&= \sum_k \alpha_1(p_{1,k}q_{k,3})e_{3,2} + \sum_k \alpha_2(p_{1,k}q_{k,3})e_{2,1} + \sum_k \alpha_2(p_{1,k}q_{k,2})e_{3,1} \\
&\quad + \sum_k \alpha_1(p_{2,k}q_{k,3})e_{3,1} + \sum_k \alpha_1(q_{1,k}p_{k,3})e_{3,2} + \sum_k \alpha_2(q_{1,k}p_{k,3})e_{2,1} \\
&\quad + \sum_k \alpha_2(q_{1,k}p_{k,2})e_{3,1} + \sum_k \alpha_1(q_{2,k}p_{k,3})e_{3,1} \\
&= [\alpha_2(p_{1,3}q_{3,2}) + \alpha_1(p_{2,1}q_{1,3}) \\
&\quad + \alpha_2(q_{1,3}p_{3,2}) + \alpha_1(q_{2,1}p_{1,3})]e_{3,1} \quad (\alpha_1(J^2) = 0 = \alpha_2(J^2))
\end{aligned}$$

are equal by *i)* and *ii)*. Then $\digamma(P \circ Q) = \digamma(P) \circ Q + P \circ \digamma(Q)$ and the proof is completed. ■

Proposition 3.14 *If $\delta_i : J \to J$, $\beta_i : J \to K$, $\theta : J \to K$ and $\gamma : J \to K$ $(i = 1, 2, 3)$ are additive maps satisfying*

$$\begin{array}{ll}
\delta_2(J^2) = 0, & \gamma(y)z = \delta_3(yz), \\
\beta_2(J) \subseteq Ann_K(J), & y\beta_3(z) = \delta_1(yz), \\
\delta_1(yz) = y\theta(z), & \delta_3(y)x + x\delta_2(y) = \gamma(yx), \\
\theta(yz) = y\beta_3(z), & x\theta(y) = \beta_2(yx) + \beta_3(xy), \\
\gamma(yz) = \gamma(y)z, & \theta(xy) = x\delta_1(y) + \delta_2(y)x, \\
\delta_3(yz) = \beta_1(y)z, & \gamma(y)x = \beta_1(yx) + \beta_2(xy), \\
\theta(yz) = y\theta(z), & z\gamma(y) + \delta_1(z)y + y\delta_2(z) = 0, \\
\gamma(yz) = \beta_1(y)z, & \delta_2(y)z + z\delta_3(y) + \theta(z)y = 0, \\
z\gamma(y) + \theta(z)y = 0, & \beta_1(yz + zy) = \beta_1(y)z + \beta_1(z)y, \\
z\gamma(y) + \beta_3(z)y = 0, & \delta_3(yz) = \delta_3(y)z + z\delta_3(y) + \beta_3(z)y, \\
\theta(y)z + y\beta_1(z) = 0, & \delta_1(yz) = z\beta_1(y) + y\delta_1(z) + \delta_1(z)y, \\
\delta_2(y)z + z\delta_2(y) = 0, & z\delta_1(y) + \delta_3(y)z = \beta_1(yz) + \beta_3(yz), \\
z\beta_1(y) + \beta_3(z)y = 0, & \delta_1(y)z + z\delta_3(y) + \delta_1(z)y + y\delta_3(z) = 0,
\end{array}$$

then the following map is a Jordan derivation of $R_3(K,J)$ *where* $x \in K$, $y, z \in J$.

$$\begin{aligned}
\Upsilon: \quad ye_{1,3} &\rightarrow \sum_{i=1}^{3}\delta_i(y)e_{i,i} \\
ye_{i,i} &\rightarrow \beta_i(y)e_{3,1} \\
ye_{2,3} &\rightarrow \theta(y)e_{2,1} \\
ye_{1,2} &\rightarrow \gamma(y)e_{3,2}
\end{aligned}$$

We assume that the images of the elementary matrices except $ye_{1,3}$, $ye_{1,2}$, $ye_{2,3}$ *and* $ye_{i,i}$ $(i = 1, 2, 3)$ *are zeros.*

Proof. Let $X = [x_{i,j}]$, $Y = [y_{i,j}]$ be arbitrary matrices in $R_3(K,J)$. We know that

$$\begin{aligned}
\Upsilon(X \circ Y) &= \Upsilon(XY + YX) \\
&= \Upsilon(XY) + \Upsilon(YX) \\
&= \sum_{i=1}^{3}\sum_{k=1}^{3}\delta_i(x_{1,k}y_{k,3})e_{i,i} + \sum_{i=1}^{3}\sum_{k=1}^{3}\delta_i(y_{1,k}x_{k,3})e_{i,i} \\
&\quad + \sum_{i=1}^{3}\sum_{k=1}^{3}\beta_i(x_{i,k}y_{k,i})e_{3,1} + \sum_{i=1}^{3}\sum_{k=1}^{3}\beta_i(y_{i,k}x_{k,i})e_{3,1} \\
&\quad + \sum_{k=1}^{3}\theta(x_{2,k}y_{k,3})e_{2,1} + \sum_{k=1}^{3}\theta(y_{2,k}x_{k,3})e_{2,1} \\
&\quad + \sum_{k=1}^{3}\gamma(x_{1,k}y_{k,2})e_{3,2} + \sum_{k=1}^{3}\gamma(y_{1,k}x_{k,2})e_{3,2}
\end{aligned}$$

and

$$\begin{aligned}
\Upsilon(X)\circ Y + X\circ \Upsilon(Y) &= \Upsilon(X)Y + Y\Upsilon(X) + X\Upsilon(Y) + \Upsilon(Y)X \\
&= \begin{pmatrix} \sum_{i=1}^{3}\delta_i(x_{1,3})e_{i,i} + \sum_{i=1}^{3}\beta_i(x_{i,i})e_{3,1} \\ +\theta(x_{2,3})e_{2,1} + \gamma(x_{1,2})e_{3,2} \end{pmatrix} Y \\
&+ Y \begin{pmatrix} \sum_{i=1}^{3}\delta_i(x_{1,3})e_{i,i} + \sum_{i=1}^{3}\beta_i(x_{i,i})e_{3,1} \\ +\theta(x_{2,3})e_{2,1} + \gamma(x_{1,2})e_{3,2} \end{pmatrix} \\
&+ X \begin{pmatrix} \sum_{i=1}^{3}\delta_i(y_{1,3})e_{i,i} + \sum_{i=1}^{3}\beta_i(y_{i,i})e_{3,1} \\ +\theta(y_{2,3})e_{2,1} + \gamma(y_{1,2})e_{3,2} \end{pmatrix} \\
&+ \begin{pmatrix} \sum_{i=1}^{3}\delta_i(y_{1,3})e_{i,i} + \sum_{i=1}^{3}\beta_i(y_{i,i})e_{3,1} \\ +\theta(y_{2,3})e_{2,1} + \gamma(y_{1,2})e_{3,2} \end{pmatrix} X \\
&= \begin{pmatrix} \sum_{k=1}^{3}\sum_{i=1}^{3}\delta_i(x_{1,3})y_{i,k}e_{i,k} + \sum_{k=1}^{3}\sum_{i=1}^{3}\beta_i(x_{i,i})y_{1,k}e_{3,k} \\ + \sum_{k=1}^{3}\theta(x_{2,3})y_{1,k}e_{2,k} + \sum_{k=1}^{3}\gamma(x_{1,2})y_{2,k}e_{3,k} \end{pmatrix} \\
&+ \begin{pmatrix} \sum_{k=1}^{3}\sum_{i=1}^{3}y_{k,i}\delta_i(x_{1,3})e_{k,i} + \sum_{k=1}^{3}\sum_{i=1}^{3}y_{k,3}\beta_i(x_{i,i})e_{k,1} \\ + \sum_{k=1}^{3}y_{k,2}\theta(x_{2,3})e_{k,1} + \sum_{k=1}^{3}y_{k,3}\gamma(x_{1,2})e_{k,2} \end{pmatrix} \\
&+ \begin{pmatrix} \sum_{k=1}^{3}\sum_{i=1}^{3}x_{k,i}\delta_i(y_{1,3})e_{k,i} + \sum_{k=1}^{3}\sum_{i=1}^{3}x_{k,3}\beta_i(y_{i,i})e_{k,1} \\ + \sum_{k=1}^{3}x_{k,2}\theta(y_{2,3})e_{k,1} + \sum_{k=1}^{3}x_{k,3}\gamma(y_{1,2})e_{k,2} \end{pmatrix} \\
&+ \begin{pmatrix} \sum_{k=1}^{3}\sum_{i=1}^{3}\delta_i(y_{1,3})x_{i,k}e_{i,k} + \sum_{k=1}^{3}\sum_{i=1}^{3}\beta_i(y_{i,i})x_{1,k}e_{3,k} \\ + \sum_{k=1}^{3}\theta(y_{2,3})x_{1,k}e_{2,k} + \sum_{k=1}^{3}\gamma(y_{1,2})x_{2,k}e_{3,k} \end{pmatrix}.
\end{aligned}$$

Then we obtain $\Upsilon(X\circ Y) = \Upsilon(X)\circ Y + X\circ\Upsilon(Y)$ by using the given conditions for additive maps δ_i, β_i and θ $(i = 1, 2, 3)$. ■

Theorem 3.15 *Every Jordan derivation of $R_n(K, J)$ for $n \geq 4$ is of the form $\Delta = \Phi + \Omega$ where Φ is a derivation of $R_n(K, J)$ and Ω is an extremal Jordan derivation of $R_n(K, J)$. Moreover, Φ is the sum of certain diagonal, inner, annihilator, ring and almost annihilator derivations.*

Before we prove the theorem, the following helpful lemmas will be given.

Lemma 3.16 *Let Δ be an arbitrary Jordan derivation of R for $n \geq 4$. Then for $1 < i < n-1$ and $x \in K$, $y \in J$, we have*

$$\Delta(xe_{i+1,i}) = \sum \Delta^{i+1,i}_{i+1,t}(x)e_{i+1,t} + \sum_{s \neq i+1} \Delta^{i+1,i}_{s,i}(x)e_{s,i} + \Delta^{i+1,i}_{n,1}(x)e_{n,1} \tag{1}$$

which can be written as

$$\begin{bmatrix}
0 & 0 & \cdots & 0 & \Delta^{i+1,i}_{1,i}(x) & 0 & \cdots & 0 \\
\vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\
0 & 0 & \cdots & 0 & \Delta^{i+1,i}_{i,i}(x) & 0 & \cdots & 0 \\
\Delta^{i+1,i}_{i+1,1}(x) & \Delta^{i+1,i}_{i+1,2}(x) & \cdots & \Delta^{i+1,i}_{i+1,i-1}(x) & \Delta^{i+1,i}_{i+1,i}(x) & \Delta^{i+1,i}_{i+1,i+1}(x) & \cdots & \Delta^{i+1,i}_{i+1,n}(x) \\
0 & 0 & \cdots & 0 & \Delta^{i+1,i}_{i+2,i}(x) & 0 & \cdots & 0 \\
\vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\
0 & 0 & \cdots & 0 & \Delta^{i+1,i}_{n-1,i}(x) & 0 & \cdots & 0 \\
\Delta^{i+1,i}_{n,1}(x) & 0 & \cdots & 0 & \Delta^{i+1,i}_{n,i}(x) & 0 & \cdots & 0
\end{bmatrix}$$

and

$$\Delta(ye_{1,n}) = \sum \Delta^{1,n}_{1,t}(y)e_{1,t} + \sum_{s \neq 1} \Delta^{1,n}_{s,n}(y)e_{s,n} + \Delta^{1,n}_{n-1,1}(y)e_{n-1,1}$$
$$+\Delta^{1,n}_{n-1,2}(y)e_{n-1,2} + \Delta^{1,n}_{n,1}(y)e_{n,1} + \Delta^{1,n}_{n,2}(y)e_{n,2} \tag{2}$$

which is equal to

$$\begin{bmatrix}
\Delta^{1,n}_{1,1}(y) & \Delta^{1,n}_{1,2}(y) & \Delta^{1,n}_{1,3}(y) & \cdots & \Delta^{1,n}_{1,n-1}(y) & \Delta^{1,n}_{1,n}(y) \\
0 & 0 & 0 & \cdots & 0 & \Delta^{1,n}_{2,n}(y) \\
\vdots & \vdots & \vdots & & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 0 & \Delta^{1,n}_{n-2,n}(y) \\
\Delta^{1,n}_{n-1,1}(y) & \Delta^{1,n}_{n-1,2}(y) & 0 & \cdots & 0 & \Delta^{1,n}_{n-1,n}(y) \\
\Delta^{1,n}_{n,1}(y) & \Delta^{1,n}_{n,2}(y) & 0 & \cdots & 0 & \Delta^{1,n}_{n,n}(y)
\end{bmatrix}.$$

Proof. Let us fix i, j and choose k, m such that $k > m$. If $k \neq j$ and $m \neq i$ then $x_{i,j}e_{i,j} \circ y_{k,m}e_{k,m} = 0$. By differentiating $x_{i,j}e_{i,j} \circ y_{k,m}e_{k,m} = 0$, we get

$$\begin{aligned}
0 &= \Delta(x_{i,j}e_{i,j}) \circ y_{k,m}e_{k,m} + x_{i,j}e_{i,j} \circ \Delta(y_{k,m}e_{k,m}) \\
&= \Delta(x_{i,j}e_{i,j})y_{k,m}e_{k,m} + y_{k,m}e_{k,m}\Delta(x_{i,j}e_{i,j}) + x_{i,j}e_{i,j}\Delta(y_{k,m}e_{k,m}) \\
&\quad + \Delta(y_{k,m}e_{k,m})x_{i,j}e_{i,j} \\
&= \sum_s \Delta^{i,j}_{s,k}(x_{i,j})y_{k,m}e_{s,m} + \sum_t y_{k,m}\Delta^{i,j}_{m,t}(x_{i,j})e_{k,t} \\
&\quad + \sum_t x_{i,j}\Delta^{k,m}_{j,t}(y_{k,m})e_{i,t} + \sum_s \Delta^{k,m}_{s,i}(y_{k,m})x_{i,j}e_{s,j}
\end{aligned}$$

Putting $y_{k,m} = 1$, the matrix on the right has zeros except $i-th$, $k-th$ rows and $j-th$, $m-th$ columns. Hence we have

$$\Delta_{s,k}^{i,j} = 0 \text{ for } m \neq j,\ s \neq i,\ s \neq k, \tag{3}$$

$$\Delta_{m,t}^{i,j} = 0 \text{ for } i \neq k,\ t \neq m,\ t \neq j. \tag{4}$$

On the other hand, for $k > s > m$ and $k \neq i, j$, $s \neq i, j$, $m \neq i, j$, the (k, m), (s, m), (k, s) coefficients of the equations $\Delta(x_{i,j}e_{i,j} \circ e_{k,m}) = 0$, $\Delta(x_{i,j}e_{i,j} \circ e_{s,m}) = 0$, $\Delta(x_{i,j}e_{i,j} \circ e_{k,s}) = 0$ are

$$\Delta_{k,k}^{i,j}(x_{i,j}) + \Delta_{m,m}^{i,j}(x_{i,j}) = 0, \tag{5}$$

$$\Delta_{s,s}^{i,j}(x_{i,j}) + \Delta_{m,m}^{i,j}(x_{i,j}) = 0, \tag{6}$$

$$\Delta_{k,k}^{i,j}(x_{i,j}) + \Delta_{s,s}^{i,j}(x_{i,j}) = 0, \tag{7}$$

respectively. Comparing (5) with (6) and (7), we get $\Delta_{k,k}^{i,j}(x_{i,j}) = \Delta_{s,s}^{i,j}(x_{i,j}) = \Delta_{m,m}^{i,j}(x_{i,j})$. By using (7), it can be easily seen that $2\Delta_{k,k}^{i,j} = 2\Delta_{s,s}^{i,j} = 0 = 2\Delta_{m,m}^{i,j}$. Now that K is a 2-torsion free ring, we obtain $\Delta_{k,k}^{i,j} = 0$ for all $k \neq i, j$. Therefore, the image of $xe_{i+1,i}$ $(x \in K)$ under Δ is the matrix with zeros outside $(i+1)-th$ row, $i-th$ column and $(n, 1)$ position and $\Delta(xe_{i+1,i})$ has the form (1) for $1 < i < n-1$. In particular, $\Delta(xe_{2,1}) = \sum \Delta_{2,t}^{2,1}(x)e_{2,t} + \sum_{s\neq 2} \Delta_{s,1}^{2,1}(x)e_{s,1} + \Delta_{n,2}^{2,1}(x)e_{n,2} + \Delta_{n,3}^{2,1}(x)e_{n,3}$ and $\Delta(xe_{n,n-1}) = \sum \Delta_{n,t}^{n,n-1}(x)e_{n,t} + \sum_{s\neq n} \Delta_{s,n-1}^{n,n-1}(x)e_{s,n-1} + \Delta_{n-1,1}^{n,n-1}(x)e_{n-1,1} + \Delta_{n-2,1}^{n,n-1}(x)e_{n-2,1}$. By (3) and (4), we get $\Delta_{n-1,1}^{1,n}(y) \neq 0$ because $k \neq n$, $k \neq 1$, $m \neq n-1$ and $\Delta_{n-1,2}^{1,n}(y) \neq 0$ while $m \neq n-1$, $k \neq 2$ for $y \in J$. Similarly, $\Delta_{n,1}^{1,n}(y) \neq 0$ since $k \neq 1$, $m \neq n$ and $\Delta_{n,2}^{1,n}(y) \neq 0$ as $m \neq n$, $k \neq 2$ for $y \in J$. Thus we get (2). ■

Lemma 3.17 *Let $\Delta : R \to R$ be a Jordan derivation. Then there can be found a diagonal derivation δ_D of R such that $(i+1, i)-th$ coefficient of $(\Delta - \delta_D)(e_{i+1,i})$ is zero.*

Proof. Let $D = \sum_{i=2}^{n} d_i e_{i,i}$ where $d_{i+1} = \sum_{k=1}^{i} a_k$ and $a_k = \Delta_{k+1,k}^{k+1,k}(1)$. Then there exists a diagonal derivation $\delta_D : X \to DX - XD$ induced by the diagonal matrix D such that $\delta_D(e_{i+1,i}) = De_{i+1,i} - e_{i+1,i}D = \Delta_{i+1,i}^{i+1,i}(1)e_{i+1,i}$. Since $(i+1, i)-th$ coefficient of the matrix Δ is equal to $\Delta_{i+1,i}^{i+1,i}(1)$, the proof is completed. ■

Lemma 3.18 *Let $\Delta : R \to R$ be a Jordan derivation and $(i+1,i)-th$ coefficient of $\Delta(e_{i+1,i})$ is zero for all $1 \leq i < n$. Then there is an inner derivation I satisfying that $(\Delta - I)\,(e_{i+1,i})$ has zero $i-th$ column and $(i+1,1)$ entry.*

Proof. Define a matrix $A = [A_{i,j}]_{n\times n}$ with $A_{v,v} = 0 = A_{j,1}, \;\; A_{u,i+1} = \varphi_{u,i}^{i+1,i}(1)$ $(u \neq i+1,\, 1 \leq i < n)$. Clearly A is equal to

$$\begin{bmatrix} 0 & \varphi_{1,1}^{2,1}(1) & \varphi_{1,2}^{3,2}(1) & . \; . \; . & \varphi_{1,n-1}^{n,n-1}(1) \\ 0 & 0 & \varphi_{2,2}^{3,2}(1) & . \; . \; . & \varphi_{2,n-1}^{n,n-1}(1) \\ 0 & \varphi_{3,1}^{2,1}(1) & 0 & . \; . \; . & \varphi_{3,n-1}^{n,n-1}(1) \\ 0 & \varphi_{4,1}^{2,1}(1) & \varphi_{4,2}^{3,2}(1) & . \; . \; . & \varphi_{4,n-1}^{n,n-1}(1) \\ . & . & . & & . \\ . & . & . & & . \\ . & . & . & & . \\ 0 & \varphi_{n-1,1}^{2,1}(1) & \varphi_{n-1,2}^{3,2}(1) & . \; . \; . & \varphi_{n-1,n-1}^{n,n-1}(1) \\ 0 & \varphi_{n,1}^{2,1}(1) & \varphi_{n,2}^{3,2}(1) & . \; . \; . & 0 \end{bmatrix}.$$

Now consider the action of the inner derivation I_A on the matrices $e_{i+1,i}$. Then $I_A(e_{i+1,i}) = Ae_{i+1,i} - e_{i+1,i}A = \sum\limits_{\substack{k=1 \\ k\neq i+1}}^{n} \Delta_{k,i}^{i+1,i}(1)e_{k,i} + \sum\limits_{\substack{m=1 \\ m\neq i-1}}^{n-1} [-\Delta_{i,m}^{m+1,m}(1)]e_{i+1,m+1}$ which is equal to

$$\begin{bmatrix} 0 & 0 & . \; . \; . & 0 & \varphi_{1,i}^{i+1,i}(1) & 0 & . \; . \; . & 0 \\ 0 & 0 & . \; . \; . & 0 & \varphi_{2,i}^{i+1,i}(1) & 0 & . \; . \; . & 0 \\ . & . & & . & . & . & & . \\ . & . & & . & . & . & & . \\ . & . & & . & . & . & & . \\ 0 & 0 & . \; . \; . & 0 & \varphi_{i,i}^{i+1,i}(1) & 0 & . \; . \; . & 0 \\ 0 & -\varphi_{i,1}^{2,1}(1) & . \; . \; . & -\varphi_{i,i-2}^{i-1,i-2}(1) & 0 & -\varphi_{i,i}^{i+1,i}(1) & . \; . \; . & -\varphi_{i,n-1}^{n,n-1}(1) \\ 0 & 0 & . \; . \; . & 0 & \varphi_{i+2,i}^{i+1,i}(1) & 0 & . \; . \; . & 0 \\ . & . & & . & . & . & & . \\ . & . & & . & . & . & & . \\ . & . & & . & . & . & & . \\ 0 & 0 & . \; . \; . & 0 & \varphi_{n,i}^{i+1,i}(1) & 0 & . \; . \; . & 0 \end{bmatrix}.$$

Therefore, i-th column of each matrix $(\Delta - I_A)(e_{i+1,i})$ is equal to zero. Now define a matrix

$$B = \begin{bmatrix} 0 & 0 & . \; . \; . & 0 & 0 \\ -b_3 & 0 & . \; . \; . & 0 & 0 \\ . & . & & . & . \\ . & . & & . & . \\ . & . & & . & . \\ -b_n & 0 & . \; . \; . & 0 & 0 \\ 0 & 0 & . \; . \; . & 0 & 0 \end{bmatrix}$$

and denote by I_B the inner derivation induced by the matrix B. It can be easily seen that $I_B(e_{i+1,i}) = Be_{i+1,i} - e_{i+1,i}B = \Delta_{i+1,1}^{i+1,i}(1)e_{i+1,1}$ for $i = 2, ..., n-1$. Hence i-th columns and (i+1,1) entries of the matrices $(\Delta - I)(e_{i+1,i})$ are zeros for $I = I_A + I_B$ and this comletes the proof. ∎

Lemma 3.19 *Let $\Delta : R \to R$ be a Jordan derivation such that $i-th$ columns and $(i+1,1)$ entries of the matrices $\Delta(e_{i+1,i})$ are all zeros. Then the following equalities are obtained for $x_{k,m} \in I_{k,m}$, $x \in K$ and $y \in J$;*

$$\Delta(e_{2,1}) = 0 \tag{1}$$

$$\Delta(e_{n,n-1}) = 0 \tag{2}$$

$$\Delta(e_{i+1,i}) = \Delta_{n,1}^{i+1,i}(1)e_{n,1} \ , \ 1 < i < n-1 \tag{3}$$

$$\Delta(e_{i,j}) = 0 \ , \ i - j > 1 \tag{4}$$

$$\Delta(xe_{i,1}) = \Delta_{i,1}^{i,1}(x)e_{i,1} + \Delta_{n,1}^{i,1}(x)e_{n,1} \ , \ 1 < i < n \tag{5}$$

$$\Delta(x_{n,j}e_{n,j}) = \Delta_{n,1}^{n,j}(x_{n,j})e_{n,1} + \Delta_{n,j}^{n,j}(x_{n,j})e_{n,j} \tag{6}$$

$$\begin{aligned} \Delta(ye_{1,i}) &= \Delta_{1,1}^{1,i}(y)e_{1,1} + \Delta_{1,2}^{1,i}(y)e_{1,2} \\ &+ \Delta_{1,i}^{1,i}(y)e_{1,i} + \Delta_{n,1}^{1,i}(y)e_{n,1} \\ &+ \Delta_{n,2}^{1,i}(y)e_{n,2} + \Delta_{n,i}^{1,i}(y)e_{n,i} \ , \ i \neq 1, n \end{aligned} \tag{7}$$

$$\begin{aligned} \Delta(ye_{i,n}) &= \Delta_{i,1}^{i,n}(y)e_{i,1} + \Delta_{n-1,1}^{i,n}(y)e_{n-1,1} \\ &+ \Delta_{n,1}^{i,n}(y)e_{n,1} + \Delta_{i,n}^{i,n}(y)e_{i,n} \\ &+ \Delta_{n-1,n}^{i,n}(y)e_{n-1,n} + \Delta_{n,n}^{i,n}(y)e_{n,n} \quad , \quad 1 < i < n-1 \end{aligned} \tag{8}$$

$$\begin{aligned} \Delta(ye_{1,n}) &= \Delta_{1,1}^{1,n}(y)e_{1,1} + \Delta_{1,2}^{1,n}(y)e_{1,2} + \Delta_{1,n}^{1,n}(y)e_{1,n} \\ &+ \Delta_{n-1,1}^{1,n}(y)e_{n-1,1} + \Delta_{n-1,2}^{1,n}(y)e_{n-1,2} + \Delta_{n-1,n}^{1,n}(y)e_{n-1,n} \\ &+ \Delta_{n,1}^{1,n}(y)e_{n,1} + \Delta_{n,2}^{1,n}(y)e_{n,2} + \Delta_{n,n}^{1,n}(y)e_{n,n} \end{aligned} \tag{9}$$

$$\begin{aligned} \Delta(x_{i,j}e_{i,j}) &= \Delta_{i,1}^{i,j}(x_{i,j})e_{i,1} + \Delta_{i,j}^{i,j}(x_{i,j})e_{i,j} \\ &+ \Delta_{n,1}^{i,j}(x_{i,j})e_{n,1} + \Delta_{n,j}^{i,j}(x_{i,j})e_{n,j} \quad , \quad 1 < i,j < n \end{aligned} \tag{10}$$

Proof. By (i+1,i) coefficient of the relation

$$\begin{aligned} 0 &= \Delta\left(e_{i+1,i} \circ e_{i+1,i}\right) \quad (i \neq 1, n-1) \\ &= \Delta(e_{i+1,i}) \circ e_{i+1,i} + e_{i+1,i} \circ \Delta\left(e_{i+1,i}\right) \end{aligned}$$

we get $2\Delta^{i+1,i}_{i+1,i+1}(1) = 0$ and this implies $\Delta^{i+1,i}_{i+1,i+1}(1) = 0$ considering that K is 2-torsion free. In addition, we obtain $\Delta^{i+1,i}_{i+1,j+1}(1) = 0$ for $j \neq i-1, i, i+1$ from (i+1,j) coefficient of the relation

$$\begin{aligned} 0 &= \Delta(e_{i+1,i} \circ e_{j+1,j}) \qquad (i \neq j-1, j, j+1) \\ &= \Delta(e_{i+1,i}) \circ e_{j+1,j} + e_{i+1,i} \circ \Delta(e_{j+1,j}). \end{aligned}$$

Then (i+1,i) coefficient of the relation

$$\begin{aligned} 0 &= \Delta(e_{i+1,i} \circ e_{i+2,i}) \\ &= \Delta(e_{i+1,i}) \circ e_{i+2,i} + e_{i+1,i} \circ \Delta(e_{i+2,i}), \end{aligned}$$

and (i,i), (i+1,i+1) coefficients of the relation

$$\begin{aligned} \Delta(e_{i+2,i}) &= \Delta(e_{i+2,i+1} \circ e_{i+1,i}) \\ &= \Delta(e_{i+2,i+1}) \circ e_{i+1,i} + e_{i+2,i+1} \circ \Delta(e_{i+1,i}) \end{aligned}$$

give $0 = \Delta^{i+1,i}_{i+1,i+2}(1) + \Delta^{i+2,i}_{i,i}(1) + \Delta^{i+2,i}_{i+1,i+1}(1)$ and $\Delta^{i+2,i}_{i,i}(1) = 0$, $\Delta^{i+2,i}_{i+1,i+1}(1) = \Delta^{i+1,i}_{i+1,i+2}(1)$. So it is obtained $2\Delta^{i+1,i}_{i+1,i+2}(1) = 0$ and we have (3). Now consider the products $e_{2,1} \circ e_{2,1} = 0$, $e_{2,1} \circ e_{3,1} = 0$ and $e_{3,2} \circ e_{2,1} = e_{3,1}$. Hence we obtain $\Delta^{2,1}_{n,2}(1) = 0$ by (n,1) coefficient of the relation

$$\begin{aligned} 0 &= \Delta(e_{2,1} \circ e_{2,1}) \\ &= \Delta(e_{2,1}) \circ e_{2,1} + e_{2,1} \circ \Delta(e_{2,1}), \end{aligned}$$

$\Delta^{2,1}_{2,3}(1) + \Delta^{3,1}_{1,1}(1) + \Delta^{3,1}_{2,2}(1) = 0$, $\Delta^{2,1}_{n,3}(1) + \Delta^{3,1}_{n,2}(1) = 0$ by (2,1) and (n,1) coefficients of the relation

$$\begin{aligned} 0 &= \Delta(e_{2,1} \circ e_{3,1}) \\ &= \Delta(e_{2,1}) \circ e_{3,1} + e_{2,1} \circ \Delta(e_{3,1}) \end{aligned}$$

and $\Delta^{2,1}_{n,3}(1) = \Delta^{3,1}_{n,2}(1)$, $\Delta^{2,1}_{2,3}(1) = \Delta^{3,1}_{2,2}(1)$, $\Delta^{3,1}_{1,1}(1) = \Delta^{3,2}_{1,2}(1) = 0$ by (n,2), (2,2) and (1,1) coefficients of the relation

$$\begin{aligned} \Delta(e_{3,1}) &= \Delta(e_{3,2} \circ e_{2,1}) \\ &= \Delta(e_{3,2}) \circ e_{2,1} + e_{3,2} \circ \Delta(e_{2,1}). \end{aligned}$$

By comparing these results obtained from the products $e_{2,1} \circ e_{2,1} = 0$, $e_{2,1} \circ e_{3,1} = 0$ and $e_{3,2} \circ e_{2,1} = e_{3,1}$, it is easy to see that $\Delta_{2,3}^{2,1}(1)$, $\Delta_{n,2}^{2,1}(1)$ and $\Delta_{n,3}^{2,1}(1)$ are zeros taking into account that K is 2-torsion free. So we have (1).

(n,n-1), (n,1) coefficients of the relation

$$\begin{aligned} 0 &= \Delta(e_{n,n-1} \circ e_{n,n-1}) \\ &= \Delta(e_{n,n-1}) \circ e_{n,n-1} + e_{n,n-1} \circ \Delta(e_{n,n-1}) \end{aligned}$$

give $\Delta_{n,n}^{n,n-1}(1) = 0$ and $\Delta_{n-1,1}^{n,n-1}(1) = 0$. Besides, we have $\Delta_{n-2,1}^{n,n-1}(1) + \Delta_{n-1,1}^{n,n-2}(1) = 0$ by (n,1) coefficient of

$$\begin{aligned} 0 &= \Delta(e_{n,n-1} \circ e_{n,n-2}) \\ &= \Delta(e_{n,n-1}) \circ e_{n,n-2} + e_{n,n-1} \circ \Delta(e_{n,n-2}) \end{aligned}$$

and $\Delta_{n-2,1}^{n,n-1}(1) = \Delta_{n-1,1}^{n,n-2}(1)$ by (n,1) coefficient of

$$\begin{aligned} \Delta(e_{n,n-2}) &= \Delta(e_{n,n-1} \circ e_{n-1,n-2}) \\ &= \Delta(e_{n,n-1}) \circ e_{n-1,n-2} + e_{n,n-1} \circ \Delta(e_{n-1,n-2}). \end{aligned}$$

Last two results $\Delta_{n-2,1}^{n,n-1}(1) = \Delta_{n-1,1}^{n,n-2}(1)$ and $\Delta_{n-2,1}^{n,n-1}(1) + \Delta_{n-1,1}^{n,n-2}(1) = 0$ give $2\Delta_{n-2,1}^{n,n-1}(1) = 0$ and we have (2) since K is a 2-torsion free ring.

It can be easily seen that $\Delta(e_{i,j}) = 0$ for $i - j > 1$ by induction on $i - j$:

If $i - j = 2$, then

$$\begin{aligned} \Delta(e_{i+2,i}) &= \Delta(e_{i+2,i+1} \circ e_{i+1,i}) \\ &= \Delta(e_{i+2,i+1}) \circ e_{i+1,i} + e_{i+2,i+1} \circ \Delta(e_{i+1,i}) \\ &= \Delta_{n,1}^{i+2,i+1}(1)e_{n,1} \circ e_{i+1,i} + e_{i+2,i+1} \circ \Delta_{n,1}^{i+1,i}(1)e_{n,1} \\ &= 0. \end{aligned}$$

Let $\Delta(e_{i,j}) = 0$ for an arbitrary appropriate number $t = i - j$ and let $k - m = t + 1$. Then we have (4) by

$$\begin{aligned} \Delta(e_{k,m}) &= \Delta(e_{k,m+1} \circ e_{m+1,m}) \\ &= \Delta(e_{k,m+1}) \circ e_{m+1,m} + e_{k,m+1} \circ \Delta(e_{m+1,m}) \\ &= 0 + e_{k,m+1} \circ \Delta_{n,1}^{m+1,m}(1)e_{n,1} \\ &= 0. \end{aligned}$$

If we consider the relation

$$
\begin{aligned}
0 &= \Delta(x_{i,1}e_{i,1} \circ e_{k,m}) \\
&= \Delta(x_{i,1}e_{i,1}) \circ e_{k,m}
\end{aligned}
$$

for $1 < i < n$, $m \neq i$, $k > m$, we get $\Delta^{i,1}_{s,k} = 0 = \Delta^{i,1}_{m,t}$. This means that each entry on $k-th$ column and $m-th$ row of $\Delta(x_{i,1}e_{i,1})$ is zero for all $k \neq 1$ and $m \neq i, n$. So we have (5). In particular, $\Delta(x_{n,1}e_{n,1}) = \Delta^{n,1}_{n,1}(x_{n,1})e_{n,1}$. Similarly, by using the relations

$$
\begin{aligned}
0 &= \Delta(x_{n,i}e_{n,i} \circ e_{k,m}) \quad (i \neq 1,\ k \neq i,\ k > m) \\
&= \Delta(x_{n,i}e_{n,i}) \circ e_{k,m} + x_{n,i}e_{n,i} \circ \Delta(e_{k,m}) \\
&= \Delta(x_{n,i}e_{n,i}) \circ e_{k,m},
\end{aligned}
$$

$$
\begin{aligned}
0 &= \Delta(y_{1,i}e_{1,i} \circ e_{k,m}) \quad (k \neq i,\ m \neq 1,\ i \neq n) \\
&= \Delta(y_{1,i}e_{1,i}) \circ e_{k,m} + y_{1,i}e_{1,i} \circ \Delta(e_{k,m}) \\
&= \Delta(y_{1,i}e_{1,i}) \circ e_{k,m},
\end{aligned}
$$

$$
\begin{aligned}
0 &= \Delta(y_{i,n}e_{i,n} \circ e_{k,m}) \quad (i \neq 1,\ i \neq n,\ k \neq n,\ m \neq i) \\
&= \Delta(y_{i,n}e_{i,n}) \circ e_{k,m} + y_{i,n}e_{i,n} \circ \Delta(e_{k,m}) \\
&= \Delta(y_{i,n}e_{i,n}) \circ e_{k,m},
\end{aligned}
$$

and

$$
\begin{aligned}
0 &= \Delta(y_{1,n}e_{1,n} \circ e_{k,m}) \quad (k \neq n,\ m \neq 1) \\
&= \Delta(y_{1,n}e_{1,n}) \circ e_{k,m} + y_{1,n}e_{1,n} \circ \Delta(e_{k,m}) \\
&= \Delta(y_{1,n}e_{1,n}) \circ e_{k,m},
\end{aligned}
$$

we get (6),(7),(8) and (9), respectively. In particular,

$$
\Delta(y_{1,1}e_{1,1}) = \Delta^{1,1}_{1,1}(y_{1,1})e_{1,1} + \Delta^{1,1}_{1,2}(y_{1,1})e_{1,2} + \Delta^{1,1}_{n,1}(y_{1,1})e_{n,1} + \Delta^{1,1}_{n,2}(y_{1,1})e_{n,2}
$$

and

$$
\Delta(y_{n-1,n}e_{n-1,n}) = \Delta^{n-1,n}_{n-1,n}(y_{n-1,n})e_{n-1,n} + \Delta^{n-1,n}_{n,n}(y_{n-1,n})e_{n,n}.
$$

If $1 < i, j < n$, then we have (10) because $\Delta^{i,j}_{sk}$ and $\Delta^{i,j}_{m,t}$ are zeros for $k \neq j,\ m \neq i$, $k > m$ by the relation

$$\begin{aligned} 0 &= \Delta(x_{i,j}e_{i,j} \circ e_{k,m}) \\ &= \Delta(x_{i,j}e_{i,j}) \circ e_{k,m} + xe_{i,j} \circ \Delta(e_{k,m}) \\ &= \Delta(x_{i,j}e_{i,j}) \circ e_{k,m}. \end{aligned}$$

■

Lemma 3.20 *Let Δ be a Jordan derivation of R satisfying the conditions (1)-(10) in Lemma 3.19. Then there exists an annihilator derivation Υ such that $(n,1)$ coefficients of $(\Delta - \Upsilon)(xe_{i+1,i})$ and $(\Delta - \Upsilon)(ye_{1,n})$ are equal to zero.*

Proof. Let $x \in K$, $y, z \in J$ be arbitrary elements. For $i \neq 1, n$, the (1,1) coefficient of the relation

$$\begin{aligned} \Delta(ye_{1,i}) &= \Delta(e_{n,i} \circ ye_{1,n}) \\ &= \Delta(e_{n,i}) \circ ye_{1,n} + e_{n,i} \circ \Delta(ye_{1,n}) \\ &= e_{n,i} \circ \Delta(ye_{1,n}) \end{aligned}$$

gives $\Delta^{1,i}_{1,1} = 0$. Besides, we have $\Delta^{1,n-1}_{1,1}(y) = \Delta^{n,n-1}_{n,1}(y) = 0$ by (n,1) coefficient of the relation

$$\begin{aligned} \Delta(ye_{n,n-1}) &= \Delta(e_{n,1} \circ ye_{1,n-1}) \\ &= \Delta(e_{n,1}) \circ ye_{1,n-1} + e_{n,1} \circ \Delta(ye_{1,n-1}) \\ &= e_{n,1} \circ \Delta(ye_{1,n-1}) \end{aligned}$$

as we know that $\Delta^{1,i}_{1,1} = 0$ for all $i \neq 1, n$. Say $\varsigma_i = \Delta^{i+1,i}_{n,1}$. Then $\varsigma_{n-1}(J) = 0$. If $k > 2$, then (n,k) coefficient of

$$\begin{aligned} \Delta(ye_{2,k}) &= \Delta(e_{2,1} \circ ye_{1,k}) \\ &= \Delta(e_{2,1}) \circ ye_{1,k} + e_{2,1} \circ \Delta(ye_{1,k}) \\ &= e_{2,1} \circ \Delta(ye_{1,k}) \end{aligned}$$

gives $\Delta^{2,k}_{n,k} = 0$. Moreover, we get $\Delta^{1,1}_{n,2}(y) = \Delta^{2,2}_{n,2}(y) = \Delta^{2,1}_{n,1}(y) = \varsigma_1(y)$ by comparing (n,1) coefficients of

$$\begin{aligned}\Delta(ye_{2,1}) &= \Delta(e_{2,1} \circ ye_{1,1}) \\ &= \Delta(e_{2,1}) \circ ye_{1,1} + e_{2,1} \circ \Delta(ye_{1,1}) \\ &= e_{2,1} \circ \Delta(ye_{1,1}),\end{aligned}$$

$$\begin{aligned}\Delta(ye_{2,1}) &= \Delta(ye_{2,2} \circ e_{2,1}) \\ &= \Delta(ye_{2,2}) \circ e_{2,1} + ye_{2,2} \circ \Delta(e_{2,1}) \\ &= \Delta(ye_{2,2}) \circ e_{2,1}\end{aligned}$$

and $\Delta^{1,1}_{n,2}(y) + \Delta^{2,2}_{n,2}(y) = 0$ by (n,2) coefficient of

$$\begin{aligned}\Delta(ye_{2,2} + ye_{1,1}) &= \Delta(e_{2,1} \circ ye_{1,2}) \\ &= \Delta(e_{2,1}) \circ ye_{1,2} + e_{2,1} \circ \Delta(ye_{1,2}) \\ &= e_{2,1} \circ \Delta(ye_{1,2}).\end{aligned}$$

Forasmuch as $\Delta^{1,1}_{n,2}(y) = \Delta^{2,2}_{n,2}(y) = \Delta^{2,1}_{n,1}(y) = \varsigma_1(y)$ and $\Delta^{1,1}_{n,2}(y) + \Delta^{2,2}_{n,2}(y) = 0$, we obtain $\Delta^{1,1}_{n,2}(y) = \Delta^{2,2}_{n,2}(y) = \Delta^{2,1}_{n,1}(y) = \varsigma_1(y) = 0$ because K is 2-torsion free.

For $i \neq 1, n-1$, (n,1) coefficient of the relation

$$\begin{aligned}\Delta(ye_{i+1,i}) &= \Delta(e_{n,i} \circ ye_{i+1,n}) \\ &= \Delta(e_{n,i}) \circ ye_{i+1,n} + e_{n,i} \circ \Delta(ye_{i+1,n}) \\ &= e_{n,i} \circ \Delta(ye_{i+1,n})\end{aligned}$$

gives $\Delta^{i+1,n}_{i,1}(y) = \Delta^{i+1,i}_{n,1}(y) = 0 = \varsigma_i(y)$. So $\varsigma_i(J)$ is zero $(i < n)$. For $2 \leq i \leq n-2$, we get $\varsigma_i(x)y = 0$, $y\varsigma_i(x) = 0$ by (1,1) and (n,n) coefficients of the relation

$$\begin{aligned}0 &= \Delta(xe_{i+1,i} \circ ye_{1,n}) \\ &= \Delta(xe_{i+1,i}) \circ ye_{1,n} + xe_{i+1,i} \circ \Delta(ye_{1,n}).\end{aligned}$$

Furthermore, we have $y\Delta^{2,1}_{n,1}(x) = 0$, $\Delta^{2,1}_{n,1}(x)y = 0$ by (n,1) coefficients of the relations

$$\begin{aligned}\Delta(yxe_{2,1}) &= \Delta(ye_{2,2} \circ xe_{2,1}) \\ &= \Delta(ye_{2,2}) \circ xe_{2,1} + ye_{2,2} \circ \Delta(xe_{2,1})\end{aligned}$$

and

$$\begin{aligned}\Delta(xye_{2,1}) &= \Delta(xe_{2,1} \circ ye_{1,1}) \\ &= \Delta(xe_{2,1}) \circ ye_{1,1} + xe_{2,1} \circ \Delta(ye_{1,1})\end{aligned}$$

since $\Delta^{2,1}_{n,1}(J) = 0$. Besides, we obtain $y\Delta^{n,n-1}_{n,1}(x) = 0$, $\Delta^{n,n-1}_{n,1}(x)y = 0$ from (1,1) and (n,1) coefficients of the relations

$$\begin{aligned}\Delta(yxe_{1,n-1}) &= \Delta(xe_{n,n-1} \circ ye_{1,n}) \\ &= \Delta(xe_{n,n-1}) \circ ye_{1,n} + xe_{n,n-1} \circ \Delta(ye_{1,n})\end{aligned}$$

and

$$\begin{aligned}0 &= \Delta(xe_{n,n-1} \circ ye_{1,1}) \\ &= \Delta(xe_{n,n-1}) \circ ye_{1,1} + xe_{n,n-1} \circ \Delta(ye_{1,1}),\end{aligned}$$

respectively. Finally, we have $\varsigma_n(J^2) = 0$ and $\varsigma_n(J) \subset Ann_K(J)$ by (n,1) coefficients of

$$\begin{aligned}\Delta(yze_{1,n}) &= \Delta(ye_{1,k} \circ ze_{k,n}) \quad (1 < k < n) \\ &= \Delta(ye_{1,k}) \circ ze_{k,n} + ye_{1,k} \circ \Delta(ze_{k,n}),\end{aligned}$$

$$\begin{aligned}\Delta(yze_{1,n}) &= \Delta(ye_{1,1} \circ ze_{1,n}) \\ &= \Delta(ye_{1,1}) \circ ze_{1,n} + ye_{1,1} \circ \Delta(ze_{1,n}),\end{aligned}$$

and

$$\begin{aligned}\Delta(yze_{1,n}) &= \Delta(ye_{1,n} \circ ze_{n,n}) \\ &= \Delta(ye_{1,n}) \circ ze_{n,n} + ye_{1,n} \circ \Delta(ze_{n,n}),\end{aligned}$$

respectively. Now we showed that $\varsigma_i : K \to Ann_K(J)$, $\varsigma_i(J) = 0$, $\varsigma_n : J \to Ann_K(J)$, $\varsigma_n(J^2) = 0$ for all $i = 1, 2, ..., n-1$. Thus $\Upsilon : [x_{i,j}] \to \left(\varsigma_n(x_{1,n}) + \sum_{i=1}^{n-1} \varsigma_i(x_{i+1,i})\right) e_{n,1}$ is an annihilator derivation and $(\Delta - \Upsilon)(e_{i+1,i}) = 0$ for all i. Say $\Theta = \Delta - \Upsilon$. Hence $(n, 1)$ coefficients of $\Theta(xe_{i+1,i})$ and $\Theta(ye_{1,n})$ are equal to zeros. This completes the proof. ■

Lemma 3.21 *Let $\Theta = \Delta - \Upsilon$ be a Jordan derivation of the ring R as in Lemma 3.20. Then there is a ring derivation $\bar{\theta}$ and (i, j) coefficient of $(\Theta - \bar{\theta})(x_{i,j}e_{i,j})$ is equal to zero.*

Proof. Let $x, x_1, x_2 \in K$ and $y \in J$ be arbitrary elements. By using (i,k) coefficient of the relation

$$\begin{aligned}\Theta(x_1x_2e_{i,k}) &= \Theta(x_1e_{i,j} \circ x_2e_{j,k}) \\ &= \Theta(x_1e_{i,j}) \circ x_2e_{j,k} + x_1e_{i,j} \circ \Theta(x_2e_{j,k})\end{aligned}$$

we have

$$\Theta^{i,k}_{i,k}(x_1x_2) = \Theta^{i,j}_{i,j}(x_1)x_2 + x_1\Theta^{j,k}_{j,k}(x_2)$$

for $i > j > k$. Besides, we have $\Theta^{i,k}_{i,k} = \Theta^{i,j}_{i,j} = \Theta^{j,k}_{j,k}$ as $\Theta^{i,k}_{i,k}(1) = \Theta^{i,j}_{i,j}(1) = \Theta^{j,k}_{j,k}(1) = 0$. So $\Theta^{i,j}_{i,j}$ is a derivation of K for $i > j$ and $\Theta^{i,j}_{i,j} = \Theta^{k,m}_{k,m}$ for every $k > m$. Moreover, (i,i+1) coefficients of the relations

$$\begin{aligned}\Theta(ye_{i,i+1}) &= \Theta(e_{i,k} \circ ye_{k,i+1}) \quad (k < i) \\ &= \Theta(e_{i,k}) \circ ye_{k,i+1} + e_{i,k} \circ \Theta(ye_{k,i+1}) \\ &= e_{i,k} \circ \Theta(ye_{k,i+1})\end{aligned}$$

and

$$\begin{aligned}\Theta(ye_{i,i+1}) &= \Theta(ye_{i,s} \circ e_{s,i+1}) \quad (s > i+1) \\ &= \Theta(ye_{i,s}) \circ e_{s,i+1} + ye_{i,s} \circ \Theta(e_{s,i+1}) \\ &= \Theta(ye_{i,s}) \circ e_{s,i+1}\end{aligned}$$

give $\Theta^{i,i+1}_{i,i+1} = \Theta^{i,s}_{i,s}$ and $\Theta^{i,i+1}_{i,i+1} = \Theta^{k,i+1}_{k,i+1}$ for $k < i < s-1$. This means $\Theta^{i,j}_{i,j} = \Theta^{s,t}_{s,t}$ for every $i < j$ and $s < t$. In addition, we can say $\Theta^{i,i}_{i,i} = \Theta^{j,j}_{j,j} = \Theta^{i,j}_{i,j}$ for all $i < j$ and $s < t$ by (i,i), (j,j) coefficients of the relation

$$\begin{aligned}\Theta(ye_{i,i} + ye_{j,j}) &= \Theta(ye_{i,j} \circ e_{j,i}) \\ &= \Theta(ye_{i,j}) \circ e_{j,i} + ye_{i,j} \circ \Theta(e_{j,i}) \\ &= \Theta(ye_{i,j}) \circ e_{j,i}\end{aligned}$$

which gives $\Theta^{i,j}_{i,j} = \Theta^{s,t}_{s,t}$ for all $i \le j$ and $s \le t$. Besides, we obtain that $\Theta^{i,j}_{i,j} = \Theta^{k,m}_{k,m}$ for any i, j, k, m because (i,1) coefficient of the relation

$$\begin{aligned}\Theta(ye_{i,1}) &= \Theta(e_{i,1} \circ ye_{1,1}) \\ &= \Theta(e_{i,1}) \circ ye_{1,1} + e_{i,1} \circ \Theta(ye_{1,1}) \\ &= e_{i,1} \circ \Theta(ye_{1,1})\end{aligned}$$

gives $\Theta^{i,1}_{i,1}(y) = \Theta^{1,1}_{1,1}(y)$ and we know that all $\Theta^{i,j}_{i,j}$ are equal for $i \leq j$ and $\Theta^{k,m}_{k,m}$ are equal for $k > m$. Then $\Theta^{i,j}_{i,j} = \theta$ is a ring derivation of K. Similarly, $\Theta^{i,j}_{i,j}(y) = \Theta^{i,i-1}_{i,i-1}(y)$ by $ye_{i,j} \circ e_{j,i-1} = ye_{i,i-1}$ for $i \leq j$. This implies that θ is a derivation of J as well. So $\bar{\theta} : [x_{i,j}] \to \sum_{i,j} \theta(x_{i,j})e_{i,j}$ is a ring derivation of R and (i,j) coefficient of $(\Theta - \bar{\theta})(x_{i,j})$ is zero for all i, j. ■

Let $\Xi = \Theta - \bar{\theta}$ for brevity. Thus (i, j) coefficients of the matrices $\Xi(x_{i,j}e_{i,j})$ are equal to zero.

Lemma 3.22 *Let Ξ be a Jordan derivation of R as in Lemma 3.21. Then $\Xi(xe_{i,j}) = 0$ for all $i > j$, $\Xi(ye_{i,i}) = \Xi^{i,i}_{n,1}(y)e_{n,1}$ for all i, $\Xi(ye_{1,j}) = \Xi^{1,j}_{n,j}(y)e_{n,j}$ for $1 < j < n-1$ and $\Xi(ye_{i,n}) = \Xi^{i,n}_{i,1}(y)e_{i,1}$ for $i \neq 1, 2$ where $x \in K$, $y \in J$ are arbitrary elements.*

Proof. Let $x, x_1, x_2 \in K$ and $y \in J$ be arbitrary elements. For $1 < i < n$, we get $\Xi^{i+1,i-1}_{n,i-1} = \Xi^{i+1,i}_{n,i} = \Xi^{i+1,i-1}_{i+1,1} = \Xi^{i,i-1}_{i,1} = 0$ by (n,i-1), (i+1,1) coefficients of

$$\begin{aligned}\Xi(x_1x_2e_{i+1,i-1}) &= \Xi(x_1e_{i+1,i} \circ x_2e_{i,i-1}) \\ &= \Xi(x_1e_{i+1,i}) \circ x_2e_{i,i-1} + x_1e_{i+1,i} \circ \Xi(x_2e_{i,i-1})\end{aligned}$$

considering that $\Xi(e_{i+1,i}) = 0$ for all i. Hence $\Xi(xe_{i,j})$ is equal to zero for $i > j$.
We have $\Xi^{i+1,i+1}_{i+1,1} = 0$ by (i+2,1) coefficient of

$$\begin{aligned}0 &= \Xi(xye_{i+2,i+1}) \quad (1 \leq i < n-1) \\ &= \Xi(xe_{i+2,i+1} \circ ye_{i+1,i+1}) \\ &= \Xi(xe_{i+2,i+1}) \circ ye_{i+1,i+1} + xe_{i+2,i+1} \circ \Xi(ye_{i+1,i+1})\end{aligned}$$

and $\Xi^{i+1,i+1}_{n,i+1} = 0$ by (n,i) coefficient of

$$\begin{aligned}0 &= \Xi(yxe_{i+1,i}) \quad (1 < i < n) \\ &= \Xi(ye_{i+1,i+1} \circ xe_{i+1,i}) \\ &= \Xi(ye_{i+1,i+1}) \circ xe_{i+1,i} + ye_{i+1,i+1} \circ \Xi(xe_{i+1,i}).\end{aligned}$$

Besides, (2,2) coefficient of the relation

$$\begin{aligned}0 &= \Xi(xye_{2,1}) \\ &= \Xi(xe_{2,1} \circ ye_{1,1}) \\ &= \Xi(xe_{2,1}) \circ ye_{1,1} + xe_{2,1} \circ \Xi(ye_{1,1})\end{aligned}$$

gives $\Xi^{1,1}_{1,2} = 0$. So we get $\Xi(ye_{i,i}) = \Xi^{i,i}_{n,1}(y)e_{n,1}$. Furthermore, for $1 < j < n-1$, we obtain $\Xi^{1,j}_{1,1} = \Xi^{1,j}_{1,2} = \Xi^{1,j}_{n,1} = \Xi^{1,j}_{n,2} = 0$ by (1,1), (1,2), (n,1), (n,2) coefficients of the relation

$$\begin{aligned}\Xi(yxe_{1,j}) &= \Xi(xe_{n,j} \circ ye_{1,n})\\ &= \Xi(xe_{n,j}) \circ ye_{1,n} + xe_{n,j} \circ \Xi(ye_{1,n})\\ &= xe_{n,j} \circ \Xi(ye_{1,n})\end{aligned}$$

and it means $\Xi(ye_{1,j}) = \Xi^{1,j}_{n,j}(y)e_{n,j}$. Finally, we can say $\Xi^{i,n}_{n-1,1} = \Xi^{i,n}_{n-1,n} = \Xi^{i,n}_{n,1} = \Xi^{i,n}_{n,n} = 0$ by (n-1,1), (n-1,n), (n,1) and (n,n) coefficients of the relation

$$\begin{aligned}\Xi(xye_{i,n}) &= \Xi(xe_{i,1} \circ ye_{1,n})\\ &= \Xi(xe_{i,1}) \circ ye_{1,n} + xe_{i,1} \circ \Xi(ye_{1,n})\\ &= xe_{i,1} \circ \Xi(ye_{1,n})\end{aligned}$$

for $2 < i < n$ and we have $\Xi(ye_{i,n}) = \Xi^{i,n}_{i,1}(y)e_{i,1}$. ■

Lemma 3.23 *Let Ξ be a Jordan derivation of R as in Lemma 3.22. Then there exists an almost annihilator derivation Γ of R such that $\Xi - \Gamma$ is an extremal Jordan derivation of R which is defined in Proposition 3.11.*

Proof. Let $n > k > m > s > 1$, $x \in K$ and $y \in J$. Then $(n,1)$ coeffiecient of the relation

$$\begin{aligned}\Xi(xye_{k,k} + yxe_{m,m}) &= \Xi(xe_{k,m} \circ ye_{m,k})\\ &= \Xi(xe_{k,m}) \circ ye_{m,k} + xe_{k,m} \circ \Xi(ye_{m,k})\\ &= xe_{k,m} \circ \Xi(ye_{m,k})\end{aligned}$$

gives $\Xi^{m,m}_{n,1}(yx) + \Xi^{k,k}_{n,1}(xy) = 0$. We can similarly find that $\Xi^{s,s}_{n,1}(yx) + \Xi^{k,k}_{n,1}(xy) = 0$ and $\Xi^{s,s}_{n,1}(yx) + \Xi^{m,m}_{n,1}(xy) = 0$. This means $\Xi^{k,k}_{n,1} = \Xi^{s,s}_{n,1} = \Xi^{m,m}_{n,1} = 0$ as K is 2-torsion free. On the other hand, considering that $\Xi^{2,2}_{n,1} = 0 = \Xi^{n-1,n-1}_{n,1}$, the $(n,1)$ coefficients of the relations

$$\begin{aligned}\Xi(yxe_{1,1} + xye_{2,2}) &= \Xi(ye_{1,2} \circ xe_{2,1})\\ &= \Xi(ye_{1,2}) \circ xe_{2,1} + ye_{1,2} \circ \Xi(xe_{2,1})\\ &= \Xi(ye_{1,2}) \circ xe_{2,1}\end{aligned}$$

and

$$\begin{aligned}\Xi(yxe_{n-1,n-1}+xye_{n,n}) &= \Xi(ye_{n-1,n}\circ xe_{n,n-1})\\ &= \Xi(ye_{n-1,n})\circ xe_{n,n-1}+ye_{n-1,n}\circ\Xi(xe_{n,n-1})\\ &= \Xi(ye_{n-1,n})\circ xe_{n,n-1}\end{aligned}$$

give $\Xi_{n,1}^{1,1}(yx)=\Xi_{n,2}^{1,2}(y)x$ and $\Xi_{n,1}^{n,n}(xy)=x\Xi_{n-1,1}^{n-1,n}(y)$, respectively. Let $\bar{\alpha}=\Xi_{1,1}^{1,n}$, $\bar{\beta}=\Xi_{n,n}^{1,n}$, $x\in K$ and $y,z\in J$. We have $\bar{\alpha}=\Xi_{i,1}^{i,n}$ and $\bar{\alpha}(xy)=x\bar{\alpha}(y)$ by $(i,1)$ coefficient of the relation

$$\begin{aligned}\Xi(xye_{i,n}) &= \Xi(xe_{i,1}\circ ye_{1,n})\\ &= \Xi(xe_{i,1})\circ ye_{1,n}+xe_{i,1}\circ\Xi(ye_{1,n})\\ &= xe_{i,1}\circ\Xi(ye_{1,n}).\end{aligned}$$

where $i\neq 1,n$. Besides, we obtain $\bar{\beta}=\Xi_{n,j}^{1,j}$ and $\bar{\beta}(yx)=\bar{\beta}(y)x$ by (n,j) coefficient of the relation

$$\begin{aligned}\Xi(yxe_{1,j}) &= \Xi(ye_{1,n}\circ xe_{n,j})\\ &= \Xi(ye_{1,n})\circ xe_{n,j}+ye_{1,n}\circ\Xi(xe_{n,j})\\ &= \Xi(ye_{1,n})\circ xe_{n,j}.\end{aligned}$$

for $j\neq 1,n$. Now it is easy to see that the additive map

$$\begin{array}{llll}\Gamma: & R_n(K,J) & \to & R_n(K,J)\\ & ye_{1,n} & \to & \bar{\alpha}(y)e_{1,1}+\bar{\beta}(y)e_{n,n}\\ & ye_{i,n} & \to & \bar{\alpha}(y)e_{i,1}\quad(1<i\leq n)\\ & ye_{1,j} & \to & \bar{\beta}(y)e_{n,j}\quad(1\leq j<n)\\ & x_{i,j}e_{i,j} & \to & 0\quad(1<i and j<n)\end{array}$$

is obviously an almost annihilator derivation of R. As the last part of the Lemma, we need to see $\Xi-\Gamma$ satisfies the conditions of the extremal Jordan derivation given in Proposition 3.11.

Firstly, the relation

$$\begin{aligned}\Xi(yxe_{1,j}) &= \Xi(xe_{n-1,j}\circ ye_{1,n-1})\\ &= xe_{n,n-1}\circ\Xi(ye_{1,n-1})\end{aligned}$$

gives $\Xi(ye_{1,j}) = 0$ for $1 < j < n-1$ and we can say

$$\Xi(ye_{1,j}) = 0$$

for $1 \leq j < n-1$ as $\Xi^{1,1}_{1,2} = \Xi^{1,1}_{n,2} = 0$ by Lemma 3.20, 3.21.

Secondly, the relation

$$\begin{aligned}\Xi(ye_{1,n-1}) &= \Xi(ye_{1,n} \circ e_{n,n-1}) \\ &= \Xi(ye_{1,n}) \circ e_{n,n-1}\end{aligned}$$

gives $\Xi(ye_{1,n-1}) = \Xi^{1,n-1}_{n,1}(y)e_{n,1} + \Xi^{1,n-1}_{n,2}(y)e_{n,2}$ because we have $\Xi(ye_{1,n}) = \Xi^{1,n}_{n,2}(y)e_{n,2} + \Xi^{1,n}_{n-1,1}(y)e_{n-1,1} + \Xi^{1,n}_{n-1,2}(y)e_{n-1,2}$ and if we consider that $\Xi(ye_{1,n-1}) = \Xi^{1,n-1}_{n,1}(y)e_{n,1} + \Xi^{1,n-1}_{n,2}(y)e_{n,2}$ and the relation

$$\begin{aligned}\Xi(xye_{i,j}) &= \Xi(xe_{i,1} \circ ye_{1,j}) \\ &= xe_{i,1} \circ \Xi(ye_{1,j}),\end{aligned}$$

we get $\Xi(ye_{i,j}) = 0$ for $1 < i < j < n$ where $(i,j) \neq (2, n-1)$.

Thirdly, if we say $\Pi = \Xi - \Gamma$ then we have

$$\begin{aligned}\Pi(ye_{1,n}) &= \Pi^{1,n}_{n-1,1}(y)e_{n-1,1} + \Pi^{1,n}_{n-1,2}(y)e_{n-1,2} + \Pi^{1,n}_{n,2}(y)e_{n,2}, \\ \Pi(ye_{1,n-1}) &= \Pi^{1,n-1}_{n,1}(y)e_{n,1} + \Pi^{1,n-1}_{n,2}(y)e_{n,2}, \\ \Pi(ye_{2,n-1}) &= \Pi^{2,n-1}_{n,1}(y)e_{n,1}, \\ \Pi(ye_{2,n}) &= \Pi^{2,n}_{n-1,1}(y)e_{n-1,1} + \Pi^{2,n}_{n,1}(y)e_{n,1} \\ \Pi(x_{i,j}e_{i,j}) &= 0 \text{ for } (i,j) \neq (1,n), (2,n), (1,n-1), (2,n-1)\end{aligned}$$

by the relations

$$\begin{aligned}\Pi(ye_{2,n-1}) &= \Pi(e_{2,1} \circ ye_{1,n-1}) \\ &= e_{2,1} \circ \Pi(ye_{1,n-1})\end{aligned}$$

and

$$\begin{aligned}\Pi(ye_{2,n}) &= \Pi(e_{2,1} \circ ye_{1,n}) \\ &= e_{2,1} \circ \Pi(ye_{1,n}).\end{aligned}$$

Finally, the relations

$$\begin{aligned}\Pi(yxe_{1,n-1}) &= \Pi(ye_{1,n} \circ xe_{n,n-1}) \\ &= \Pi(ye_{1,n}) \circ xe_{n,n-1},\end{aligned}$$

$$\Pi(xye_{2,n-1}) = \Pi(xe_{2,1} \circ ye_{1,n-1})$$
$$= xe_{2,1} \circ \Pi(ye_{1,n-1}),$$

$$\Pi(xye_{2,n}) = \Pi(xe_{2,1} \circ ye_{1,n})$$
$$= xe_{2,1} \circ \Pi(ye_{1,n})$$

and

$$\Pi(yze_{1,n}) = \Pi(ye_{1,3} \circ ze_{3,n})$$
$$= \Pi(ye_{1,3}) \circ ze_{3,n} + ye_{1,3} \circ \Pi(ze_{3,n})$$
$$= 0$$

give $\alpha = \Pi^{1,n}_{n-1,1} = \Pi^{1,n-1}_{n,1}$, $\beta = \Pi^{1,n}_{n-1,2} = \Pi^{1,n-1}_{n,2} = \Pi^{2,n-1}_{n,1} = \Pi^{2,n}_{n-1,1}$, $\gamma = \Pi^{1,n}_{n,2} = \Pi^{2,n}_{n,1}$, $\alpha(yx) = x\alpha(y)$, $\beta(yx) = x\beta(y)$, $\beta(xy) = \beta(y)x$, $\gamma(xy) = \gamma(y)x$ and $\alpha(J^2) = \beta(J^2) = \gamma(J^2) = 0$. Also we can say

$$\alpha, \beta, \gamma : J \to Ann_K(J)$$

by the relations

$$0 = \Pi(yze_{1,n}) = \Pi(ye_{1,n-1} \circ ze_{n-1,n}) = \Pi(ye_{1,1} \circ ze_{1,n}),$$
$$0 = \Pi(yze_{1,n}) = \Pi(ye_{1,2} \circ ze_{2,n}) = \Pi(ye_{1,n} \circ ze_{n-1,n-1})$$
$$0 = \Pi(ye_{1,n} \circ ze_{3,n}).$$

Now we have all conditions of the extremal Jordan derivation given in Proposition 3.11 for $n > 4$.

Let $n = 4$. Then we get $\alpha(J^2) = 0$, $\beta(J^2) = 0$, $\gamma(J^2)$ by (4,2), (3,1), (3,2) coefficients of the relations $\Pi(ye_{1,2} \circ ze_{2,4}) = \Pi(yze_{1,4})$, $\Pi(ye_{1,1} \circ ze_{1,4}) = \Pi(yze_{1,4})$, $\Pi(ye_{1,3} \circ ze_{3,4}) = \Pi(yze_{1,4})$, respectively. In addition, we have $\alpha, \beta, \gamma : J \to Ann_K(J)$ by (3,1), (3,2)-(4,2), (3,1)-(3,2), (3,2) coefficients of the relations $\Pi(ye_{1,1} \circ ze_{1,4}) = \Pi(yze_{1,4})$, $\Pi(ye_{1,2} \circ ze_{2,4}) = \Pi(yze_{1,4})$, $\Pi(ye_{1,3} \circ ze_{3,4}) = \Pi(yze_{1,4})$, $\Pi(ye_{1,4} \circ ze_{3,4}) = 0$, respectively. This completes the proof. ■

Theorem 3.15 follows by Lemma 3.16 - 3.23. In other words, any Jordan derivation Δ of $R_n(K, J)$ can be written as

$$\Delta = \delta_D + I + \Upsilon + \bar{\theta} + \Gamma + \Omega$$

where δ_D, I, Υ, $\bar{\theta}$, Γ and Ω are diagonal, inner, annihilator, ring, almost annihilator derivations and extremal Jordan derivation, respectively.

Let **n=3.** After applying Lemma 3.16 - 3.21, it is obtained that Ξ is equal to the sum of the Jordan derivations described in Proposition 3.13, 3.14:

- The conditions of the Jordan derivation given in Proposition 3.13 can be obtained by using the relations $ye_{1,3} \circ xe_{3,2} = yxe_{1,2}$, $xe_{2,1} \circ ye_{1,3} = xye_{2,3}$, $ye_{1,2} \circ ze_{2,3} = yze_{1,3}$, $ye_{1,3} \circ ze_{2,3} = 0$, $ye_{1,3} \circ ze_{1,2} = 0$.
- The conditions of the Jordan derivation given in Proposition 3.14 can be obtained by using the relations

$$\begin{aligned}
&ye_{1,2} \circ ze_{2,3} = 0, \quad ye_{1,3} \circ ze_{1,3} = 0,\\
&ye_{2,2} \circ ze_{1,3} = 0, \quad ye_{1,1} \circ ze_{1,3} = yze_{1,3},\\
&ye_{1,3} \circ ze_{3,3} = 0, \quad ye_{2,3} \circ ze_{3,3} = yze_{2,3},\\
&ye_{1,1} \circ ze_{3,3} = 0, \quad ye_{1,1} \circ ze_{1,2} = yze_{1,2},\\
&ye_{1,2} \circ ze_{2,2} = 0, \quad xe_{2,1} \circ ye_{1,3} = xye_{2,3},\\
&ye_{2,3} \circ ze_{1,1} = 0, \quad ye_{1,1} \circ ze_{1,1} = yze_{1,1} + zye_{1,1},\\
&ye_{1,2} \circ ze_{3,3} = 0, \quad ye_{1,2} \circ ze_{2,1} = yze_{1,1} + zye_{2,2},\\
&ye_{1,2} \circ ze_{1,3} = 0, \quad ye_{1,3} \circ ze_{3,1} = yze_{1,1} + zye_{3,3},\\
&ye_{1,3} \circ ze_{2,3} = 0, \quad ye_{2,3} \circ xe_{3,2} = yxe_{2,2} + xye_{3,3}.
\end{aligned}$$

4 RESULTS

In this thesis, some elementary matrix operations are utilized to classify the derivations of $R_n(K, J)$, and therefore the Jordan derivations of the same ring. Firstly, it is proved that any derivation of $R_n(K, J)$ can be written as a sum of diagonal, inner, annihilator and almost annihilator derivations and some of this proof is published ([23]). After describing the derivations of $R_n(K, J)$, we characterized all Jordan derivations of $R_n(K, J)$ and we showed that any Jordan derivation of $R_n(K, J)$ can be written as a sum of a derivation and an extremal Jordan derivation.

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Oral and Poster Presentations

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