THEORY OF ORTHOGONALLY ADDITIVE OPERATORS

DİKEY TOPLAMSAL OPERATÖRLERİN TEORİSİ

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ABSTRACT

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An orthogonally additive operator is a map that satisfies the additivity property under the disjointness condition.

This thesis focuses on the theory of orthogonally additive operators. The concept of fragments plays a significant role in constructing the theory of orthogonally additive operators, and it is also studied in this thesis.

The first chapter is dedicated to the study of fragments. The concept is explored in the context of vector lattices and lattice-normed vector spaces. The conclusions derived from Sections 2.2 and 2.3 of Chapter 2 can be found in [1].

The second chapter introduces the classes of orthogonally additive operators defined on vector lattices and a novel class of vector lattice known as C-complete. This chapter also addresses the extension problems associated with orthogonally additive maps. Various examples and conclusions are provided to support the findings.

In the last chapter, orthogonally additive operators are examined in the context of lattice normed spaces.

Keywords: Orthogonally additive operators, Fragments, (bo)-Fragments, C-complete vector lattices, Lattice normed spaces, Dominated orthogonally additive operators

ÖZET

DİKEY TOPLAMSAL OPERATÖRLERİN TEORİSİ

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Diklik koşulu altında toplamsallık özelliğini sağlayan operatörlere dikey toplamsal denir.

Bu tez dikey toplamsal operatörlerin teorisine odaklanmaktadır. Ayrıca bu tezde, dikey toplamsal operatörlerin teorisi içinde önemli bir yere sahip olan "fragment" kavramı da incelenmiştir.

Birinci bölüm, fragment kavramının çalışmasına odaklanmıştır. Fragment kavramı hem vektör latisler hem de latis-normlu vektör uzaylar üzerinde incelenmiştir. 2. Bölümün 2.2 ve 2.3 kısımlarında elde edilen sonuçlar, kaynakça kısmındaki [1] makalesinde bulunabilir.

İkinci bölüm, dikey toplamsal operatör sınıflarını ve "C-tam" olarak bilinen yeni bir vektör latis sınıfını tanıtmaktadır. Bu bölüm ayrıca bir dikey toplamsal dönüşümün genişletilmesi problemini de ele almaktadır. Birçok örnek ve sonuç ile bulgular desteklenmiştir.

Son bölümde, dikey toplamsal operatör kavramı latis normlu uzaylar üzerinde incelenmiştir.

Anahtar Kelimeler: Dikey toplamsal operatörler, Fragmentlar, (bo)-Fragmentlar, C-tam vektör latisler, Latis normlu uzaylar, Domine edilmiş dikey toplamsal operatörler

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1. Introduction

The field of operator theory plays a crucial role in various branches of mathematics. One noteworthy area within the operator theory is the theory of orthogonally additive operators. These operators possess remarkable properties and have been the subject of intense study in recent years, see [2–11]. It is also worth noting that the study of orthogonally additive operators has valuable applications across various fields of modern mathematics, see [11, 12].

The concept of orthogonally additive operators was initially introduced in [13, 14] by Mazón and Segura de León in the early 1990s. Since then, the theory of these operators has become a dynamic field within modern analysis. An orthogonally additive operator is defined as an additive map that satisfies the additivity property under the disjointness condition. Consequently, linear operators can be considered as a special subclass of orthogonally additive operators.

The main objective of this thesis is to provide a comprehensive study of orthogonally additive operators, exploring their fundamental properties and characterizations. Because of that, we study the concept of fragments, which is an essential tool for orthogonally additive operators. After that, we will investigate different classes of orthogonally additive operators and analyze their relationships with each other.

The structure of this thesis is organized as follows.

In Chapter 2, we examine the articles [1, 15, 16]. We conduct a preliminary study on vector lattices, providing essential definitions. We introduce the concept of fragments and establish their properties. Notably, we demonstrate that the set of fragments forms a Boolean algebra. Besides we explore the general properties of fragments on arbitrary vector lattices, we specifically investigate this concept on the vector lattice $C[0,1]$. Our findings reveal that if an element in $C[0,1]$ is a fragment of another element, it must have a root within the open interval $(0,1)$. Furthermore, we demonstrate that if an element in $C[0,1]$ has a proper fragment, i.e., a fragment that is not zero or the element itself, it must also possess a root within the open interval $(0,1)$.

Subsequently, we extend this concept to lattice normed spaces, where fragments are referred to as (bo)-fragments. Although there are similarities between fragments and (bo)-fragments, significant differences exist. Notably, the set of (bo)-fragments does not generally form a Boolean algebra. However, we show that under specific conditions, the set of (bo)-fragments forms a Boolean algebra. We thoroughly investigate this concept, presenting numerous properties that hold importance for the theory of orthogonally additive operators defined on lattice normed spaces.

At the main part of Chapter 2, we present a collection of lemmas, propositions, and theorems about fragments. These results have been carefully derived and selected due to their significance and utility in the subsequent chapters of this thesis. They serve as essential tools for our further investigations into orthogonally additive operators and contribute to the overall understanding of fragments within the context of vector lattices and lattice normed spaces.

In Chapter 3, we examine the articles [2, 4, 5, 13, 16]. We begin by introducing orthogonally additive operators and providing their characterization through a significant theorem. Furthermore, we present several important examples to illustrate the concept. We define two important classes of orthogonally additive operators: positive operators and order-bounded operators. An order-bounded orthogonally additive operator is called an abstract Urysohn operator. Notably, we demonstrate that the set of all abstract Urysohn operators from a vector lattice E to a Dedekind complete vector lattice F forms a Dedekind complete vector lattice.

Moving forward, we introduce the concept of C-bounded orthogonally additive operators, which is a generalization of order-bounded operators. Additionally, we define regular orthogonally additive operators in a similar manner to linear operator theory. We establish that C-bounded operators and regular operators coincide under a specific condition. Furthermore, we define disjointness-preserving and non-expanding operators and provide various propositions and examples to illustrate their relationships.

We then delve into the exploration of a new class of vector lattices known as C-complete vector lattices. We clarify that this class represents a generalization of Dedekind complete vector lattices. Moreover, we introduce important definitions such as horizontal convergence of a net, horizontally-to-norm continuous and horizontally-to-order continuous orthogonally additive operators.

Next, we introduce the notion of compact-like operators, such as narrow operators, within the context of orthogonally additive operators. We investigate their behaviour in C-complete vector lattices.

The chapter also addresses the extension problem for orthogonally additive operators. We demonstrate that an orthogonally additive map defined on an arbitrary subset of a vector lattice generally does not extend to an orthogonally additive operator. Thus, we explore the proper conditions for extending an orthogonally additive map to an operator. To enable this, we introduce the definitions of lateral ideals and lateral bands, which play a crucial role in the extension process. Finally, we provide various propositions and theorems to construct the extension of orthogonally additive operators.

In the final part of Chapter 3, we introduce the concept of projection bands within the context of orthogonally additive operators, presenting significant results and essential definitions.

The study of orthogonally additive operators defined on lattice normed spaces remains an active and dynamic field of research, as evidenced by notable studies, see [17–19]. In the final chapter of this thesis, our objective is to provide a foundational understanding of this topic. We aim to present key understandings and essential concepts related to orthogonally additive operators in the context of lattice normed spaces. One of the central focuses of this chapter is the introduction of dominated operators and their characterization.

2. On Fragments

In this chapter, we introduce the concept of fragments, which is a required notion for orthogonally additive operators. In the first section, we demonstrate several properties of fragments in vector lattices, with one of the most significant being that they form a Boolean algebra. In the second section, we discuss the concept of fragments in a specific vector lattice $C[0, 1]$. We choose this space because it requires the consideration of various special conditions for the fragments to preserve the continuity of functions. In the third section, we introduce a new type of fragment called (bo)-fragments, which are defined on lattice normed spaces. This new concept is essential for the orthogonally additive operators defined on lattice normed spaces. We investigate the relationship between fragments and (bo)-fragments and show that (bo)-fragments do not, in general, form a Boolean algebra. Moreover, we demonstrate several important properties of (bo)-fragments. In the first section, fragments are initially defined as positive elements of a vector lattice. However, for the purpose of structuring the theory of orthogonally additive operators, it is necessary to consider arbitrary elements of a vector lattice rather than just positive elements. Hence, in the last section, we redefine the concept of fragments as (bo)-fragments on the lattice normed vector lattice $(E, |\cdot|, E)$, where $|\cdot|$ denotes the modulus on E. We present several important propositions, lemmas, and theorems that are crucial for constructing orthogonally additive operators, in this section.

2.1. Fragments in Vector Lattices

In this section, we introduce the concept of fragments in vector lattices. For a more in-depth understanding, readers may refer to any functional analysis or Banach lattice books, such as those listed in the references [20–25].

Definition 2.1.1. A real vector space E is said to be an ordered vector space whenever it is equipped with an order relation $\leq$ that is compatible with the vector space operations in the sense that it satisfies the following two axioms:

(1) if $x \le y$, then $x + z \le y + z$ holds for all $z \in E$.

(2) if $x \le y$, then $\alpha x \le \alpha y$ holds for all $\alpha \ge 0$.

An alternative notation for $x \le y$ is $y \ge x$. A vector x in an ordered vector space E is called positive whenever $x \ge 0$ holds. We denote by $E_+ := \{x \in E \colon x \ge 0\}$ the set of all positive vectors of E. The set E_+ is called the positive cone of E.

Definition 2.1.2. An ordered vector space E is called a vector lattice if for each pair of vectors $x, y \in E$ the supremum and the infimum of the set $\{x, y\}$ both exist in E. We shall use the following notations for supremum and infimum:

$$x \vee y := \sup\{x, y\} \quad \text{and} \quad x \wedge y = \inf\{x, y\}.$$

The following information provides an overview of vector lattices. For a more comprehensive understanding, please see the [20]. In a vector lattice, we define the elements $x^+ := x \vee 0$, $x^- := (-x) \vee 0$ and $|x| := x \vee (-x)$. The element x^+ is called the positive part, x^- is called the negative part, and $|x|$ is called the absolute value of x. In a vector lattice, two elements x and y are said to be disjoint (in symbols $x \perp y$) whenever $|x| \wedge |y| = 0$ holds. The equality $x = y \sqcup z$ means that $x = y + z$ and $y \perp z$. If A is a nonempty subset of a vector lattice E, then its disjoint complement A^d is defined by $A^d := \{x \in E \colon x \perp y \text{ for all } y \in A\}$. A subset A of a vector lattice is called solid whenever $|x| \le |y|$ and $y \in A$ imply $x \in A$. A solid vector subspace of a vector lattice is referred to as an ideal. A net $(x_\alpha)_{\alpha \in \Delta}$ of a vector lattice is said to be order convergent to a vector x (in symbols $x_\alpha \xrightarrow{o} x$ or $o - \lim_{\alpha \in \Delta} x_\alpha = x$) whenever there exists another net $(y_\alpha)_{\alpha \in \Delta}$ with the same index set satisfying $y_\alpha \downarrow 0$ and $|x_\alpha - x| \le y_\alpha$ for all $\alpha \in \Delta$. A subset A of a vector lattice is said to be order closed whenever $(x_\alpha)_{\alpha \in \Delta} \subseteq A$ and $x_\alpha \xrightarrow{o} x$ imply $x \in A$. An order closed ideal is referred to as a band. A band B in a vector lattice E that satisfies $E = B \oplus B^d$ is referred to as a projection band. We note that not every band is a projection band. A vector lattice in which every band is a projection band is said to be a vector lattice with the projection property. Let B be a projection band in a vector lattice E. Thus, $E = B \oplus B^d$ holds, and so every vector

$x \in E$ has a unique decomposition $x = x_1 + x_2$, where $x_1 \in B$ and $x_2 \in B^d$. A projection $P_B : E \to E$ defined by the formula

$$P_B(x) := x_1$$

is said to be an order projection (or a band projection).

Theorem 2.1.3. *If B is a projection band of a vector lattice E, then*

$$P_B(x) = \sup\{y \in B \colon 0 \leq y \leq x\} = \sup B \cap [0, x]$$

holds for all $x \in E_+$

For detailed proof, please refer to [20, Theorem 1.43].

A vector x in a vector lattice E is said to be a projection vector whenever band B_x generated by x (i.e., $B_x = \{y \in E \colon \ |y| \wedge n\,|x| \uparrow |y|\}$) is a projection band. If every vector in a vector lattice is a projection vector, then the vector lattice is said to have the principal projection property. For a projection vector x we shall write P_x for the order projection onto the band B_x.

Theorem 2.1.4. *A vector x in a vector lattice is a projection vector if and only if* $\sup\{y \wedge n\,|x| : n \in \mathbb{N}\}$ *exists for each $y \geq 0$. In this case*

$$P_x(y) = \sup\{y \wedge n\,|x| : n \in \mathbb{N}\}$$

holds for all $y \geq 0$.

For the detailed proof, please refer to [20, Theorem 1.47].

Now we introduce the structure of Boolean algebra. To do so, we remind the concept of lattices. A lattice is a partially ordered set in which every two elements have a supremum and infimum. We shall consider a lattice with the largest and smallest elements. In a lattice, the element with the greatest value is known as the unit (denoted by $\mathbf{1}$), while the element with the least value is known as the zero (denoted by $\mathbf{0}$).

Definition 2.1.5. Let X be a lattice. If $x \in X$ and there exists an $\neg x$ satisfying the properties: $x \vee \neg x = \mathbf{1}$ and $x \wedge \neg x = \mathbf{0}$, then $\neg x$ is called the complement of the element x.

Definition 2.1.6. A Boolean algebra is a distributive lattice with the unit and zero for which the complement of each element exists.

For further information on Boolean algebras, please refer to [21].

Definition 2.1.7. Let V be a real vector space and C be a convex subset of V. An element $x \in C$ is called an extreme point of C if it cannot be expressed as a non-trivial convex combination of other points in C. More specifically, x is an extreme point if and only if any representation of x as a convex combination $x = \lambda y + (1 - \lambda)z$, with $y, z \in C$ and $0 < \lambda < 1$, implies that $x = y = z$.

Definition 2.1.8. Let V be a vector lattice. An element $z \in V_+$ is called a fragment of $x \in V_+$ if $z \wedge (x - z) = 0$. We denote by $\mathcal{C}_x = \{z \in V_+ \colon z \wedge (x - z) = 0\}$ the set of all fragments of x.

We note that, if $z \in \mathcal{C}_x$, then $x - z \in \mathcal{C}_x$. Moreover, by definition, $z \leq x$. We observe that the set $\mathcal{C}_x$ forms a Boolean algebra when it is equipped with the partial order induced by V. The lattice operations defined on $\mathcal{C}_x$ coincide with those of V, and the Boolean complement is given by $\neg z = x - z$. The details are given in the theorem below.

Theorem 2.1.9. *For a positive vector x in a vector lattice V we have:*

(1) If $y, z \in \mathcal{C}_x$ and $z \leq y$ holds, then $y - z \in \mathcal{C}_x$.

(2) If $y_1, y_2, z_1, z_2 \in \mathcal{C}_x$ satisfy the inequalities $y_1 \leq y_2 \leq z_1 \leq z_2$, then $y_2 - y_1 \perp z_2 - z_1$.

(3) If $y, z \in \mathcal{C}_x$, then $y \vee z$ and $y \wedge z$ both belong to $\mathcal{C}_x$ (and so $\mathcal{C}_x$ is a Boolean algebra with smallest element 0 and largest element x).

(4) If V is Dedekind complete, then for every non-empty subset of C of $\mathcal{C}_x$ the element $\sup C$ *and* $\inf C$ *both belong to $\mathcal{C}_x$ (and so in this case $\mathcal{C}_x$ is a Dedekind complete Boolean algebra).*

(5) The set of fragments $\mathcal{C}_x$ of x is precisely the set of all extreme points of the convex set $[0, x]$.

Proof. (1) The proof follows from the following inequalities:

$$\begin{aligned}0 \le (y-z) \wedge (x-(y-z)) &= (y-z) \wedge ((x-y)+z) \\ &\le (y-z) \wedge (x-y) + (y-z) \wedge z \\ &\le y \wedge (x-y) + (x-z) \wedge z \\ &= 0.\end{aligned}$$

(2) We note that $y_2 - y_1 \le z_1 - y_1 \le z_1$ and $z_2 - z_1 \le x - z_1$. Thus, we have

$$(y_2 - y_1) \wedge (z_2 - z_1) \le z_1 \wedge (x - z_1) = 0.$$

(3) Let $y, z \in \mathcal{C}_x$. Then, we have

$$\begin{aligned}0 \le (y \vee z) \wedge (x - y \vee z) &= (y \vee z) \wedge ((x-y) \wedge (x-z)) \\ &= (y \wedge (x-y) \wedge (x-z)) \vee (z \wedge (x-y) \wedge (x-z)) \\ &= 0,\end{aligned}$$

and hence $y \vee z \in \mathcal{C}_x$. By employing a similar technique, it can be shown that $y \wedge z \in \mathcal{C}_x$.

(4) Let V be a Dedekind complete vector lattice and let C be a nonempty subset of $\mathcal{C}_x$. Put by definition, $u = \sup C$. It follows from the infinite distributive laws that

$$\begin{aligned} 0 \leq u \wedge (x-u) &= \sup C \wedge (x-u) \\ &= \sup\{c \wedge (x-u) \colon c \in C\} \\ &\leq \sup\{c \wedge (x-c) \colon c \in C\} \\ &= 0, \end{aligned}$$

and hence $\sup C \in \mathcal{C}_x$. The same thing holds true for $\inf C \in \mathcal{C}_x$.

(5) Let z be an element of the extreme points of $[0,x]$. Put by definition $y = z \wedge (x-z)$. We claim that $y = 0$. Indeed, we first note that

$$0 \leq z - y \leq x - y \leq x$$

and

$$0 \leq z + y \leq z + (z \wedge (x-z)) \leq z + (x-z) = x.$$

Considering the convex combination $z = \frac{1}{2}(z-y) + \frac{1}{2}(z+y)$, we have $z - y = z + y$, and hence $y = 0$ and $z \in \mathcal{C}_x$.

On the other hand, let us assume that $v \in \mathcal{C}_x$ and let $v = \lambda y + (1-\lambda)z$, where $y, z \in [0,x]$ and $\lambda \in (0,1)$. We claim that $v = y = z$. It follows from $v \wedge (x-v) = 0$ that $y \wedge (x-v) = 0$. Thus, we have

$$y = y \wedge x = y \wedge (v + (x-v)) \leq y \wedge v + y \wedge (x-v) = y \wedge v \leq v.$$

Similarly, $z \leq v$. Now assume that $y < v$ or $z < v$ holds true. Then we have

$$v = \lambda y + (1-\lambda)z < \lambda v + (1-\lambda)v = v,$$

which leads to a contradiction. Hence $v = y = z$ holds true, and so v is an extreme point of $[0,x]$.

$\square$

Proposition 2.1.10. *Let V be a vector lattice and $v \in V$. If v is a projection vector, then $P_v x \in \mathcal{C}_x$.*

Proof. First, we note that an element $v \in V_+$ is a fragment of x if and only if $2v \wedge x = v$ by definition. Therefore we have the following:

$$\begin{aligned} 2P_v x \wedge x &= \sup\{2x \wedge 2n\,|v| : n \in \mathbb{N}\} \wedge x \\ &= \sup\{2x \wedge 2n\,|v| \wedge x \colon n \in \mathbb{N}\} \\ &= \sup\{x \wedge 2n\,|v| : n \in \mathbb{N}\} \\ &= P_v x, \end{aligned}$$

and hence $P_v x \in \mathcal{C}_x$. $\square$

Remark 2.1.11. *Let V be a vector lattice with the principal projection property. Consider an element $x \in V_+$ and order interval $[0, x]$. We assume that $\mathcal{C}_x = \{0, x\}$. Take disjoint elements $u, v \in [0, x]$. We claim that in this case, $u = 0$ or $v = 0$. Before starting the proof, we note that since u and v are disjoint, then $P_{u+v} = P_u + P_v$(This is a conclusion of [20, Theorem 1.45]. Since $P_{u+v}(x) \in \mathcal{C}_x$, there are two cases now, either $P_{u+v}(x) = 0$ or $P_{u+v}(x) = x$. If $P_{u+v}(x) = 0$ holds true, then $P_u x = 0$ and $P_v x = 0$, which implies $u = 0$ and $v = 0$. On the other hand, if $P_{u+v}(x) = x$, then $P_u(x) = 0$ or $P_v(x) = 0$. Therefore $u = 0$ or $v = 0$. This finishes the proof.*

2.2. Fragments in C[0,1]

In this section, we study the fragments in the vector lattice $C[0, 1]$. Determining the fragments of a positive element f in $C[0, 1]$ is not always a straightforward task, especially when it comes to preserving continuity. Therefore, we present some significant findings regarding fragments in $C[0, 1]$. Additionally, the majority of the results presented in this section can be found in the article [1].

Definition 2.2.1. Let V be a vector lattice and $f \in V_+$. Fragments of f except the zero and itself are called proper fragments of f.

Lemma 2.2.2. *Let f be a positive element of the vector lattice $C[0,1]$. If $z \in C[0,1]$ is a proper fragment of f, then z has at least one root in the open interval $(0,1)$.*

Proof. Suppose there is no root of z in (0,1). Then

$$z(t) \neq 0, \forall t \in (0,1). \tag{2.2.1}$$

Since z is a proper fragment of f, we have

$$z(t) \wedge (f(t) - z(t)) = 0, \forall t \in [0,1]. \tag{2.2.2}$$

Considering (2.2.1) and (2.2.2), we obtain

$$f(t) - z(t) = 0, \forall t \in (0,1) \Rightarrow f(t) = z(t), \forall t \in (0,1).$$

Since both f and z are continuous functions and $\overline{(0,1)} = [0,1]$, f must be equal to z for all elements of [0,1]. However it is a contradiction because z is a proper fragment of f. Therefore the assumption $z(t) \neq 0, \forall t \in (0,1)$ is wrong. And hence z has at least one root in (0,1). $\square$

Theorem 2.2.3. *Let f be a positive element of the vector lattice $C[0,1]$. If f has a proper fragment, then f has at least one root in the open interval $(0,1)$.*

Proof. Let z be a proper fragment of f. Also suppose that there is no root of f in (0,1). Then

$$f(t) \neq 0, \forall t \in (0,1). \tag{2.2.3}$$

Let us define

$$A = \{t \in (0,1) \colon z(t) \neq 0\}$$

and

$$B = \{t \in (0,1) \colon f(t) = z(t)\}$$

and show that $A = B$. Let $t \in A$. Then $z(t) \neq 0, \forall t \in (0,1)$. Considering that z is a proper fragment of f, we obtain

$$f(t) - z(t) = 0 \Rightarrow f(t) = z(t) \Rightarrow x \in B$$

and hence $A \subseteq B$. Conversely, let $t \in B$. Then $f(t) = z(t)$. Considering the (2.2.3), we have $z(t) \neq 0 \Rightarrow t \in A$, hence $B \subseteq A$.

Now, we investigate the topological properties of A and B.

$$\begin{aligned} A &= \{t \in (0,1) \colon z(t) \neq 0\} \\ &= (0,1) \cap z^{-1}(\mathbb{R} \setminus \{0\}) \end{aligned}$$

Hence, A is an open set in the $\tau_{(0,1)}$ subspace of standard topology τ_s because $\mathbb{R} \setminus \{0\} \in \tau_s$ and $z(t)$ is a continuous function.

$$\begin{aligned} B &= \{t \in (0,1) \colon f(t) = z(t)\} \\ &= (0,1) \cap (f - z)^{-1}(\{0\}) \end{aligned}$$

In this case B is a closed set in $\tau_{(0,1)}$ because $\{0\}$ is a closed set in τ_s and $f - z$ is a continuous function.

Since $(0,1)$ is connected set, we can say that $A = (0,1)$ or $A = \emptyset$. We know that from Lemma 2.2.2 if z is a proper fragment of f in $C[0,1]$, then z has at least one root in $(0,1)$. Hence there exists $t \in (0,1)$ such that $z(t) = 0$ and hence the set A can not be equal $(0,1)$. Therefore $A = \emptyset$ and hence $z(t) = 0$ for all $t \in (0,1)$. Since z is a continuous function and $\overline{(0,1)} = [0,1]$, we obtain $z(t) = 0$ for all $t \in [0,1]$. However it is a contradiction because z is a proper fragment of f. Hence our assumption that f has no root in $(0,1)$ is wrong. $\square$

Corollary 2.2.4. *In $C[0,1]$, the fragments of those functions having no roots in the open interval $(0,1)$ are either themselves or zero.*

It may be asked whether the converse of Theorem 2.2.3 holds. If a function $f(t) \in C[0,1]$ has at least one root in the open interval $(0,1)$, does it have a proper fragment? The subsequent example serves as a counterexample.

Example 2.2.5. Let $f \in C[0,1]$, $c \in (0,1)$. Let us assume $f(t) = 0$ whenever $t \in [0,c]$ and $f(t) > 0$ whenever $t \in (c,1)$. We show that $\mathcal{C}_f = \{0, f\}$. Let us consider $h \in C[0,1]$ and assume $h(t) \wedge (f(t) - h(t)) = 0$. Since $f(t) = 0$ for all elements of $[0,c]$, we obtain $h(t) \wedge -h(t) = 0$ and hence $h(t) = 0$. Now we want to understand h in the interval $(c,1]$. Let us define $g = f|_{[c,1]}$, which is a restriction of f. We can see that $g \in C[c,1]$ and $g(t) > 0$ for all $t \in (c,1)$. Similar to the idea in Theorem 2.2.3, we can see that g has no proper fragment. Therefore fragments of g are only 0 and itself. Since $g = f|_{[c,1]}$, we conclude that the fragments of g correspond to $h|_{[c,1]}$. Therefore $h(x) = 0$ or $h(x) = g(x)$ for all elements of $[c,1]$. Considering the above results we get $h = 0$ or $h = f$ for all elements of $[0,1]$ and hence $\mathcal{C}_f = \{0, f\}$.

2.3. (bo)-Fragments

In this section, we begin by introducing the notion of lattice-normed spaces and lattice-normed vector lattices. We then proceed to explore the concept of fragments in the case of lattice-normed spaces called (bo)-fragments. This new concept holds significance and interest both by itself and for its implications in the theory of orthogonally additive operators defined on lattice-normed spaces. Within this section, we present various properties of (bo)-fragments and explain the relationship between fragments and (bo)-fragments. Furthermore, within the context of the discussed topic, this section consists of several examples and in-depth discussions covering various cases. The majority of the results presented in this section can be found in the article [1].

Definition 2.3.1. Consider a vector space V and a real vector lattice E. A mapping $\|\cdot\| : V \to E_+$ is a vector (E-valued) norm if it satisfies the following axioms:

(1) $\|x\| = 0 \Leftrightarrow x = 0$ for $x \in V$;

(2) $\|\lambda x\| = |\lambda|\|x\|$ for $\lambda \in \mathbb{R}$ and $x \in V$;

(3) $\|x + y\| \leq \|x\| + \|y\|$ for $x, y \in V$.

A triple $(V, \|\cdot\|, E)$ ((V, E) or V for short) is called a lattice normed space (over E) if $\|\cdot\|$ is an E-valued norm in the vector space V. A vector norm is called a decomposable or Kantorovich norm, see [21], if

(4) for all $e_1, e_2 \in E_+$ and $x \in V$, from $\|x\| = e_1 + e_2$ it follows that there exist $x_1, x_2 \in V$ such that $x = x_1 + x_2$ and $\|x_k\| = e_k,\ k = 1, 2$.

Two elements x, y of a lattice-normed space V is said to be disjoint (notation $x \perp y$), if $\|x\| \wedge \|y\| = 0$.

Not every vector norm needs to be decomposable, and we can provide an example of a non-decomposable vector norm, as shown below:

Example 2.3.2. Consider the vector lattice $\mathbb{R}^2$ with respect to pointwise order. We observed that the following map, $\|\cdot\| : \mathbb{R} \to \mathbb{R}^2$ defined by $\|x\| = (|x|, |x|)$, is a vector norm. However, we claim that it is not decomposable. Considering the definition of the norm, we observe that the map is not onto; therefore, we can choose numbers that are not in the range set. Consider the elements $e_1 = (\frac{1}{3}, \frac{1}{4})$ and $e_2 = (\frac{2}{3}, \frac{3}{4})$. Obviously, we have $\|1\| = (1, 1) = (\frac{1}{3} + \frac{2}{3}, \frac{1}{4} + \frac{3}{4}) = e_1 + e_2$. In contrast, no elements $x, y \in \mathbb{R}$ satisfy the conditions $\|x\| = e_1$ and $\|y\| = e_2$.

Proposition 2.3.3. *Every lattice normed space $(V, \|\cdot\|, \mathbb{R})$ is decomposable.*

Proof. Let $x \in V$ and $\|x\| = e_1 + e_2$. Put by definition $x_1 = e_1 \frac{x}{\|x\|}$ and $x_2 = x - x_1$. One can see that $\|x_1\| = e_1$. On the other hand, we have

$$\begin{aligned} \|x_2\| = \|x - x_1\| &= \left\| x - e_1 \frac{x}{\|x\|} \right\| \\ &= \left\| x(1 - \frac{e_1}{\|x\|}) \right\| \\ &= \|x\| \left| 1 - \frac{e_1}{\|x\|} \right| \\ &= (e_1 + e_2) \left| 1 - \frac{e_1}{e_1 + e_2} \right| \\ &= e_2. \end{aligned}$$

Thus, the proof is finished. $\square$

Definition 2.3.4. Suppose that V is a vector lattice. The vector norm $\|\cdot\|$ is called monotone if $|x| \leq |y|$ implies $\|x\| \leq \|y\|$ for $x, y \in V$. In this case $(V, \|\cdot\|, E)$ is called lattice normed vector lattices, briefly LNVL.

It is worth noting that on lattice normed vector lattices, the inequality

$$\big\| |x| - |y| \big\| \leq \|x - y\|$$

holds for all $x, y \in V$.

Example 2.3.5. Consider the lattice normed space $(\mathbb{R}, |\cdot|, \mathbb{R})$ where $|\cdot|$ denotes the absolute value on $\mathbb{R}$. By applying the triangle inequality for the absolute value, we can conclude that it is monotone. As a result, $(\mathbb{R}, |\cdot|, \mathbb{R})$ is lattice normed vector lattice.

We provide an example of a lattice-normed space that is not monotonic. Therefore it is not a lattice normed vector lattice.

Example 2.3.6. Consider the vector lattice $\mathbb{R}^2$ with respect to pointwise order. We claim that the mapping $(\mathbb{R}^2, \|\cdot\|, \mathbb{R}^2)$ defined by

$$\big\| (x, y) \big\| = (|x + y|, |x - y|)$$

is a lattice normed space. However, since the norm is not monotone, we conclude that $(\mathbb{R}^2, \|\cdot\|, \mathbb{R}^2)$ is not LNVL. To begin with, we demonstrate that it is a lattice normed space. Indeed, we have the following:

(1) Given any element $(x, y) \in \mathbb{R}^2$, we obtain,

$$\begin{aligned}\big\|(x,y)\big\| = (0,0) &\iff (|x+y|, |x-y|) = (0,0)\\ &\iff x+y=0 \text{ and } x-y=0\\ &\iff x=y=0\\ &\iff (x,y)=(0,0).\end{aligned}$$

(2) Let $\lambda \in \mathbb{R}$. Thus we have,

$$\begin{aligned}\big\|\lambda(x,y)\big\| &= \big\|(\lambda x, \lambda y)\big\|\\ &= (|\lambda x + \lambda y|, |\lambda x - \lambda y|)\\ &= |\lambda| (|x+y|, |x-y|)\\ &= |\lambda| \big\|(x,y)\big\|.\end{aligned}$$

(3) Given any other element $(z, w) \in \mathbb{R}^2$, we obtain,

$$\begin{aligned}\big\|(x,y)+(z,w)\big\| &= \big\|(x+z, y+w)\big\|\\ &= (\big|(x+z)+(y+w)\big|, \big|(x+z)-(y+w)\big|)\\ &\leq (|x+y| + |z+w|, |x-y| + |z-w|)\\ &= (|x+y|, |x-y|) + (|z+w|, |z-w|)\\ &= \big\|(x,y)\big\| + \big\|(z+w)\big\|.\end{aligned}$$

Therefore, we showed that it is a lattice normed space. We now prove that the norm is not monotone. Consider the elements $(1,-1)$ and $(1,1)$. We see that $\big|(1,-1)\big| = \big|(1,1)\big| = (1,1)$. Therefore we have $\big|(1,-1)\big| \leq \big|(1,1)\big|$. However, for the norms of these elements, we have $\big\|(1,-1)\big\| = (0,2)$ and $\big\|(1,1)\big\| = (2,0)$. Clearly $(0,2) \leq (2,0)$ is invalid. It follows that the norm is not monotone, and hence $(\mathbb{R}^2, \|\cdot\|, \mathbb{R}^2)$ is not a lattice normed vector lattice.

Definition 2.3.7. Let $(V, \|\cdot\|, E)$ be a lattice normed vector lattice. The vector norm $\|\cdot\|$ is order-semicontinuous if $\sup\|x_\alpha\| = \|\sup x_\alpha\|$ for each increasing net $(x_\alpha) \subset V$ with a least upper bound $x \in V$. The vector norm $\|\cdot\|$ is order-continuous if $\inf\|x_\alpha\| = 0$ for any decreasing net $(x_\alpha) \subset V$ with $\inf x_\alpha = 0$.

Definition 2.3.8. Let $(V, \|\cdot\|, E)$ be a lattice normed space. An element $z \in V$ is called a (bo)-fragment of $x \in V$ if $\|z\| \wedge \|x - z\| = 0$. We denote by $\mathcal{C}_x^{bo} = \{z \in V\colon \|z\| \wedge \|x - z\| = 0\}$ the set of all (bo)-fragments of x.

Analogous to the case of fragments, it is worth noting that for any $x \in V$, $\{0, x\} \subseteq \mathcal{C}_x^{bo}$ holds. We note that the element x is not necessarily positive.

Proposition 2.3.9. *Let $(V, \|\cdot\|, \mathbb{R})$ be lattice normed space and $x \in V$. Then $\mathcal{C}_x^{bo} = \{0, x\}$.*

Proof. Let $z \in C_x^{bo}$. It follows from $\|z\|, \|x - z\| \in \mathbb{R}$ that

$$\|z\| \wedge \|x - z\| = 0 \implies \|z\| = 0 \text{ or } \|x - z\| = 0,$$

and hence $z = 0$ or $x = z$. This finishes the proof. $\square$

Proposition 2.3.10. *Let V be a vector lattice and consider the lattice normed vector lattice $(V, |\cdot|, V)$ where $|\cdot|$ is the modulus on V. Let $x, z \in V_+$. If z is a fragment of x then z is a (bo)-fragment of x.*

Proof. Let z be a fragment of x. Since z and $x - z$ are elements of V_+, we have $z \wedge (x - z) = 0$ and so $|z| \wedge |x - z| = 0$. Hence z is a (bo)-fragment of x. $\square$

Proposition 2.3.11. *Let $(V, \|\cdot\|, E)$ be a lattice normed vector lattice and $x \in V_+$, $z \in V$. If z is a (bo)-fragment of x then z is a fragment of x.*

Proof. Let z be a (bo)-fragment of x so that

$$\|z\| \wedge \|x - z\| = 0$$

holds. We show that $\big\|z \wedge (x-z)\big\| \leq \|z\|$ and $\big\|z \wedge (x-z)\big\| \leq \|x-z\|$. For the first inequality we have

$$\begin{aligned}
\big\|z \wedge (x-z)\big\| &= \left\|\frac{1}{2}(z + (x-z) - \big|z - (x-z)\big|)\right\| \\
&= \frac{1}{2}\big\|x - |2z - x|\big\| \\
&= \frac{1}{2}\big\||x| - |2z - x|\big\| \quad \text{(as } x \in V_+\text{)} \\
&\leq \frac{1}{2}\big\|x - (2z - x)\big\| \quad \text{(as the norm is monotone)} \\
&= \frac{1}{2}\|2x - 2z\| \\
&= \|x - z\|\,.
\end{aligned}$$

For the second inequality, we have

$$\begin{aligned}
\big\|z \wedge (x-z)\big\| &= \big\|-(z \wedge (x-z))\big\| \\
&= \big\|-z \vee (z-x)\big\| \\
&= \left\|\frac{1}{2}((-z + (z-x) + \big|-z - (z-x)\big|)\right\| \\
&= \frac{1}{2}\big\|-x + |-2z + x|\big\| \\
&= \frac{1}{2}\big\||-2z + x| - |x|\big\| \quad \text{(as } x \in V_+\text{)} \\
&\leq \frac{1}{2}\|-2z + x - x\| \quad \text{(as the norm is monotone)} \\
&= \|z\|\,.
\end{aligned}$$

It follows that

$$0 \leq \big\|z \wedge (x-z)\big\| \leq \|z\| \wedge \|x - z\| = 0.$$

Hence $z \wedge (x-z) = 0$, and, z is a fragment of x. □

Corollary 2.3.12. *Let $(V, \|\cdot\|, E)$ be a lattice normed vector lattice and $x \in V_+$. Then every (bo)-fragment of x is a positive element of V.*

Proof. Let z be a (bo)-fragment of x. Then z is also a fragment of x. It follows from

$$0 = z \wedge (x - z) \leq z$$

that z is positive. □

Corollary 2.3.13. *Let $(V, |\cdot|, V)$ be a lattice normed vector lattice where $|\cdot|$ is the modulus on V. Then $\mathcal{C}_x = \mathcal{C}_x^{bo}$ whenever $x \in V_+$.*

Proof. The proof follows from Proposition 2.3.10 and Proposition 2.3.11. □

Proposition 2.3.11 shows that $\mathcal{C}_x^{bo} \subseteq \mathcal{C}_x$ holds in lattice normed vector lattices whenever $x \in V_+$. However, there exists a counterexample to show that $\mathcal{C}_x$ is not necessarily a subset of $\mathcal{C}_x^{bo}$ in general.

Example 2.3.14. Consider the vector lattice $C[0, 2\pi]$ with pointwise ordering. We define the function $f(t) = |\sin t|$, and the function $z(t) = \sin t$ whenever $t \in [0, \pi]$ and $z(t) = 0$ whenever $t \in [\pi, 2\pi]$. Both $f(t)$ and $z(t)$ are positive elements of $C[0, 2\pi]$. Moreover $z(t)$ is a fragment of $f(t)$. Indeed, if $t \in [0, \pi]$, then $f(t) = z(t)$, which implies that $z(t) \wedge (f(t) - z(t)) = 0$. On the other hand, if $t \in [\pi, 2\pi]$, then $z(t) = 0$, and therefore $z(t) \wedge (f(t) - z(t)) = 0$. However $z(t)$ is not a (bo)-fragment of $f(t)$ in the lattice normed vector lattice $(C[0, 2\pi], \|\cdot\|_1, \mathbb{R})$ with the integral norm. To see this we note that

$$\|z(t)\| = \int_0^{2\pi} |z(t)| \, dt = \int_0^{\pi} \sin t \, dt = 2$$

and

$$\|f(t) - z(t)\| = \int_0^{2\pi} |f(t) - z(t)| \, dt = \int_{\pi}^{2\pi} |\sin t| \, dt = 2$$

Hence, $\|z(t)\| \wedge \|f(t) - z(t)\| \neq 0$.

The importance of the monotonicity of the E-valued norm $\|\cdot\| : V \to E$ is evident in Proposition 2.3.11. Additionally, it is important to note that Proposition 2.3.11 does not hold for lattice normed spaces in general. The following example demonstrates this fact.

Example 2.3.15. Consider Example 2.3.6. We note that the lattice normed space $(\mathbb{R}^2, \|\cdot\|, \mathbb{R}^2)$ with norm

$$\big\|(x, y)\big\| = (|x + y|, |x - y|)$$

is not a lattice normed vector lattice because the norm is not monotone. Let us consider the element $x = (1, 2) \in \mathbb{R}^2$. Firstly, we observe that

$$\mathcal{C}_x = \{(0, 0), (1, 0), (0, 2), (1, 2)\}.$$

Secondly, we claim that $(-\frac{1}{2}, \frac{1}{2})$ is a (bo)-fragment of x. Indeed, one has

$$\left\|(-\frac{1}{2}, \frac{1}{2})\right\| \wedge \left\|(1, 2) - (-\frac{1}{2}, \frac{1}{2})\right\| = (0, 1) \wedge (3, 0) = (0, 0).$$

However $(-\frac{1}{2}, \frac{1}{2})$ is not a fragment of x. Hence, Proposition 2.3.11 may fail in the case of lattice normed spaces, in general.

Proposition 2.3.16. *Let $(V, \|\cdot\|, E)$ be a lattice normed space and $x \in V$. Then $z \in \mathcal{C}_x^{bo}$ if and only if $-z \in \mathcal{C}_{-x}^{bo}$.*

Proof. Let $z \in \mathcal{C}_x^{bo}$. It follows from

$$\big\|-x - (-z)\big\| \wedge \|-z\| = \|x - z\| \wedge \|z\| = 0$$

that $-z \in \mathcal{C}_{-x}^{bo}$. The same method can be done for the other side. □

Theorem 2.3.17. *Let $(V, \|\cdot\|, E)$ be a lattice normed vector lattice and $x \in V$. Then the following statements hold:*

(1) If $z \in \mathcal{C}_x^{bo}$, then $|z| \in \mathcal{C}_{|x|}^{bo}$;

(2) If $z \in \mathcal{C}_x^{bo}$, then $z^+ \in \mathcal{C}_{x^+}^{bo}$ and $z^- \in \mathcal{C}_{x^-}^{bo}$.

Proof. Firstly, we demonstrate the proof of assertion (1). Let $z \in \mathcal{C}_x^{bo}$. It follows from

$$\big\| |x| - |z| \big\| \leq \|x - z\|$$

that

$$\big\| |x| - |z| \big\| \wedge \big\| |z| \big\| \leq \|x - z\| \wedge \|z\| = 0$$

and hence $|z| \in \mathcal{C}_{|x|}^{bo}$. We now begin the proof of assertion (2). In vector lattices we have

$$\left| x^+ - z^+ \right| \leq |x - z|$$

and

$$0 \leq z^+ \leq |z| \,.$$

Therefore using monotonicity we obtain,

$$\left\| x^+ - z^+ \right\| \leq \|x - z\|$$

and

$$\left\| z^+ \right\| \leq \|z\| \,.$$

It follows that

$$0 \leq \left\| x^+ - z^+ \right\| \wedge \left\| z^+ \right\| \leq \|x - z\| \wedge \|z\| = 0,$$

and hence, $z^+ \in \mathcal{C}_{x^+}^{bo}$. Similar steps can be applied in the case of z^-.

$\square$

We remark that Example 2.3.6 demonstrates that Theorem 2.3.17 does not hold in the more general settings of lattice normed spaces. Indeed, in Example 2.3.6 we showed that $(-\frac{1}{2}, \frac{1}{2}) \in \mathcal{C}_x^{bo}$ for $x = (1, 2)$. However, one can observe that $(-\frac{1}{2}, \frac{1}{2})^+ = (0, \frac{1}{2}) \notin \mathcal{C}_{x^+}^{bo}$.

Proposition 2.3.18. *Let $(V, \|\cdot\|, E)$ be a lattice normed vector lattice and $x \in V$. Then $\mathcal{C}_x^{bo}$ is an order bounded set and furthermore $\mathcal{C}_x^{bo} \subseteq [-x^-, x^+]$.*

Proof. Let $z \in \mathcal{C}_x^{bo}$. According to Theorem 2.3.17, we know that $z^+ \in \mathcal{C}_{x^+}^{bo}$ and $z^- \in \mathcal{C}_{x^-}^{bo}$. Since both x^+ and x^- are positive, by Proposition 2.3.11 we have $z^+ \in \mathcal{C}_{x^+}$ and $z^- \in \mathcal{C}_{x^-}$. Therefore $0 \leq z^- \leq x^-$ and $0 \leq z^+ \leq x^+$. Because

$$-x^- \leq z^+ - z^- = z \leq x^+$$

we have $\mathcal{C}_x^{bo} \subseteq [-x^-, x^+]$. □

The subsequent example provides evidence that Proposition 2.3.18 may not hold true in the more general settings of lattice normed spaces.

Example 2.3.19. Let us consider Example 2.3.6. For element $x = (1, -2) \in \mathbb{R}^2$, we have

$$\mathcal{C}_x^{bo} = \{(0,0), (\frac{3}{2}, -\frac{3}{2}), (-\frac{1}{2}, -\frac{1}{2}), (1, -2)\}$$

and

$$[-x^-, x^+] = [(0, -2), (1, 0)].$$

We can see that $(\frac{3}{2}, -\frac{3}{2}) \notin [(0, -2), (1, 0)]$ and hence $\mathcal{C}_x^{bo}$ is not a subset of the order interval $[-x^-, x^+]$.

It follows from Theorem 2.1.9 that in the case of vector lattices $\mathcal{C}_x = \{z \in V_+ \colon z \wedge (x - z) = 0\}$ is a Boolean algebra for $x > 0$. In the present case it can be asked whether the set $\mathcal{C}_x^{bo} = \{z \in V \colon \|z\| \wedge \|x - z\| = 0\}$ consisting of all (bo)-fragments of x is a Boolean algebra. The following example illustrates that if x is not positive, the answer to this question is negative.

Example 2.3.20. Consider the lattice normed space $(\mathbb{R}^2, \|\cdot\|, \mathbb{R}^2)$ with the norm

$$\big\|(x, y)\big\| = (|x| + |y|, |x| + |y|).$$

We note that this norm is monoton and hence $(\mathbb{R}^2, \|\cdot\|, \mathbb{R}^2)$ is a lattice normed vector lattice. For $x = (-1,2) \in \mathbb{R}^2$, we have $\mathcal{C}_x^{bo} = \{(0,0),(-1,2)\}$. In this case $\mathcal{C}_x^{bo}$ is not a Boolean algebra, since $(0,0) \wedge (-1,2) = (-1,0) \notin \mathcal{C}_x^{bo}$. This shows that $\mathcal{C}_x^{bo}$ is not necessarily a Boolean algebra even in the case of lattice normed vector lattices.

The following result shows that if x is positive, then $\mathcal{C}_x^{bo}$ is a Boolean algebra in the case of lattice normed vector lattices.

Theorem 2.3.21. *Let $(V, \|\cdot\|, E)$ be a lattice normed vector lattice and $x \in V_+$. Then $\mathcal{C}_x^{bo}$ is a Boolean subalgebra of $\mathcal{C}_x$. In particular, $\mathcal{C}_x^{bo}$ is a Boolean algebra in itself.*

Proof. Since $0 \in \mathcal{C}_x^{bo}$, the set $\mathcal{C}_x^{bo} \neq \emptyset$. Additionally, by Proposition 2.3.11, if $x \in V_+$, then $\mathcal{C}_x^{bo} \subseteq \mathcal{C}_x$. Let $z_1, z_2 \in \mathcal{C}_x^{bo}$. Then z_1 and z_2 are also fragments of x. Hence $z_1, z_2, (x - z_1)$ and $(x - z_2)$ are positive elements of V. Now, using this and monotonicity we have the following inequalities;

$$\big\|(x - z_1) \wedge (x - z_2)\big\| \leq \|x - z_1\| \wedge \|x - z_2\| \tag{2.3.1}$$

and

$$\|z_1 \wedge z_2\| \leq \|z_1\| \wedge \|z_2\| . \tag{2.3.2}$$

By using (2.3.1) and (2.3.2) let us show that $z_1 \wedge z_2$ and $z_1 \vee z_2$ belong to $\mathcal{C}_x^{bo}$. First,

$$\begin{aligned}
&\|z_1 \vee z_2\| \wedge \big\|x - (z_1 \vee z_2)\big\| \\
&= \|z_1 \vee z_2\| \wedge \big\|(x - z_1) \wedge (x - z_2)\big\| \\
&\leq \|z_1 \vee z_2\| \wedge \|x - z_1\| \wedge \|x - z_2\| \\
&\leq (\|z_1\| + \|z_2\| + \|z_1 \wedge z_2\|) \wedge \|x - z_1\| \wedge \|x - z_2\| \\
&\leq (\|z_1\| \wedge \|x - z_1\| \wedge \|x - z_2\|) \\
&+ (\|z_2\| \wedge \|x - z_1\| \wedge \|x - z_2\|) \\
&+ (\|z_1 \wedge z_2\| \wedge \|x - z_1\| \wedge \|x - z_2\|) \\
&= 0.
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\|z_1 \wedge z_2\| \wedge \|x - (z_1 \wedge z_2)\| \\
&= \|z_1 \wedge z_2\| \wedge \|(x - z_1) \vee (x - z_2)\| \\
&\leq \|z_1 \wedge z_2\| \wedge (\|x - z_1\| + \|x - z_2\| + \|x - z_1\| \wedge \|x - z_2\|) \\
&\leq (\|z_1 \wedge z_2\| \wedge \|x - z_1\|) + (\|z_1 \wedge z_2\| \wedge \|x - z_2\|) \\
&+ (\|z_1 \wedge z_2\| \wedge \|x - z_1\| \wedge \|x - z_2\|) \\
&= 0,
\end{aligned}$$

and hence, the result. □

According to Theorem 2.1.9, it is true that

$$\mathcal{C}_x = \{z \in V_+ \colon z \wedge (x - z) = 0\} = \text{ext}[0, x]$$

in the case of vector lattices. However, in the present case, the question arises whether a similar statement holds true for $\mathcal{C}_x^{bo}$. Example 2.3.20 provides a counterexample, demonstrating that this claim does not hold true in either lattice normed spaces or lattice normed vector lattices.

Now, we define the set $\|\mathcal{C}_x^{bo}\| = \{\|z\| \in E \colon z \in \mathcal{C}_x^{bo}\}$ which consists of the norms of the elements of $\mathcal{C}_x^{bo}$.

Proposition 2.3.22. *Let $(V, \|\cdot\|, E)$ be a lattice normed space and $x \in V$. Then the set $\|\mathcal{C}_x^{bo}\|$ is order bounded and $\|\mathcal{C}_x^{bo}\| \subseteq [0, \|x\|]$.*

Proof. To prove this, take an element $\|z\| \in \|\mathcal{C}_x^{bo}\|$. It follows from

$$\|z\| \wedge \|x - z\| = 0$$

that

$$\|z + x - z\| = \|z\| + \|x - z\|,$$

and hence,

$$0 \leq \|z\| = \|x\| - \|x - z\| \leq \|x\| .$$

We conclude that $\|z\| \in [0, \|x\|]$. $\square$

The knowledge that $\mathcal{C}_x^{bo}$ is an order bounded set on lattice normed vector lattices plays a central role in the next theorem.

Theorem 2.3.23. *Let $(V, \|\cdot\|, E)$ be an order continuous lattice normed vector lattice where V and E are order complete vector lattices. Then $\mathcal{C}_x^{bo}$ is order complete for every $x \in V_+$.*

Proof. Consider any non empty subset A of $\mathcal{C}_x^{bo}$. Since $\mathcal{C}_x^{bo}$ is order bounded by Proposition 2.3.18, A is order bounded in V and $\alpha = \sup A$ exists in V. We need to prove that α is an element of $\mathcal{C}_x^{bo}$. For every $y \in A$, we have $0 \leq y \leq \alpha \leq x$ and hence $0 \leq x - \alpha \leq x - y$. Since norm is monotone, we get

$$\|x - \alpha\| \leq \|x - y\|$$

and

$$0 \leq \|y\| \wedge \|x - \alpha\| \leq \|y\| \wedge \|x - y\| .$$

It follows that $\|y\| \wedge \|x - \alpha\| = 0$ for every y in A. Consider the set $\|A\| = \{\|y\| : y \in A\} \subset E$. It is bounded above by $\|x\|$ and since E is order complete, $\sup\|A\|$ exists. As the norm is order-semicontinuous $\sup\|A\| = \|\sup A\| = \|\alpha\|$. Since

$$\|\alpha\| \wedge \|x - \alpha\| = \sup\{\|y\| \wedge \|x - \alpha\| : y \in A\} = 0$$

we obtain that $\alpha \in \mathcal{C}_x^{bo}$. $\square$

Proposition 2.3.24. *Let $(V, \|\cdot\|, E)$ be lattice normed space and $x \in V$. Then $\left\|\mathcal{C}_x^{bo}\right\| \subseteq \mathcal{C}_{\|x\|}$.*

Proof. Let $\|z\| \in \left\|\mathcal{C}_x^{bo}\right\|$. Since $0 \leq \|z\| \leq \|x\|$, we obtain

$$0 \leq \|x\| - \|z\| \leq \|x - z\|$$

and

$$0 \leq \|z\| \wedge (\|x\| - \|z\|) \leq \|z\| \wedge \|x - z\| \,.$$

It follows that $\|z\| \wedge (\|x\| - \|z\|) = 0$, and hence, $\|z\| \in \mathcal{C}_{\|x\|}$. $\square$

Proposition 2.3.25. *Let $(V, \|\cdot\|, E)$ be lattice normed space and $x \in V$. Then $\|\mathcal{C}_x^{bo}\| \subseteq$ ext$[0, \|x\|]$.*

Proof. Since $\mathcal{C}_{\|x\|} = \text{ext}[0, \|x\|]$, the proof follows from Proposition 2.3.24. $\square$

We know that $\mathcal{C}_x = \text{ext}[0, x]$ holds in vector lattices. Because of that, it can be asked whether $\|\mathcal{C}_x^{bo}\| = \text{ext}[0, \|x\|]$. So many examples like Example 2.3.6, Example 2.3.14 and Example 2.3.20 show us that if the norm on V is surjective on E_+, then $\|\mathcal{C}_x^{bo}\| = \text{ext}[0, \|x\|]$; otherwise, $\|\mathcal{C}_x^{bo}\| \neq \text{ext}[0, \|x\|]$. However, the following example shows that even if the norm on V is not surjective on E_+, $\|\mathcal{C}_x^{bo}\|$ can be equal to ext$[0, \|x\|]$.

Example 2.3.26. Consider the lattice normed space $(\mathbb{R}^3, \|\cdot\|, \mathbb{R}^3)$ with the vector norm

$$\|(x, y, z)\| = (|x + y| + |z|\,, |x - y|\,, 0)$$

for $(x, y, z) \in \mathbb{R}^3$. We note that, in this case, $\|\cdot\|$ is not surjective on the positive cone of $\mathbb{R}^3$. However, for $x = (1, 0, -2)$, we have $\|\mathcal{C}_x^{bo}\| = \text{ext}[0, \|x\|]$.

After seeing Example 2.3.26, we searched again for the necessary conditions that make $\|\mathcal{C}_x^{bo}\|$ equal to ext$[0, \|x\|]$. In the view of some examples, we decide the norm on V must be surjective on the positive cone of an ideal of E. This claim is still open.

The following is an open question: Which conditions do we need to make $\|\mathcal{C}_x^{bo}\|$ equal to ext$[0, \|x\|]$?

Remark 2.3.27. *Let us consider $(C[0, 1], |\cdot|, C[0, 1])$ lattice normed vector lattice where $|\cdot|$ is the modulus on $C[0, 1]$. For taken positive element $f \in C[0, 1]$, we have $\mathcal{C}_f = \mathcal{C}_f^{bo}$ from*

Corollary 2.3.13. Therefore the results stated in this chapter for fragments of f are also valid for (bo)-fragments of f.

Remark 2.3.28. *Let us consider $(C[0,1], \|\cdot\|, E)$ lattice normed vector lattice. For positive element $f \in C[0,1]$, we have $\mathcal{C}_f^{bo} \subseteq \mathcal{C}_f$ from Proposition 2.3.11. In this case assuming $f(t) > 0$, $\forall t \in (0,1)$ and using Theorem 2.2.3, we obtain $\mathcal{C}_f = \{0, f\}$ and hence we have $\mathcal{C}_f^{bo} = \mathcal{C}_f = \{0, f\}$.*

Remark 2.3.29. *Let us consider $(V, \|\cdot\|, C[0,1])$ lattice normed space. For an arbitrary element $f \in V$, we have $\left\|\mathcal{C}_f^{bo}\right\| \subseteq \mathcal{C}_{\|f\|}$ from Proposition 2.3.24. This idea can help us to determine the set $\left\|\mathcal{C}_f^{bo}\right\|$.*

The following example shows how Remark 2.3.29 is vital for determining the set $\left\|\mathcal{C}_f^{bo}\right\|$.

Example 2.3.30. Consider the lattice normed space $(C[0,1], \|\cdot\|, C[0,1])$ with the norm

$$\|f\| = \int_0^u \left|f(t)\right| \, dt$$

for $f \in C[0,1]$. Let us assume that $f \neq 0$. Assign $\|f\| = g(u)$ and notice that $g(0) = 0$ and $g(u)$ is non-decreasing function. Now we have two situations. In the first case, the function g is zero up to a certain point of the interval $(0,1)$ and positive after that. In the second case, there is no root of g in $(0,1)$. Thanks to Example 2.2.5 in the first case and Theorem 2.2.3 in the second case, we can see that there is no proper fragment of g. Now considering Remark 2.3.29, we obtain

$$\left\|\mathcal{C}_f^{bo}\right\| = \mathcal{C}_{\|f\|} = \{0, \|f\|\}.$$

2.4. A specific case of (bo)-fragments

Fragments are commonly defined as positive elements of a vector lattice in many studies (see [20, 21]). However, to understand the theory of orthogonally additive operators, we must redefine this concept in the following section. When defining orthogonally additive operators on vector lattices or C-complete vector lattices, it is preferable to choose fragments as arbitrary elements, rather than restricting them to only positive elements of the corresponding vector lattice. This approach allows us to avoid the problem of expanding an orthogonally additive operator only along the positive cone of the corresponding vector lattice, as well as other related issues. Because of that, we give a new definition of fragments for the following sections. Let E be a vector lattice and $x \in E$. An element $z \in E$ is called a fragment of x, if $z \perp (x - z)$, that is, $|z| \wedge |x - z| = 0$. We use the notation $z \sqsubseteq x$ to say that z is a fragment of x. We denote by $\mathcal{C}_x = \{z \in V\colon\ |z| \wedge |x - z| = 0\}$ the set of all fragments of x. We say that $y, z \in \mathcal{C}_x$ are mutually complemented, if $x = y \sqcup z$. It should be noted that the revised definition of fragments corresponds to the concept of (bo)-fragments on the lattice normed vector lattice $(E, |\cdot|, E)$, where $|\cdot|$ denotes the modulus on E. Furthermore, this new definition is a specific case of (bo)-fragments. Consequently, all the findings regarding (bo)-fragments of a lattice normed vector lattice are applicable to the fragments. In this section, we discuss some details about fragments which have a significant role in the theory of orthogonally additive operators. In this subsection, we refer to the [1, 15, 16]

Theorem 2.4.1. *Let E be a vector lattice and $x \in E$. Then the following statements hold:*

(1) If $z \in \mathcal{C}_x$ then $|z| \in \mathcal{C}_{|x|}$;

(2) $z \in \mathcal{C}_x$ if and only if $z^+ \in \mathcal{C}_{x^+}$ and $z^- \in \mathcal{C}_{x^-}$.

Proof. We note that assertion (1) is proved in Theorem 2.3.17. Moreover, by revisiting Theorem 2.3.17, one can see that it is enough to demonstrate only the sufficiency aspect of the proof of assertion (2). Let $z^+ \sqsubseteq x^+$ and $z^- \sqsubseteq x^-$. It follows that $z^+ \leq x^+$ and $z^- \leq x^-$. Then $0 \leq z^+ \wedge x^- \leq x^+ \wedge x^- = 0$, and hence $z^+ \perp x^-$. Therefore one has

$$\begin{aligned} 0 \leq \left|z^+\right| \wedge |x - z| &= z^+ \wedge \left|x^+ - x^- - z^+ + z^-\right| \\ &\leq z^+ \wedge \left|x^+ - z^+\right| + z^+ \wedge \left|x^-\right| + z^+ \wedge \left|z^-\right| \\ &= 0, \end{aligned}$$

and hence $z^+ \sqsubseteq (x-z)$. The same holds true for $z^- \sqsubseteq (x-z)$. It directly follows that

$$|z| \wedge |x-z| = (z^+ + z^-) \wedge |x-z| \leq z^+ \wedge |x-z| + z^- \wedge |x-z| = 0.$$

Thus, the proof is complete. $\square$

Proposition 2.4.2. *Let E be a vector lattice. Then the binary relation $\sqsubseteq$ is a partial order in E.*

Proof. It's clear that $x \sqsubseteq x$. Assume $x \sqsubseteq y$ and $y \sqsubseteq x$. It follows from $|x| \wedge |y-x| = 0$ and $|y| \wedge |x-y| = 0$ that

$$\begin{aligned} |x-y| &\leq |x| + |y| \\ &\leq (|x| + |y|) \wedge |x-y| \\ &\leq (|x| \wedge |x-y|) + (|y| \wedge |x-y|) \\ &= 0, \end{aligned}$$

and hence $x = y$. Now we show that transitivity. We first show that the transitivity holds for positive elements of E. So let us take $x, y, z \in E_+$ with $z \sqsubseteq y$ and $y \sqsubseteq x$. It follows from $z \wedge (y-z) = 0$, $y \wedge (x-y) = 0$ and $z \leq y$ that

$$\begin{aligned} (x-z) \wedge z = ((x-y) + (y-z)) \wedge z &\leq ((x-y) \wedge z) + ((y-z) \wedge z)) \\ &\leq (x-y) \wedge y \\ &= 0, \end{aligned}$$

and hence $z \sqsubseteq x$. Take any arbitrary $x, y, z \in E$ with $z \sqsubseteq y$ and $y \sqsubseteq x$. Because of the above and Theorem 2.4.1, we get

$$z^+ \sqsubseteq y^+ \quad \text{and} \quad y^+ \sqsubseteq x^+ \implies z^+ \sqsubseteq x^+$$

and

$$z^- \sqsubseteq y^- \quad \text{and} \quad y^- \sqsubseteq x^- \implies z^- \sqsubseteq x^-.$$

Thus we obtain $z \sqsubseteq x$ $\square$

Lemma 2.4.3. *Let E be a vector lattice and $x \in E$. Then the set $\mathcal{C}_x$ is order closed.*

Proof. Take a net $(y_\alpha)_{\alpha \in \Delta}$ in $\mathcal{C}_x$. Assume $(y_\alpha) \xrightarrow{o} y$ and $y \in E$. We claim that $y \sqsubseteq x$. Indeed, by the order continuity of the lattice operations, we have

$$0 = |y_\alpha| \wedge |x - y_\alpha| \xrightarrow{o} |y| \wedge |x - y|,$$

and hence $|y| \wedge |x - y| = 0$. $\square$

Proposition 2.4.4. *Let E be a vector lattice and $x \in E$. Then the set $\mathcal{C}_x$ equipped with the partial order $\sqsubseteq$ is a Boolean algebra with the least element 0, maximal element x and with the respect of Boolean operations: $z \cup y := (z^+ \vee y^+) - (z^- \vee y^-)$, $z \cap y := (z^+ \wedge y^+) - (z^- \wedge y^-)$, $\neg z = x - z$.*

Proof. We note that by Theorem 2.1.9, the structures $\mathcal{C}_{x^+}$ and $\mathcal{C}_{x^-}$ are Boolean algebras equipped with zero 0, units x^+ and x^- respectively. Additionally, we note that the cartesian product $\mathcal{C}_{x^+} \times \mathcal{C}_{x^-}$ is a Boolean algebra equipped with zero $(0, 0)$, unit (x^+, x^-) and lattice operations $(y_1, z_1) \vee (y_2, z_2) = (y_1 \vee y_2, z_1 \vee z_2)$, $(y_1, z_1) \wedge (y_2, z_2) = (y_1 \wedge y_2, z_1 \wedge z_2)$ and $\neg(y_1, z_1) = (x^+ - y_1, x^- - z_1)$. Consider the mapping $\Phi : \mathcal{C}_{x^+} \times \mathcal{C}_{x^-} \to \mathcal{C}_x$ defined by the formula $\Phi((a, b)) = a - b$. We claim that Φ is a bijection and it induces the Boolean algebra on $\mathcal{C}_x$. Firstly, we show that $\Phi((a, b)) \in \mathcal{C}_x$. It follows from $a \sqsubseteq x^+$ and $b \sqsubseteq x^-$ that

$0 \leq a \wedge b \leq x^+ \wedge x^- = 0$. Thus, we have

$$\begin{aligned}
&|a-b| \wedge \big|x-(a-b)\big| \\
&\qquad = |a-b| \wedge \big|x^+ - x^- - a + b\big| \\
&\qquad \leq (|a|+|b|) \wedge \big(\big|x^+ - a\big| + \big|x^- - b\big|\big) \\
&\qquad \leq |a| \wedge \big|x^+ - a\big| + |a| \wedge \big|x^- - b\big| + |b| \wedge \big|x^+ - a\big| + |b| \wedge \big|x^- - b\big| \\
&\qquad = |a| \wedge \big|x^- - b\big| + |b| \wedge \big|x^+ - a\big| \\
&\qquad \leq |a| \wedge \big|x^-\big| + |a| \wedge |b| + |b| \wedge \big|x^+\big| + |b| \wedge |a| \\
&\qquad \leq \big|x^+\big| \wedge \big|x^-\big| + \big|x^-\big| \wedge \big|x^+\big| \\
&\qquad = 0.
\end{aligned}$$

Secondly, we show that Φ is a bijection. Take an element $y \in \mathcal{C}_x$. Clearly, $y = y^+ - y^-$, $y^+ \sqsubseteq x^+$ and $y^- \sqsubseteq x^-$. Therefore there exist a pair $(y^+, y^-) \in \mathcal{C}_{x^+} \times \mathcal{C}_{x^-}$ such that $\Phi((y^+, y^-)) = y$. Moreover, let $\Phi((a,b)) = \Phi((c,d))$ for taken elements $(a,b), (c,d) \in \mathcal{C}_{x^+} \times \mathcal{C}_{x^-}$. It directly follows that

$$\begin{aligned}
a - b = c - d \implies |a-c| &= |b-d| \wedge |a-c| \\
&\leq (|b|+|d|) \wedge (|a|+|c|) \\
&\leq |b| \wedge |a| + |b| \wedge |c| + |d| \wedge |a| + |d| \wedge |c| \\
&= |b| \wedge |c| + |d| \wedge |a| \\
&\leq \big|x^-\big| \wedge \big|x^+\big| + \big|x^-\big| \wedge \big|x^+\big| \\
&= 0,
\end{aligned}$$

and hence $a = c$. The same method can be used to show that $b = d$. Thus, $(a,b) = (c,d)$. Consider the lattice operations $\cup$, $\cap$ and $\neg$ defined by $z \cup y := (z^+ \vee y^+) - (z^- \vee y^-)$, $z \cap y := (z^+ \wedge y^+) - (z^- \wedge y^-)$ and $\neg z = x - z$ for all $y, z \in \mathcal{C}_x$. For taken any $y, z \in \mathcal{C}_x$, we have the following:

$$\begin{aligned} z \cup y &= (z^+ \vee y^+) - (z^- \vee y^-) \\ &= \Phi((z^+ \vee y^+, z^- \vee y^-)) \\ &= \Phi((z^+, z^-) \vee (y^+, y^-)), \end{aligned}$$

and hence $z \cup y \in \mathcal{C}_x$. The same thing holds true for $z \cap y \in \mathcal{C}_x$. Moreover, we note that $\Phi((0,0)) = 0$ and $\Phi((x^+, x^-)) = x$. Hence $\mathcal{C}_x$ is a Boolean algebra with the least element 0 and maximal element x. $\square$

Proposition 2.4.5. *Let E be a vector lattice, $x, y, z, v \in E$ and $z \sqcup v = x \sqcup y$. Then there exist elements $z_1, z_2, v_1, v_2 \in E$ such that*

(1) $z = z_1 \sqcup z_2; v = v_1 \sqcup v_2$;

(2) $x = z_1 \sqcup v_1; y = z_2 \sqcup v_2$.

For the detailed proof, please refer to [16, Lemma 3.5].

3. On Orthogonally Additive Operators

In this chapter, we introduce the concept of orthogonally additive operators in vector lattices. We provide important examples and discuss significant properties of orthogonally additive operators. Additionally, we introduce several new classes of orthogonally additive operators, including C-bounded, regular, disjointness preserving, non-expanding, narrow, and strictly narrow operators. We investigate the relationships among these operators and support our findings with various examples, propositions, lemmas, and theorems. Subsequently, we explore the extension of orthogonally additive operators. Consequently, we analyze the suitable conditions for extending orthogonally additive maps, focusing specifically on the extension problem in C-complete vector lattices. Finally, we introduce the concept of projection lateral bands and examine this topic in relation to orthogonally additive maps.

3.1. Preliminaries

In this section, we provide an introduction to orthogonally additive operators and cover fundamental topics related to this area. The majority of the results presented in this section can be found in the [4, 13].

Definition 3.1.1. Let E be a vector lattice and X be a real vector space. An operator $T : E \to X$ is called orthogonally additive if $T(x + y) = Tx + Ty$ for every disjoint elements $x, y \in E$.

It is important to note that, for any orthogonally additive operator T, the condition $T(0) = 0$ holds true. Indeed, since $0 \perp 0$, one has $T(0 + 0) = T(0) + T(0)$, and hence $T(0) = 0$.

Example 3.1.2. Let $T : \mathbb{R} \to \mathbb{R}$ be a function with $T(0) = 0$. Then it is an orthogonally additive operator. Indeed, take disjoint elements $x, y \in \mathbb{R}$. It directly follows that $x = 0$ or $y = 0$. Let $y = 0$. Therefore one has

$$T(x + y) = Tx = Tx + 0 = Tx + Ty.$$

Corollary 3.1.3. *Let $T : \mathbb{R} \to \mathbb{R}$ be a function. Then T is an orthogonally additive operator if and only if $T(0) = 0$.*

Remark 3.1.4. *Let $\mathcal{OA}(E, X)$ denote the set of all orthogonally additive operators from E to X. It is noteworthy that $\mathcal{OA}(E, X)$ forms a real vector space.*

The following proposition presents the structures of orthogonally additive operators defined on finite-dimensional spaces.

Proposition 3.1.5. *Let $T : \mathbb{R}^n \to \mathbb{R}^m$. The operator T is orthogonally additive if and only if there exist real functions $T_{ij} : \mathbb{R} \to \mathbb{R}$, $1 \leq i \leq m$, $1 \leq j \leq n$ satisfying $T_{ij}(0) = 0$ such that*

$$(T(x_1, ..., x_n))_i = \sum_{j=1}^{n} T_{ij}(x_j)$$

Proof. To begin, we shall establish the validity of the "if" implication. Take disjoint $x = (x_1, ..., x_n), y = (y_1, ..., y_n) \in \mathbb{R}^n$. We show that $T(x + y) = Tx + Ty$. Let us define the following sets:

$$I = \{k \in \{1, ..., n\} \colon y_k = 0\}$$

and $J = \{1, ..., n\} \setminus I$. By definition one has $J \cap I = \emptyset$ and $J \cup I = \{1, ..., n\}$. We note that if $k \in J$, then $x_k = 0$. Indeed, if we consider the disjointness of x and y, we obtain the conclusion $x_i = 0$ or $y_i = 0$ for all $i \in \{1, ..., n\}$. It directly follows that

$$k \in J \implies k \notin I \implies y_k \neq 0 \implies x_k = 0.$$

Considering the definition of T, we have

$$T(x + y) = (\sum_{j=1}^{n} T_{1j}(x_j + y_j), ..., \sum_{j=1}^{n} T_{mj}(x_j + y_j)).$$

Taken any $i \in \{1, ..., m\}$, we have the following:

$$\begin{aligned}
\sum_{j=1}^{n} T_{ij}(x_j + y_j) &= \sum_{j \in I} T_{ij}(x_j + y_j) + \sum_{j \in J} T_{ij}(x_j + y_j) \\
&= \sum_{j \in I} T_{ij}(x_j) + \sum_{j \in J} T_{ij}(y_j) \\
&= \sum_{j \in I} T_{ij}(x_j) + \sum_{j \in J} T_{ij}(x_j) + \sum_{j \in J} T_{ij}(y_j) + \sum_{j \in I} T_{ij}(y_j) \\
&= \sum_{j=1}^{n} T_{ij}(x_j) + \sum_{j=1}^{n} T_{ij}(y_j),
\end{aligned}$$

and hence $T(x + y) = Tx + Ty$ and T is an orthogonally additive operator. We are now ready to give proof for the "only if" implication of the proposition. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be an orthogonally additive operator. By definition $Tx = (\tilde{T}_1 x, ..., \tilde{T}_m x)$. Take any $i \in \{1, ..., m\}$ and $x = (x_1, ..., x_n) \in \mathbb{R}^n$. Since T is an orthogonally additive operator, we have the following:

$$\tilde{T}_i(x_1, ..., x_n) = \tilde{T}_i(x_1, 0, 0, ..., 0) + \tilde{T}_i(0, x_2, 0, ..., 0) + ... + \tilde{T}_i(0, 0, 0, ..., x_n).$$

Take any $j \in \{1, ..., n\}$. Consider the function $T_j : \mathbb{R} \to \mathbb{R}^n$ defined by $T_j(x) = (0, 0, 0, ..., x, ..., 0)$, whose j^{th} term is x and every other zero. Put by definition $T_{ij} := \tilde{T}_i \circ T_j$. We note that $T_{ij} : \mathbb{R} \to \mathbb{R}$ and

$$\begin{aligned}
\tilde{T}_i(x_1, ..., x_n) &= \tilde{T}_i(x_1, 0, 0, ..., 0) + \tilde{T}_i(0, x_2, 0, ..., 0) + ... + \tilde{T}_i(0, 0, 0, ..., x_n) \\
&= T_{i1}(x_1) + T_{i2}(x_2) + ... + T_{in}(x_n) \\
&= \sum_{j=1}^{n} T_{ij}(x_j).
\end{aligned}$$

In that case, in order to complete the proof it is enough to show that $T_{ij}(0) = 0$. Since T is an orthogonally additive operator, one has $T(0) = 0$. It follows that $\tilde{T}_i(0) = 0$. Thus, $T_{ij}(0) = \tilde{T}_i(0) = 0$. This finishes the proof. $\square$

Let (A, Σ, μ) be a finite measure space. For a given $f \in L_0(\mu)$ by supp$f$, we denote the measurable set

$$\mathrm{supp} f := \{t \in A \colon f(t) \neq 0\}.$$

The characteristic function of a set D is denoted by $\mathbf{1}_D$.

Definition 3.1.6. Let (A, Σ, μ) and $(B, \mho, \upsilon)$ be finite measure space and $f \in L_0(\mu)$. By $(A \times B, \mu \otimes \upsilon)$ we denote the completion of their product measure space. A map $K : A \times B \times \mathbb{R} \to \mathbb{R}$ is said to be a Carathêodory function if it satisfies the following conditions:

(C_1) $K(\cdot, \cdot, r)$ is $\mu \otimes \upsilon$-measurable for all $r \in \mathbb{R}$;

(C_2) $K(s, t, \cdot)$ is continuous on $\mathbb{R}$ for $\mu \otimes \upsilon$-almost all $(s, t) \in A \times B$.

A Carathêodory function K is called normalized if $K(\cdot, \cdot, 0) = 0$ for $\mu \otimes \upsilon$-almost all $(s, t) \in A \times B$.

Proposition 3.1.7. *Let E be an order ideal of $L_0(\upsilon)$, $K : A \times B \times \mathbb{R} \to \mathbb{R}$ be a normalized Carathêodory function and for every $f \in E$ the inequality*

$$\int_B \big|K(s, t, f(t))\big| \, d\upsilon(t) < \infty$$

holds for almost all $s \in A$. Then an orthogonally additive operator $T : E \to L_0(\mu)$ is defined by setting

$$Tf(s) = \int_B K(s, t, f(t)) \, d\upsilon(t).$$

Proof. In the first step of the proof, we show that for every $f \in E$ and μ-almost all $s \in A$ the following equality holds

$$\int_B K(s, t, f\mathbf{1}_{\mathrm{supp} f}(t)) \, d\upsilon(t) = \int_B K(s, t, f(t))\mathbf{1}_{\mathrm{supp} f} \, d\upsilon(t).$$

Let us define the set $D := \{(s, t) \in A \times B \colon K(s, t, 0) \neq 0\}$. By definition of D, we have $(\mu \otimes \upsilon)(D) = 0$. Given any $(s, t) \notin D$, there are two cases: either $t \in \mathrm{supp} f$ or $t \notin \mathrm{supp} f$.

If $t \in \text{supp} f$ we have

$$K(s,t,f\mathbf{1}_{\text{supp} f}(t)) = K(s,t,f(t)) = K(s,t,f(t))\mathbf{1}_{\text{supp} f}.$$

For the opposite side, if $t \notin \text{supp} f$ we have

$$K(s,t,f\mathbf{1}_{\text{supp} f}(t)) = 0 = K(s,t,f(t))\mathbf{1}_{\text{supp} f}.$$

Thus $K(s,t,f\mathbf{1}_{\text{supp} f}(t)) = K(s,t,f(t))\mathbf{1}_{\text{supp} f}$ holds true for all $(s,t) \notin D$. Let us define the set

$$H := \{s \in A\colon \int_B K(s,t,f\mathbf{1}_{\text{supp} f}(t))\, d\upsilon(t) \neq \int_B K(s,t,f(t))\mathbf{1}_{\text{supp} f}\, d\upsilon(t)\},$$

and assume $\mu(H) > 0$. Consider the υ-measurable set $H_s := \{t \in B\colon (s,t) \in D\}$. One can see that $\upsilon(H_s) > 0$ for all $s \in H$. Additionally, for the set $H' = \{(s,t)\colon s \in H, t \in H_s\} \subseteq D$, we have $(\mu \otimes \upsilon)(H') = 0$. However, by [26, Theorem 3.4.1]

$$(\mu \otimes \upsilon)(H') = \int_B H_s\, d\mu(s) > 0,$$

which is a contradiction. In the next step, we show that T is an orthogonally additive operator. Take any disjoint elements $f, g \in E$. Then for almost all $t \in B$, we have

$$\begin{aligned}
f(t) = 0 \;\text{ or }\; g(t) = 0 &\iff t \notin \text{supp} f \;\text{ or }\; t \notin \text{supp} g \\
&\iff t \in (\text{supp} f)^c \;\text{ or }\; t \in (\text{supp} g)^c \\
&\iff t \in (\text{supp} f)^c \cup (\text{supp} g)^c \\
&\iff t \notin \text{supp} f \cap \text{supp} g,
\end{aligned}$$

and hence $\upsilon\{t \in \text{supp} f \cap \text{supp} g\} = 0$. It directly follows that

$$\mathbf{1}_{\text{supp} f \cup \text{supp} g} = \mathbf{1}_{\text{supp} f} + \mathbf{1}_{\text{supp} g}.$$

Therefore, we have the following:

$$\begin{aligned}
T(f+g) &= \int_B K(s,t,(f+g)(t))\,dv(t) \\
&= \int_B K(s,t,(f+g)(t))\mathbf{1}_{\text{supp}(f+g)}(t)\,dv(t) \\
&= \int_B K(s,t,(f+g)(t))(\mathbf{1}_{\text{supp}f}+\mathbf{1}_{\text{supp}g})(t)\,dv(t) \\
&= \int_B K(s,t,(f+g)(t))(\mathbf{1}_{\text{supp}f})(t)\,dv(t) + \int_B K(s,t,(f+g)(t))(\mathbf{1}_{\text{supp}g})(t)\,dv(t) \\
&= \int_B K(s,t,(f+g)\mathbf{1}_{\text{supp}f}(t))\,dv(t) + \int_B K(s,t,(f+g)\mathbf{1}_{\text{supp}g}(t))\,dv(t) \\
&= \int_B K(s,t,f(t))\,dv(t) + \int_B K(s,t,g(t))\,dv(t) \\
&= Tf + Tg
\end{aligned}$$

□

In the theory of linear operators, positive operators are defined as mappings that transform positive elements into positive elements. However, in the context of orthogonally additive operators, it is more suitable to define positive operators as mappings that transform any element into a positive element. The motivation behind this will become more obvious in subsequent sections.

Definition 3.1.8. Let E and F be vector lattices. An orthogonally additive operator $T : E \to F$ is called:

(1) positive if $Tx \geq 0$ for all $x \in E$;

(2) order bounded if it maps order bounded subsets of E to order bounded subsets of F.

In the context of linear operators, every positive operator is order bounded. However, this does not hold true in the context of orthogonally additive operators. The following example shows that a positive orthogonally additive operator may not be order bounded.

Example 3.1.9. Consider the mapping $T : \mathbb{R} \to \mathbb{R}$ defined by

$$T(x) = \begin{cases} 0 & \text{if } x = 0 \\ \frac{1}{x^2} & \text{if } x \neq 0. \end{cases}$$

As a consequence of Corollary 3.1.3, T is an orthogonally additive operator. Additionally, since $T(x) \geq 0$ for all $x \in \mathbb{R}$, it is positive. However, it is not order bounded. To see this, consider the order bounded interval $[0, 1]$ and its image under T, which is $T([0, 1]) = [1, \infty] \cup \{0\}$. As this set is not order bounded, we conclude that T is not order bounded.

We note that order boundedness of an orthogonally additive operator does not imply its positivity. The operator $T : \mathbb{R} \to \mathbb{R}$ defined by $T(x) = x$ shows this fact.

Definition 3.1.10. Let E and F be vector lattices and $T : E \to F$. The operator T is called an abstract Urysohn operator if it is orthogonally additive and order bounded. We denote by $\mathcal{U}(E, F)$ the set of all abstract Urysohn operators from E into F.

Remark 3.1.11. *We define the following order on $\mathcal{U}(E, F)$: $T \geq S$ whenever $T - S$ is positive. In that case $\mathcal{U}(E, F)$ becomes an ordered vector space. It is worth noting that this bears similarity to the case of linear operators, where the set of all order bounded linear operators forms an ordered vector space.*

We recall from Chapter 2 that an element $z \in E$ is called a fragment of x, if $z \perp (x - z)$, that is, $|z| \wedge |x - z| = 0$. We use the notation $z \sqsubseteq x$ to say that z is a fragment of x. We denote by $\mathcal{C}_x = \{z \in V \colon\ |z| \wedge |x - z| = 0\}$ the set of all fragments of x. We say that $y, z \in \mathcal{C}_x$ are mutually complemented, if $x = y \sqcup z$.

Theorem 3.1.12. *Let E and F be vector lattices with F Dedekind complete. Then $\mathcal{U}(E, F)$ is a Dedekind complete vector lattice. Moreover, for $T, S \in \mathcal{U}(E, F)$ and for $x \in E$ the following hold:*

(1) $(T \vee S)(x) = \sup\{Ty + Sz \colon x = y \sqcup z\}$

(2) $(T \wedge S)(x) = \inf\{Ty + Sz \colon x = y \sqcup z\}$

(3) $T^{+}(x) = \sup\{Ty \colon y \sqsubseteq x\}$

(4) $T^{-}(x) = -\inf\{Ty \colon y \sqsubseteq x\}$

(5) $|Tx| \leq |T|(x)$

Proof. Let $T, S \in \mathcal{U}(E, F)$. For every $x \in E$ we define the mapping

$$Hx := \sup\{Ty + Sz \colon x = y \sqcup z\}.$$

We note that since the operators T and S are order bounded and the space F is Dedekind complete, the operator H is well defined and order bounded. We now claim that the operator H is orthogonally additive. To see this, let us take disjoint elements $x, y \in E$. Assume $x = x_1 \sqcup x_2$ and $y = y_1 \sqcup y_2$. In order to complete the proof, we need to show that $x_1 \perp y_1$, $x_2 \perp y_2$ and $(x_1 + y_1) \perp (x_2 + y_2)$. It follows from $x \perp y$ that

$$\begin{aligned} |x_1| \wedge |y_1| &\leq (|x_1| + |x_2|) \wedge (|y_1| + |y_2|) \\ &= |x_1 + x_2| \wedge |y_1 + y_2| \\ &= |x| \wedge |y| \\ &= 0, \end{aligned}$$

and hence $x_1 \perp y_1$. The same holds true for $x_2 \perp y_2$. On the other hand, we have

$$\begin{aligned} |x_1 + y_1| \wedge |x_2 + y_2| &= (|x_1| + |y_1|) \wedge (|x_2| + |y_2|) \\ &\leq |x_1| \wedge |x_2| + |x_1| \wedge |y_2| + |y_1| \wedge |x_2| + |y_1| \wedge |y_2| \\ &= |x_1| \wedge |y_2| + |y_1| \wedge |x_2| \\ &= |x| \wedge |y| + |y| \wedge |x| \\ &= 0. \end{aligned}$$

Thus, we have

$$\begin{aligned}(Tx_1 + Sx_2) + (Ty_1 + Sy_2) &= T(x_1 + y_1) + S(x_2 + y_2)\\ &\le \sup\{Tz + Sw\colon x + y = z \sqcup w\}\\ &= H(x + y).\end{aligned}$$

By taking the supremum over all x_1, x_2 and y_1, y_2 on the left side of the previous inequality, we obtain

$$Hx + Hy \le H(x + y).$$

We now prove the converse inequality. Let z, v be any elements of E such that $x + y = z \sqcup v$. It follows that $x \sqcup y = z \sqcup v$. By Proposition 2.4.5, there exist elements $z_1, z_2, v_1, v_2 \in E$ such that

(1) $z = z_1 \sqcup z_2; v = v_1 \sqcup v_2$;

(2) $x = z_1 \sqcup v_1; y = z_2 \sqcup v_2$.

Therefore we have the following:

$$\begin{aligned}Tz + Sv &= T(z_1 + z_2) + S(v_1 + v_2)\\ &= (Tz_1 + Sv_1) + (Tz_2 + Sv_2)\\ &\le Hx + Hy.\end{aligned}$$

By taking the supremum over all z, v on the left side of the previous inequality, we obtain $H(x + y) \le Hx + Hy$. Therefore, we have $H(x + y) = Hx + Hy$, which implies that H is an orthogonally additive operator. Hence the operator H is an abstract Urysohn operator. Next, we show that $H = T \vee S \in \mathcal{U}(E, F)$. We note that $T \le H$ and $S \le H$. Let us take $R \in \mathcal{U}(E, F)$ such that $T \le R$ and $S \le R$. Given any $x = y \sqcup z$, we have $Ty + Tz \le Ry + Rz = Rx$. By taking the supremum over all y, z, we obtain $Hx \le Rx$. Thus, we have $H = T \vee S$, and as a result, $\mathcal{U}(E, F)$ becomes a vector lattice. Consequently, we can establish the proof by demonstrating the validity of assertions (1)-(5).

(1) $(T \vee S)(x) = \sup\{Ty + Sz \colon x = y \sqcup z\}$

(2) $(T \wedge S)(x) = -[(-T) \vee (-S)] = -\sup\{-Ty - Sz \colon x = y \sqcup z\}$
$= \inf\{Ty + Sz \colon x = y \sqcup z\}$

(3) $T^+(x) = (T \vee 0)(x) = \sup\{Ty + 0z \colon x = y \sqcup z\} = \sup\{Ty \colon y \sqsubseteq x\}$

(4) $T^-(x) = (-T)^+(x) = \sup\{-Ty \colon y \sqsubseteq x\} = -\inf\{Ty \colon y \sqsubseteq x\}$

(5) $|T|\, x = (T^+ \vee T^-)(x) = \sup\{T^+y - T^-z \colon x = y \sqcup z\}$.

Therefore we have $Tx \leq |T|\, x$ and $-Tx \leq |T|\, x$. Hence $|Tx| \leq |T|\, x$. For the last part of the proof, we need to show that $\mathcal{U}(E, F)$ is Dedekind complete. So, let us take a net $(T_\alpha)_{\alpha\in\Delta} \in \mathcal{U}(E, F)$ such that $0 \leq T_\alpha \uparrow\leq T$. We claim that $\sup T_\alpha$ exists in $\mathcal{U}(E, F)$. Consider the element $Sx := \sup_{\alpha\in\Delta} T_\alpha(x)$ for every $x \in E$. Since F is Dedekind complete, the element Sx exists in F. Additionally, taken disjoint $x, y \in E$, we have

$$\begin{aligned} S(x + y) = \sup_{\alpha\in\Delta} T_\alpha(x + y) &= \sup_{\alpha\in\Delta}\{T_\alpha(x) + T_\alpha(y)\} \\ &= \sup_{\alpha\in\Delta} T_\alpha(x) + \sup_{\alpha\in\Delta} T_\alpha(y) \\ &= Sx + Sy, \end{aligned}$$

and hence the operator S is orthogonally additive. Finally, considering the $S \leq T$, we deduce S is order bounded. Hence $S \in \mathcal{U}(E, F)$ and $S = \sup_{\alpha\in\Delta} T_\alpha$. The proof is finished. □

We denote by $\mathcal{U}^+(E, F)$ the set of all positive abstract Urysohn operators. The following example demonstrates that a positive abstract Urysohn operator can be constructed using positive linear operators.

Example 3.1.13. Let $T : E \to F$ be a positive linear operator. Consider the mapping $G_T : E \to F$ defined by $G_T(x) = T\,|x|$. One can see that the mapping G_T is well defined. We claim that G_T is an orthogonally additive operator. Indeed, consider disjoint elements

$x, y \in E$. It follows from T is a linear operator that

$$G_T(x+y) = T(|x+y|) = T(|x| + |y|) = T\,|x| + T\,|y| = G_T(x) + G_T(y).$$

Additionally, by the definition of G_T, it is positive and order bounded. Hence $G_T \in \mathcal{U}^+(E, F)$.

Here we introduce another significant example of a non-linear abstract Urysohn operator.

Example 3.1.14. Consider the operator $T : \ell_2 \to \mathbb{R}$ defined by

$$T((x_n)) = \sum_{n \in I_x} n(|x_n| - 1)$$

where $I_x := \{n \in \mathbb{N} \colon\ |x_n| \geq 1\}$. We show that $T \in \mathcal{U}^+(\ell_2, \mathbb{R})$. To begin with, we claim that T is well-defined. Consider any sequence $(x_n)_{n\in\mathbb{N}} \in \ell_2$. By definition of ℓ_2, we have the following:

$$\begin{aligned}
(\sum_{n=1}^{\infty} |x_n|^2)^{1/2} < \infty &\implies {x_n}^2 \to 0 \\
&\implies x_n \to 0 \\
&\implies \exists N \in \mathbb{N} : \forall n > N \in \mathbb{N} : |x_n| \leq 1,
\end{aligned}$$

and therefore the set I_x contains a finite number of elements. Hence $\sum_{n\in I_x} n(|x_n| - 1) < \infty$, which implies that T is well defined. In the next step, we show that T is an orthogonally additive operator. Take any disjoint $x = (x_n), y = (y_n) \in \ell_2$. It follows that $x_i = 0$ or $y_i = 0$ for all $i \in \mathbb{N}$. Also, we note that if $x_n \neq 0$ (or $y_n \neq 0$), then $y_n = 0$ (or $x_n = 0$). In order to complete the proof, we need to show that $I_x \cup I_y = I_{x+y}$. Indeed, let $n \in I_x \cup I_y$. Thus, we have the following:

$$\begin{aligned} n \in I_x \text{ or } n \in I_y &\implies |x_n| \geq 1 \text{ or } |y_n| \geq 1 \\ &\implies |x_n + y_n| \geq 1 \text{ or } |y_n + x_n| \geq 1 \\ &\implies n \in I_{x+y}. \end{aligned}$$

On the other hand, let $n \in I_{x+y}$. Then, we have $|x_n + y_n| \geq 1$. Considering the disjointness of x and y, we obtain $|x_n| \geq 1$ or $|y_n| \geq 1$, and hence $n \in I_x \cup I_y$. Additionally, one can easily see that $I_x \cap I_y = \emptyset$. By considering the last results, we have the following:

$$\begin{aligned} T(x+y) &= \sum_{n \in I_{x+y}} n(|x_n + y_n| - 1) \\ &= \sum_{n \in I_x} n(|x_n + y_n| - 1) + \sum_{n \in I_y} n(|x_n + y_n| - 1) \\ &= \sum_{n \in I_x} n(|x_n| - 1) + \sum_{n \in I_y} n(|y_n| - 1) \\ &= Tx + Ty, \end{aligned}$$

and hence T is an orthogonally additive operator. It follows from the definition of T that it is positive. In the last step, we show that the operator T is order bounded. Consider the order interval $[z, w]$, where $z = (z_n), w = (w_n) \in \ell_2$. Given any $x = (x_n) \in [z, w]$, we have $|x| \leq |z| \vee |w| = \eta$, where $\eta = (\eta_n) \in \ell_2$. We note that $I_x \subseteq I_\eta$. Thus, we have

$$Tx = \sum_{n \in I_x} n(|x_n| - 1) \leq \sum_{n \in I_x} n(|\eta_n| - 1) \leq \sum_{n \in I_\eta} n(|\eta_n| - 1) = T\eta.$$

Hence $T \in \mathcal{U}^+(\ell_2, \mathbb{R})$.

3.2. C-bounded and Regular OAO

In this section, we introduce two classes of orthogonally additive operators: C-bounded and regular operators. Some of the results discussed in this section can be found in the [16].

Definition 3.2.1. Let E, F be vector latices. An orthogonally additive operator $T : E \to F$ is called:

(1) regular if $T = S_1 - S_2$, where S_1, S_2 are positive orthogonally additive operator from E to F;

(2) C-bounded if $T(\mathcal{C}_x)$ is an order bounded subset of F for every $x \in E$.

We denote by $\mathcal{P}(E, F)$ the set of all C-bounded orthogonally additive operators from E to F. We note that $\mathcal{P}(E, F)$ is known in the literature as the space of Popov operators. Furthermore, one can show that if F is Dedekind complete, then all the conclusions of Theorem 3.1.12 regarding abstract Urysohn operators also hold for C-bounded operators (see [16, Theorem 3.6]). That is, if F is Dedekind complete, then $\mathcal{P}(E, F)$ is a Dedekind complete vector lattice with the same order as on $\mathcal{U}(E, F)$. Additionally, we note that $\mathcal{U}(E, F) \subseteq \mathcal{P}(E, F)$ holds true, by definition. However, the converse side is not valid.

Example 3.2.2. Consider Example 3.1.9. Take any $x \in \mathbb{R}$. Since the set $\mathcal{C}_x = \{0, x\}$, we obtain that $T(\mathcal{C}_x) = \{0, Tx\}$, which is order bounded. Therefore the operator T is C-bounded but not order bounded.

Proposition 3.2.3. *Let E, F be vector lattices with F Dedekind complete and $T : E \to F$ be an orthogonally additive operator. Then T is C-bounded if and only if T is regular.*

Proof. Let $T : E \to F$ be a regular orthogonally additive operator. We need to show that $T(\mathcal{C}_x)$ is order bounded for every $x \in E$. Take any $y \in \mathcal{C}_x$. Since T is regular, there exist positive orthogonally additive operators S_1, S_2 from E to F such that $T = S_1 - S_2$. It follows that

$$\begin{aligned}
|Ty| = |S_1 y - S_2 y| &\leq |S_1 y| + |S_2 y| \\
&= S_1 y + S_2 y \\
&= (S_1 x - S_1(x - y)) + (S_2 x - S_2(x - y)) \\
&\leq S_1 x + S_2 x,
\end{aligned}$$

and hence the set $T(\mathcal{C}_x)$ is order bounded and T is C-bounded.

On the other hand, Let $T : E \to F$ be a C-bounded orthogonally additive operator. Consider the operator $G : E \to F$ defined by

$$Gx = \sup\{|T|\, y \colon y \sqsubseteq x\}.$$

We note that since $|T|$ is C-bounded and F is Dedekind complete, the operator G is well defined. Let us take disjoint elements $x, y \in E$. For an arbitrary $z \sqsubseteq x + y$, by the decomposition property (see [20, Theorem 1.13]), there exist z_1, z_2 of E such that $z = z_1 + z_2$ and $|z_1| \leq |x|$ and $|z_2| \leq |y|$. One can see that $z_1 \perp z_2$ is valid. Now we claim that $z_1 \sqsubseteq x$ and $z_2 \sqsubseteq y$. To prove that, we must show $(x - z_1) \perp (y - z_2)$. Indeed,

$$\begin{aligned}
|x - z_1| \wedge |y - z_2| &\leq (|x| + |z_1|) \wedge (|y| + |z_2|) \\
&\leq |x| \wedge |y| + |x| \wedge |z_2| + |z_1| \wedge |y| + |z_1| \wedge |z_2| \\
&= |x| \wedge |z_2| + |z_1| \wedge |y| \\
&\leq |x| \wedge |y| + |x| \wedge |y| \\
&= 0.
\end{aligned}$$

We can now show that $z_1 \sqsubseteq x$. Because of the above, we have

$$\begin{aligned}
|z_1| \wedge |x - z_1| &\leq (|z_1| + |z_2|) \wedge (|x - z_1| + |y - z_2|) \\
&\leq |z_1 + z_2| \wedge |x - z_1 + y - z_2| \\
&= |z| \wedge |x + y - z| \\
&= 0.
\end{aligned}$$

One can apply the same steps for z_2. Thus, we have

$$\begin{aligned}
|T|\,(z) = |T|\, z_1 + |T|\, z_2 &\leq \sup\{|T|\, w \colon w \sqsubseteq x\} + \sup\{|T|\, w \colon w \sqsubseteq y\} \\
&= Gx + Gy.
\end{aligned}$$

Taking supremum over z, we obtain $G(x+y) \leq Gx + Gy$. Now we prove the other side. We claim that if $z_1 \sqsubseteq x$ and $z_2 \sqsubseteq y$ then $z_1 + z_2 \sqsubseteq x + y$. Because

$$
\begin{aligned}
|z_1 + z_2| \wedge |x - z_1 + y - z_2| &\leq (|z_1| + |z_2|) \wedge (|x - z_1| + |y - z_2|) \\
&\leq |z_1| \wedge |y - z_2| + |z_2| \wedge |x - z_1| \\
&\leq |z_1| \wedge |y| + |z_1| \wedge |z_2| + |z_2| \wedge |x| + |z_2| \wedge |z_1| \\
&\leq |x| \wedge |y| + |z_1| \wedge |z_2| + |y| \wedge |x| + |z_2| \wedge |z_1| \\
&= 0,
\end{aligned}
$$

and hence $z_1 + z_2 \sqsubseteq x + y$. It follows that

$$|T|\, z_1 + |T|\, z_2 = |T|\, (z_1 + z_2) \leq \sup\{|T|\, w \colon w \sqsubseteq x + y\} = G(x+y).$$

Taking supremum over z_1 and z_2, we conclude $Gx + Gy \leq G(x+y)$. Additionally, G is positive by definition. Therefore there exist a positive orthogonally additive operator such that $T \leq G$, which implies T is a regular operator. This finishes the proof. $\square$

Proposition 3.2.4. *Let E, F be vector lattices. Then every positive orthogonally additive operator $T : E \to F$ is a C-bounded operator.*

Proof. Let $T : E \to F$ be a positive orthogonally additive operator. Take an arbitrary element $x \in E$. Then for every $y \in \mathcal{C}_x$, we have

$$Tx = T(x-y) + Ty \implies Ty = Tx - T(x-y) \leq Tx.$$

Therefore Tx is an upper bound for $T(\mathcal{C}_x)$. Hence T is a C-bounded operator. $\square$

3.3. Disjointness Preserving OAO

Some of the results presented in this section can be found in reference [4].

Definition 3.3.1. Let E and F be vector lattices. An orthogonally additive operator $T : E \to F$ is called disjointness preserving if $Tx \perp Ty$ for every disjoint $x, y \in E$. We denote by $OA_{dpo}(E, F)$ the set of all disjointness preserving orthogonally additive operators from E to F.

The following provides an example of a non-disjointness preserving orthogonally additive operator.

Example 3.3.2. Consider the operator $T : \mathbb{R}^2 \to \mathbb{R}$ defined by $T(x, y) = x + y$. We observe that T is an orthogonally additive operator. However, T is not disjointness preserving. Because for taken disjoint elements $(0, 1)$ and $(1, 0)$, one can see that $T(0, 1) = T(1, 0) = 1$, and hence $T(0, 1) \wedge T(1, 0) \neq 0$.

Definition 3.3.3. Let E be a vector lattice. An orthogonally additive operator $T : E \to E$ is called non-expanding if $Tx \in \{x\}^{dd}$ for every $x \in E$.

Example 3.3.4. Let (A, Σ, μ) be a finite measure space. We say that $N : A \times \mathbb{R} \to \mathbb{R}$ is a superpositionally measurable function, if $N(\cdot, f(\cdot))$ is μ-measurable for every $f \in L_0(\mu)$. A superpositionally measurable function N is called normalized if $N(s, 0) = 0$ for μ-almost all $s \in A$. Now we define the operator

$$\begin{aligned} \mathcal{N} :& L_0(\mu) \to L_0(\mu) \\ & f(s) \to \mathcal{N}(f)(s) = N(s, f(s)). \end{aligned}$$

We claim that $\mathcal{N}$ is a non-expanding orthogonally additive operator. Take any disjoint elements $f, g \in L_0(\mu)$. By applying the same method used in Proposition 3.1.7, we can conclude that $\mathbf{1}_{\text{supp}f \cup \text{supp}g} = \mathbf{1}_{\text{supp}f} + \mathbf{1}_{\text{supp}g}$. Therefore one has

$$\begin{aligned}
\mathcal{N}(f+g)(s) = N(s, f+g) &= N(s, (f+g)\mathbf{1}_{\text{supp}f \cup \text{supp}g}) \\
&= N(s, f+g)\mathbf{1}_{\text{supp}f \cup \text{supp}g} \\
&= N(s, f+g)\mathbf{1}_{\text{supp}f} + N(s, f+g)\mathbf{1}_{\text{supp}g} \\
&= N(s, f)\mathbf{1}_{\text{supp}f} + N(s, f)\mathbf{1}_{\text{supp}g} \\
&= N(s, f) + N(s, f) \\
&= \mathcal{N}(f)(s) + \mathcal{N}(g)(s).
\end{aligned}$$

Thus, $\mathcal{N}$ is an orthogonally additive operator. Let us now show that $\mathcal{N}$ is a non-expanding operator. First, we have supp$\mathcal{N}(f) \subseteq$ supp$f$ since $\mathcal{N}(f)(s) = N(s, f)\mathbf{1}_{\text{supp}f}$. It follows that

$$\begin{aligned}
h \in \{f\}^d &\Rightarrow \mu(\text{supp}f \cap \text{supp}h) = 0 \\
&\Rightarrow \mu(\text{supp}\mathcal{N}(f) \cap \text{supp}h) = 0 \\
&\Rightarrow \mathcal{N}(f) \perp h \\
&\Rightarrow \mathcal{N}(f) \in \{f\}^{dd}.
\end{aligned}$$

We also note that the operator $\mathcal{N}$ is known in the literature as the nonlinear superposition operator or Nemytskii operator.

Proposition 3.3.5. *Let E be a vector lattice. Then every non-expanding orthogonally additive operator from E to E preserves disjointness.*

Proof. Let $T : E \to E$ be a non-expanding operator and let x, y be disjoint elements of E. Considering the $y \in \{x\}^d$ and $Tx \in \{x\}^{dd}$, we obtain $y \perp Tx$. Therefore $Tx \in \{y\}^d$. Since $Ty \in \{y\}^{dd}$, then $Tx \perp Ty$. $\square$

The following example shows that not every disjoint preserving operator has to be a non-expanding operator.

Example 3.3.6. Consider the orthogonally additive operator $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(x, y) = (x^2, x)$. Let (a, b) and (c, d) be disjoint elements of $\mathbb{R}^2$. Starting from this, one can obtain $a = 0$ or $c = 0$, and hence it follows from $T(a, b) = 0$ or $T(c, d) = 0$ that T is a disjointness preserving operator. However, T is not a non-expanding operator. Because for taken element $(1, 0) \in \mathbb{R}^2$, one may see that $T(1, 0) = (1, 1) \notin \{(1, 0)\}^{dd} = \{(x, 0) \in \mathbb{R}^2 \colon x \in \mathbb{R}\}$.

Proposition 3.3.7. *Let E and F be vector lattices with F Dedekind complete. Then $OA_{dpo}(E, F) \subseteq OA_r(E, F)$.*

Proof. Let $T \in OA_{dpo}(E, F)$ and $x \in E$. Given $y \in \mathcal{C}_x$, we have $y \perp (x - y)$. Considering the $T \in OA_{dpo}(E, F)$, one has

$$\left|T(x)\right| = \left|T(x - y \sqcup y)\right| = \left|T(x - y) \sqcup T(y)\right| = \left|T(x - y)\right| \sqcup \left|T(y)\right| .$$

Thus $\left|T(y)\right| \leq \left|T(x)\right|$. Hence the set $T(\mathcal{C}_x)$ is upper bounded by $\left|T(x)\right|$, and T is a C-bounded operator. Considering Proposition 3.2.3 and that F is Dedekind complete, T is a regular operator. $\square$

Remark that $OA_r(E, F) \subseteq OA_{dpo}(E, F)$ does not generally hold. Indeed, in Example 3.3.2, we have seen that the operator T is not disjointness preserving. Despite this, T is C-bounded. Because for taken $x = (a, b) \in \mathbb{R}^2$, one can observe that $\mathcal{C}_x = \{(0, 0), (0, b), (a, 0), (a, b)\}$ and hence $T(\mathcal{C}_x)$ is order bounded. Since $\mathbb{R}$ is Dedekind complete, T is a regular operator.

Remark that if F is Dedekind complete, then $OA_{dpo}(E, F)$ is a solid subset of $OA_r(E, F)$. Let $S \in OA_r(E, F)$, $T \in OA_{dpo}(E, F)$, and $|S| \leq |T|$. We show that $S \in OA_{dpo}(E, F)$. Given disjoint elements $x, y \in E$, we have $|Tx| \wedge |Ty| = 0$. It follows that $|Sx| \wedge |Sy| \leq |Tx| \wedge |Ty| = 0$, and hence $S \in OA_{dpo}(E, F)$.

Proposition 3.3.8. *Let E, F be vector lattices and $T \in OA_{dpo}(E, F)$. Then the module $|T| = T \vee (-T)$ of T exists and $|T| \in OA_{dpo}(E, F)$. Moreover*

$$|T|\, x = |Tx|\,, \;\; x \in E.$$

Proof. Consider the operator $R : E \to F$ defined by $R(x) = |Tx|$. Given disjoint elements $x, y \in E$, we have $R(x+y) = \big|T(x+y)\big| = \big|T(x) + T(y)\big|$. Since $T \in OA_{dpo}(E, F)$, one has $Tx \perp Ty$. It follows that

$$R(x+y) = |Tx| + |Ty| = Rx + Ry.$$

Thus R is an orthogonally additive operator. Also, one can see that R is a positive disjointness preserving operator. Let $L \in OA_+(E, F)$ such that $Lx \geq Tx$ and $Lx \geq (-Tx)$ for all $x \in E$. Then,

$$Lx \geq Tx \vee (-Tx) = Rx, \;\; x \in E.$$

It follows from $T, -T \leq R$ and $L \geq R$ that $R = T \vee (-T)$. □

Example 3.3.9. Every orthogonally additive operator $T : \mathbb{R} \to \mathbb{R}$ is non-expanding and therefore preserves disjointness. To see this, take an element $x \in \mathbb{R}$. If $x = 0$, then $\{x\}^{dd} = \{0\}$. It follows from $Tx = T0 = 0$ that $Tx \in \{x\}^{dd}$. If $x \neq 0$, then $\{x\}^{dd} = \mathbb{R}$. It follows that $Tx \in \{x\}^{dd}$.

3.4. C-complete Vector Lattices and Relationship with OAO

In this section, we begin by introducing a new class of vector lattices known as C-complete. We explore the relationship between C-completeness and both the Archimedean property and Dedekind completeness. Furthermore, we present two significant classes of orthogonally additive operators: horizontally-to-norm continuous operators and horizontally-to-order continuous operators. Some of the results presented in this section can be found in the [2, 4].

Definition 3.4.1. A vector lattice E is called C-complete, if for each $x \in E_+$ every subset $D \subseteq \mathcal{C}_x$ has a supremum $z = \sup D$.

Remark that considering Lemma 2.4.3, we observe that $z = \sup D \in \mathcal{C}_x$. Besides if E is a C-complete vector lattice, $x \in E_+$, and $D \subseteq \mathcal{C}_x$, then $v = \inf D$ exists. Indeed, consider the set $\bar{D} = \{x - z \colon z \in D\}$. Then $\bar{D}$ has a supremum $\bar{z} = \sup \bar{D}$ and $v = x - \bar{z}$.

Example 3.4.2. Every Dedekind complete vector lattice E is C-complete.

We present an example of a non-C-complete vector lattice, as a consequence it is not Dedekind complete.

Example 3.4.3. Consider the vector lattice c, which is the space of all convergent (real) sequences, i.e., $\mathbf{c} = \{(x_n) \subseteq \mathbb{R} \colon \lim x_n \text{ exists in } \mathbb{R}\}$. We claim that it is not C-complete. To see this consider the sequence e_n, whose n^{th} term is one and every other zero. Put by definition, $D := \{e_{2n} \colon n \in \mathbb{N}\}$. Now let us define the sequence $\mathbf{1} = (1, 1, 1, ...)$, where each term is one. We observe that $D \subseteq \mathcal{C}_{\mathbf{1}}$. However, since the sequence $\sup D = (0, 1, 0, 1, ...)$ is not convergent, then c is not C-complete.

The following shows us that the C-completeness of a vector lattice does not imply the Dedekind completeness of it. We note that $C[0, 1]$ is not a Dedekind complete vector lattice.

Example 3.4.4. The vector lattice $C[0, 1]$ of all continuous functions on the interval [0,1] is C-complete.

For detailed proof, please refer to [2, Prop. 4.2]

The relationship between Archimedean vector lattices and C-complete vector lattices is a natural question to ask. Example 3.4.3 shows that Archimedean property does not imply C-completeness. In addition, we present an example of a vector lattice in this context that is C-complete but not Archimedean. This example serves as evidence that the property of C-completeness does not necessarily imply the Archimedean property in vector lattices.

Example 3.4.5. Consider the vector lattice $(\mathbb{R}^2, \leq_l)$ equipped with the lexicographic order. Notice that $(\mathbb{R}^2, \leq_l)$ is not Archimedean. However, it is C-complete. Indeed, take a positive element $x = (x_1, x_2) \in \mathbb{R}^2$. Since there exist only two fragments of x, that is, $\mathcal{C}_x = \{0, (x_1, x_2)\}$, then $(\mathbb{R}^2, \leq_l)$ is C-complete.

Definition 3.4.6. Let E be a vector lattice. A net $(e_\alpha)_{\alpha\in\Delta}$ in E horizontally converges to an element $e \in E$ (notation $e_\alpha \xrightarrow{h} e$) if the net $(e_\alpha)_{\alpha\in\Delta}$ order converges to e and $e_\alpha \sqsubseteq e_\beta \sqsubseteq e$ for all $\alpha, \beta \in \Delta$ with $\alpha \leq \beta$.

Example 3.4.7. Consider the vector lattice c and its element $\mathbf{1} = (1, 1, 1, ...)$. We find a sequence of c that horizontally converges to $\mathbf{1}$. Put by definition, $a_n = (1, 1, 1, ..., 1, 0, 0, 0, ...)$, whose first n^{th} term is one and every other zero. Now one can see that

$$|\mathbf{1} - a_n| = (0, 0, 0, ..., 0, 1, 1, 1, ...) = b_n.$$

It follows from $b_n \downarrow 0$ that $a_n \xrightarrow{o} \mathbf{1}$. Besides for taken $n, m \in \mathbb{N}$ with $n \leq m$, we have $a_n \sqsubseteq a_m \sqsubseteq \mathbf{1}$. Hence $a_n \xrightarrow{h} \mathbf{1}$.

Of course, not every order convergent net is horizontally convergent. For example, the net $e_\alpha = \frac{1}{\alpha}$ for $\alpha \in \mathbb{R}^+$ order converges to zero. However, the convergence is not horizontal. Because for given $\alpha, \beta \in \mathbb{R}^+$ with $\alpha \leq \beta$, the statement $\frac{1}{\alpha} \sqsubseteq \frac{1}{\beta} \sqsubseteq 0$ is not valid.

Remark that in any vector lattice E, the only net that horizontally converges to zero is the sequence $(x_n)_{n\in\mathbb{N}} = (0)_{n\in\mathbb{N}}$, where each element is zero. The reason is that there is no fragment of zero except itself.

Proposition 3.4.8. *Every sequence of* $\mathbb{R}$ *that horizontally converges to* $x \in \mathbb{R}$ *is an element of* $\ell^\infty_c = \{(x_n)_{n\in\mathbb{N}} \colon (\exists k \in \mathbb{N})\ (\forall n \geq k)\ (x_n = x_k)\}$, *which is the space of eventually constant sequences.*

Proof. Let the sequence $(x_n) \subseteq \mathbb{R}$ horizontally converges to $x \in \mathbb{R}$. Therefore for all $n \in \mathbb{N}$ we have $x_n \sqsubseteq x$. It follows from $\mathcal{C}_x = \{0, x\}$ for every $x \in \mathbb{R}$ that $x_n = 0$ or $x_n = x$. Considering the sequence (x_n) order converges to x, one can see that there exists $N \in \mathbb{N}$ such that $x_n = x$ for all $n \geq N$. $\square$

Definition 3.4.9. Let E be a vector lattice and let X be a normed space. An orthogonally additive operator $T : E \to X$ is called horizontally-to-norm continuous if for every net $(x_\alpha)_{\alpha\in\Delta}$ in E horizontally convergent to $x \in E$ the net $(Tx_\alpha)_{\alpha\in\Delta}$ norm converges to Tx.

Corollary 3.4.10. *Let X be a normed space. Then every orthogonally additive operator $T : \mathbb{R} \to X$ is horizontally-to-norm continuous.*

Proof. The proof follows from Proposition 3.4.8. $\square$

Definition 3.4.11. Let E, F be vector lattices. An orthogonally additive operator $T : E \to F$ is called horizontally-to-order continuous if for every net $(x_\alpha)_{\alpha\in\Delta}$ in E horizontally convergent to $x \in E$ the net $(Tx_\alpha)_{\alpha\in\Delta}$ order converges to Tx.

The Nemytskii operator is an example of the horizontally-to-order continuous orthogonally additive operator. The following example confirms this.

Example 3.4.12. Let (A, Σ, μ) be a finite measure space and let $N : A \times \mathbb{R} \to \mathbb{R}$ be a superpositionally measurable function. Consider the Nemytskii operator $\mathcal{N} : L_0(\mu) \to L_0(\mu)$ associated with N. Take a net $(f_\alpha)_{\alpha\in\Delta} \subseteq L_0(\mu)$ which horizontally converges to $f \in L_0(\mu)$. We notice that suppf_α is a measurable subset of A for all $\alpha \in \Delta$, and supp$f_\alpha \subseteq$ suppf_β where $\alpha, \beta \in \Delta$, $\alpha \leq \beta$. Indeed, since $f\alpha \sqsubseteq f_\beta$ is equivalent to $f_\alpha \perp (f_\beta - f_\alpha)$, we have $\mu\{t \in \text{supp}f_\alpha \cap \text{supp}(f_\beta - f_\alpha)\} = 0$. It directly follows that supp$f_\alpha \subseteq$ suppf_β. Also, we note that $f_\alpha = f\mathbf{1}_{\text{supp}f_\alpha}$ for all $\alpha \in \Delta$. Now we claim that the net $(\mathbf{1}_{\text{supp}f_\alpha})_{\alpha\in\Delta}$ order converges to $\mathbf{1}_{\text{supp}f}$. One can see that

$$\left|\mathbf{1}_{\text{supp}f_\alpha} - \mathbf{1}_{\text{supp}f}\right| \leq \mathbf{1}_{\text{supp}(f_\alpha - f)} \text{ and } \mathbf{1}_{\text{supp}(f_\alpha - f)} \downarrow 0,$$

thus $\mathbf{1}_{\text{supp}f_\alpha} \xrightarrow{o} \mathbf{1}_{\text{supp}f}$. After all these inferences, we get the following for the Nemytskii operator;

$$\mathcal{N}f_\alpha = N(\cdot, f_\alpha(\cdot)) = N(\cdot, f\mathbf{1}_{\text{supp}f_\alpha}(\cdot)) = N(\cdot, f_\alpha(\cdot))\mathbf{1}_{\text{supp}f_\alpha}(\cdot)$$

order converges to $N(\cdot, f(\cdot))\mathbf{1}_{\text{supp}f}(\cdot) = N(\cdot, f\mathbf{1}_{\text{supp}f}(\cdot)) = N(\cdot, f(\cdot)) = \mathcal{N}(f)$. Hence the Nemytskii operator is a horizontally-to-order continuous orthogonally additive operator.

3.5. Compact Like OAO

In this section, we focus on the study of compact-like operators, such as AM-compact and Narrow operators, within the framework of orthogonally additive operators. We study their properties and characteristics in this context. It is worth noting that some of the results presented in this section can be found in the [2, 4].

Definition 3.5.1. Let E be a vector lattice and let X be a normed space. An orthogonally additive operator $T : E \to X$ is called:

(1) AM-compact if T maps order bounded subsets of E into relatively compact sets in X,

(2) C-compact if for every $x \in E$ the set $T(\mathcal{C}_x)$ is relatively compact in X.

Definition 3.5.2. A vector $e > 0$ in a vector lattice E is called an atom if $x \wedge y$ and $x, y \in [0, e]$ imply either $x = 0$ or $y = 0$.

Definition 3.5.3. A vector lattice E is called atom-filled if each element of E_+ is an atom.

Example 3.5.4. $(\mathbb{R}, \leq)$ is an example of atom-filled vector lattice.

Example 3.5.5. Consider the set $C_c[0,1] = \{f(x) \in C[0,1] \colon (\exists c \in \mathbb{R})\ (\forall x \in [0,1])\ (f(x) = c)\}$. We observed that the set $C_c[0,1]$ is a vector sublattice of $C[0,1]$. Besides, every element of its positive cone is an atom. Hence $C_c[0,1]$ is an atom-filled vector lattice.

Remark 3.5.6. *We remark that if $e > 0$ is an atom in a vector lattice E, then $\mathcal{C}_e = \{0, e\}$. Indeed, let $x \sqsubseteq e$. Since $x \wedge (e - x) = 0$ and $x, e - x \in [0, e]$, then we have $x = 0$ or $e - x = 0$. This finishes the proof.*

Theorem 3.5.7. *Let E be an atom-filled vector lattice. Then the followings are valid for all $x, y \in E_+$.*

(1) $x \wedge y = 0 \iff x = 0$ or $y = 0$,

(2) $x \wedge y = x$ or $x \wedge y = y$, and $x \vee y = x$ or $x \vee y = y$,

(3) E_+ *is a chain,*

(4) $\mathcal{C}_e = \{0, e\}$ *is valid for all* $e \in E$.

Proof. (1) Let $x \wedge y = 0$. Since E is an atom-filled vector lattice, then $x \vee y$ is also an atom. It follows from $x, y \in [0, x \vee y]$ and $x \wedge y = 0$ that $x = 0$ or $y = 0$.

(2) Note that $(x - x \wedge y) \wedge (y - x \wedge y) = x \wedge y - x \wedge y = 0$. It follows from the previous conclusion that $x - x \wedge y = 0$ or $y - x \wedge y = 0$. The same is true for the other side.

(3) It follows directly from the conclusion (2).

(4) Take an arbitrary $e \in E$. Since $|e|$ is an atom, then we have $\mathcal{C}_{|e|} = \{0, |e|\}$. Let $0 \neq y \sqsubseteq e$. By Theorem 2.4.1, we have $|y| \sqsubseteq |e|$, $y^+ \sqsubseteq e^+$ and $y^- \sqsubseteq e^-$. It follows from $y \neq 0$ that $|e| = |y|$. Considering the e^- is an atom, we have two cases: $y^- = 0$ or $y^- = e^-$. If $y^- = 0$, then we have the following:

$$\begin{aligned} |y| = |e| &\implies y^+ + y^- = e^+ + e^- \\ &\implies y^+ - e^+ = e^- \\ &\implies e^- \leq 0 \\ &\implies e \geq 0 \\ &\implies \mathcal{C}_e = \{0, e\}. \end{aligned}$$

On the other hand, let us assume $y^- = e^-$. It follows from $y^+ + y^- = e^+ + e^-$ that $y^+ = e^+$. Thus we obtain $e = y$. This finishes the proof.

□

Proposition 3.5.8. *Let* E *be a vector lattice with the principal projection property and* $x \in E_+$. *Then* x *is an atom if and only if* $\mathcal{C}_x = \{0, x\}$.

Proof. The proof follows from Remark 2.1.11 and Remark 3.5.6. □

Definition 3.5.9. A vector lattice E is called atomless if there is no atom in E.

Example 3.5.10. The vector lattice $C[0,1]$ is an example of an atomless vector lattice.

Proposition 3.5.11. *Let E be an atomless Dedekind complete vector lattice and $0 \neq x \in E$. Then the set $\mathcal{C}_x$ has infinite cardinality.*

Proof. It follows from $x \neq 0$ that $x^+ \neq 0$ or $x^- \neq 0$. Let $x^+ \neq 0$. We note that $x^+ \sqsubseteq x$ is valid. Since $0 \neq x^+ \in E$ is not an atom, by Proposition 3.5.8 we have $\mathcal{C}_{x^+} \neq \{0, x^+\}$. Therefore there exists a proper fragment $y \in E$ of x^+, that is, $y \sqsubseteq x^+$ and $0 \neq y \neq x^+$. Note that y is not an atom either. Thus, similarly, there exists a proper fragment $z \in E$ of y such that $z \sqsubseteq y$ and $0 \neq z \neq y$. Considering the Proposition 2.4.2, one has $z \sqsubseteq y \sqsubseteq x^+ \sqsubseteq x$. Using the same steps, we obtain that x has infinitely many proper fragments. $\square$

Proposition 3.5.12. *Let E be an atom-filled vector lattice and let X be a normed space. Then every orthogonally additive operator $T : E \to X$ is C-compact.*

Proof. Let $x \in E$. Since E is an atom-filled vector lattice, then $\mathcal{C}_x = \{0, x\}$. We observe that the set $T(\mathcal{C}_x) = \{0, Tx\}$ is compact in any normed space. Hence T is C-compact. $\square$

Remark that every AM-compact orthogonally additive operator $T : E \to X$ is C-compact because the set $\mathcal{C}_x$ is order bounded. However, the converse of this statement is not valid.

Example 3.5.13. Consider the orthogonally additive operator $T : \mathbb{R} \to \mathbb{R}$ defined by

$$T(x) = \begin{cases} 0 & \text{if } x = 0 \\ \frac{1}{x} & \text{if } x \neq 0 \end{cases}$$

Although the set $[0,1]$ is order bounded, $T([0,1]] = [1, \infty] \cup \{0\}$ is not relatively compact. Hence T is not AM-compact. Additionally, by Proposition 3.5.12, we can see that T is C-compact.

Definition 3.5.14. Let E be a vector lattice and X be a normed space. An orthogonally additive operator $T : E \to X$ is called narrow if for every $x \in E$ and $\epsilon > 0$ there exists a pair x_1, x_2 of mutually complemented fragments of x such that $\|Tx_1 - Tx_2\| < \epsilon$.

We observed that if $T : E \to F$ is a narrow orthogonally additive operator and $x \in E$ is an atom, then $Tx = 0$. Indeed, since $\mathcal{C}_x = \{0, x\}$, we obtain $\|Tx - T0\| < \epsilon$ for all $\epsilon > 0$. Hence $Tx = 0$. For this reason, we choose the vector space E as atomless in the following several propositions and theorems. Additionally, starting from this point until Definition 3.5.22, it is assumed that every vector lattice E has the principal projection property.

Theorem 3.5.15. *Let E be an atomless C-complete vector lattice and let X be a Banach space. Then every orthogonally additive horizontally-to-norm continuous C-compact operator $T : E \to X$ is narrow.*

In order to prove Theorem 3.5.15, we require the following results.

The next auxiliary proposition is well known (see e.g. [27, Lemma 10.20]).

Proposition 3.5.16. *Let $(v_i)_{i=1}^n$ be a finite subset of elements in a finite-dimensional normed space V and $(\lambda_i)_{i=1}^n$ be non-negative numbers such that $0 \le \lambda_i \le 1$ for each i. Then there exists a set $(\theta_i)_{i=1}^n$ of numbers such that $\theta_i \in \{0, 1\}$, $i \in \{1, 2, ..., n\}$ and*

$$\left\| \sum_{i=1}^{n} (\lambda_i - \theta_i) v_i \right\| \le \frac{dimV}{2} \max_{i \in \{1,2,...,n\}} \|v_i\| .$$

Proposition 3.5.17. *Let E be an atomless vector lattice, $x \in E$, X a Banach space and $T : E \to X$ an orthogonally additive horizontally-to-norm continuous operator. Then for any $\epsilon > 0$ there exists a decomposition $x = y \sqcup z$, where y, z are nonzero fragments of x such that $\|Tz\| < \epsilon$.*

Proof. Since E is an atomless vector lattice, by Proposition 3.5.11, $\mathcal{C}_x$ has infinitely many elements. We observed that the net $(x_\alpha)_{\alpha \in \mathcal{C}_x} = \alpha \subseteq \mathcal{C}_x$ horizontally converges to x. Since T is a horizontally-to-norm continuous operator, then $\forall \epsilon > 0, \exists \alpha_0 \in \mathcal{C}_x$ such that $\|Tx - Tx_\alpha\| \le \epsilon$ for all $\alpha \ge \alpha_0$. Note that $x_{\alpha_0} \sqsubseteq x$. It follows that $x = (x - x_{\alpha_0}) \sqcup x_{\alpha_0}$. Thus one has

$$Tx = T((x - x_{\alpha_0}) \sqcup x_{\alpha_0}) = T(x - x_{\alpha_0}) + T(x_{\alpha_0}).$$

It follows that $Tx - T(x_{\alpha_0}) = T(x - x_{\alpha_0})$, and hence $\big\|T(x - x_{\alpha_0})\big\| < \epsilon$. Assign $y := x_{\alpha_0}$ and $z := x - x_{\alpha_0}$. Then $\|Tz\| < \epsilon$ and $x = y \sqcup z$ is the desirable disjoint decomposition. $\square$

Proposition 3.5.18. *Let E be an atomless vector lattice, $x \in E$, X a Banach space and $T : E \to X$ an orthogonally additive horizontally-to-norm continuous operator. Assume that $(y_n)_{n\in\mathbb{N}}$ is a sequence of fragments of x such that $y_1 = x, y_n \sqsubseteq y_m$ for $n \geq m; m, n \in \mathbb{N}$ and $\bigcap\limits_{n\in\mathbb{N}} \mathcal{C}_{y_n} = \{0\}$. Then*

$$\lim_{n\to\infty} \|Ty_n\| = 0.$$

Proof. Put by definition $x_n := x - y_n$ for all $n \in \mathbb{N}$. Firstly, we show $x_m \sqsubseteq x_n$ for all $m \leq n$. It follows from $y_m \sqsubseteq x$ and $|y_n| \leq |y_m|$ that

$$\begin{aligned}
|x_m| \wedge |x_n - x_m| &= |x - y_m| \wedge \big|(x - y_n) - (x - y_m)\big| \\
&= |x - y_m| \wedge |y_m - y_n| \\
&\leq |x - y_m| \wedge |y_m| + |x - y_m| \wedge |y_n| \\
&\leq |x - y_m| \wedge |y_m| + |x - y_m| \wedge |y_m| \\
&= 0.
\end{aligned}$$

Secondly, using the hypothesis, one has $\bigcap\limits_{n\in\mathbb{N}} \mathcal{C}_{x-x_n} = \bigcap\limits_{n\in\mathbb{N}} \mathcal{C}_{y_n} = \{0\}$. It follows that $x_n \xrightarrow{h} x$. Now notice that $x_n \perp (x - x_n)$ for all $n \in \mathbb{N}$. It follows that $x = x_n \sqcup (x - x_n), n \in \mathbb{N}$. Therefore we obtain $Tx = T(x_n) + T(x - x_n)$. And finally, considering the T is a horizontally-to-norm continuous operator, one has

$$\begin{aligned}
\lim_{n\to\infty} \|Ty_n\| &= \lim_{n\to\infty} \big\|T(x - x_n)\big\| \\
&= \lim_{n\to\infty} \|Tx - Tx_n\| \\
&= 0.
\end{aligned}$$

$\square$

Proposition 3.5.19. *Let E be an atomless C-complete vector lattice, X a Banach space, $T : E \to X$ an orthogonally additive horizontally-to-norm continuous operator, $x \in E$ and $\epsilon > 0$. Then for some $n \in \mathbb{N}$ there exists a decomposition*

$$x = \bigsqcup_{i=1}^{n} x_i,$$

where x_i are nonzero fragments of x such that $\|Tx_i\| < \epsilon$ for any $i \in \{1, 2, ..., n\}$.

Proof. Let us define the set $D_{x,T,\epsilon} := \{z \in \mathcal{C}_x \colon z \neq 0, \|Tz\| < \epsilon\}$. By Proposition 3.5.17 there is a decomposition of $x = y \sqcup z$ such that $\|Tz\| < \epsilon$. Therefore $D_{x,T,\epsilon} \neq \emptyset$. Besides $D_{x,T,\epsilon}$ is a partially ordered set with respect to relation $\sqsubseteq$. Let $\Delta \subseteq D_{x,T,\epsilon}$ be a totally ordered set. Put by definition $(u_\alpha)_{\alpha \in \Delta} = \alpha$. Clearly $(u_\alpha)_{\alpha\in\Delta} \subseteq D_{x,T,\epsilon}$ is a chain and $u_\alpha \sqsubseteq u_\beta$ for all $\alpha, \beta \in \Delta,\ \alpha \sqsubseteq \beta$. Since E is a C-complete vector lattice, then there exists $u \in \mathcal{C}_x$ such that $u_\alpha \xrightarrow{o} u$. Thus we obtain $u_\alpha \xrightarrow{h} u$. It follows from T is a horizontally-to-norm continuous operator that $Tu_\alpha \xrightarrow{\|\cdot\|} Tu$. Therefore one has

$$\begin{aligned} \|Tu\| &\leq \|Tu - Tu_\alpha\| + \|Tu_\alpha\| \\ &\leq \epsilon' + \epsilon, \end{aligned}$$

and hence $u \in D_{x,T,\epsilon}$. By Zorn's Lemma, $D_{x,T,\epsilon}$ has a maximal element $z \in D_{x,T,\epsilon}$. If $\big\|T(x-z)\big\| < \epsilon$, then the proof is done. Let us define $y = x - z$. Assume $\big\|T(y)\big\| > \epsilon$. By Proposition 3.5.17 there is a decomposition of $y = y_1 \sqcup y_2$, where y_1 is a maximal element in $D_{y,T,\epsilon}$ with $\|Ty_1\| < \epsilon$. Similarly, if we assume $\|Ty_2\| > \epsilon$, then we can construct a sequence of decompositions $y_{2k} = y_{2k+1} + y_{2k+2}$, where y_{2k+1} is a maximal element in $D_{y_{2k},T,\epsilon}$, and satisfying the conditions $\|Ty_{2k+1}\| < \epsilon$ and $\|Ty_{2k+2}\| > \epsilon,\ k \in \mathbb{N}$. Now we claim that there exists $l \in \mathbb{N}$ such that $y_{2l} = y_{2l+1} \sqcup y_{2l+2}$ and both $\|Ty_{2l+1}\|, \|Ty_{2l+2}\| < \epsilon$. Assume the contrary. Then $\|Ty_{2k}\| > \epsilon$ for all $k \in \mathbb{N}$. Now we show that $\bigcap\limits_{n\in\mathbb{N}} \mathcal{C}_{y_{2n}} = \{0\}$. Assume on the contrary, there exists a nonzero element $v \in \bigcap\limits_{n\in\mathbb{N}} \mathcal{C}_{y_{2n}}$. Put by definition $y'_{2k} = y_{2k} - v$ for all

$k \in \mathbb{N}$. It follows from $y_{2n} \sqsubseteq y_{2m}$ for all $m \leq n$ that

$$\begin{aligned}
\left|y'_{2n}\right| \wedge \left|y'_{2m} - y'_{2n}\right| &= |y_{2n} - v| \wedge \left|y_{2m} - v - (y_{2n} - v)\right| \\
&= |y_{2n} - v| \wedge |y_{2m} - y_{2n}| \\
&\leq |y_{2n}| \wedge |y_{2m} - y_{2n}| + |v| \wedge |y_{2m} - y_{2n}| \\
&= |v| \wedge |y_{2m} - y_{2n}| \\
&\leq |v| \wedge |y_{2m} - v| + |v| \wedge |v - y_{2m}| \\
&= 0,
\end{aligned}$$

and hence $y'_{2n} \sqsubseteq y'_{2m}$ and $\bigcap\limits_{n\in\mathbb{N}} \mathcal{C}_{y'_{2n}} = \{0\}$. By Proposition 3.5.18 there exists $n_0 \in \mathbb{N}$ such that $\left\|Ty'_{2n_0}\right\| < \epsilon$. Thus $y_{2n_0} = y'_{2n_0} \sqcup v,\ y'_{2n_0} \in D_{y_{2n_0},T,\epsilon}$ and y_{2n_0+1} is a maximal element of $D_{y_{2n_0},T,\epsilon}$. Therefore we have

$$y_{2n_0} = y_{2n_0+1} \sqcup y_{2n_0+2} = y'_{2n_0} \sqcup v.$$

Now let us show that $y'_{2n_0} = y_{2n_0+1}$. In the first part of the proof, we show that $y_{2n_0+1} \sqsubseteq y'_{2n_0}$. It follows from $y_{2n_0+1} \sqsubseteq y_{2n_0}$ that

$$\begin{aligned}
|y_{2n_0+1}| \wedge \left|y'_{2n_0} - y_{2n_0+1}\right| &= |y_{2n_0+1}| \wedge \left|y_{2n_0} - v - (y_{2n_0+1})\right| \\
&\leq |y_{2n_0+1}| \wedge |y_{2n_0} - y_{2n_0+1}| + |y_{2n_0+1}| \wedge |v| \\
&= |y_{2n_0+1}| \wedge |v| \\
&= |y_{2n_0} - y_{2n_0+2}| \wedge |v| \\
&\leq |y_{2n_0} - v| \wedge |v| + |v - y_{2n_0+2}| \wedge |v| \\
&= 0.
\end{aligned}$$

In the second part, let us consider the maximality of y_{2n_0+1} on $D_{y_{2n_0},T,\epsilon}$ and $y'_{2n_0} \in D_{y_{2n_0},T,\epsilon}$. It diretly follows that $y'_{2n_0} = y_{2n_0+1}$. Since $y'_{2n_0} = y_{2n_0+1}$, then we obtain that $v = y_{2n_0+2}$.

Considering the construction of the sequence $(y_{2k})_{k\in\mathbb{N}}$, we have

$$..., \sqsubseteq y_{2n_0+6} \sqsubseteq y_{2n_0+4} \sqsubseteq v = y_{2n_0+2}.$$

It directly follows that $v = y_{2n}$ for all $n \geq n_0+1$. This conclusion implies that $y_{2n+1} = 0$ for all $n \geq n_0 + 1$, but it is a contradiction caused by Proposition 3.5.17. Hence $\bigcap\limits_{n\in\mathbb{N}} \mathcal{C}_{y_{2n}} = \{0\}$ and so, by applying Proposition 3.5.18 we get

$$\lim_{n\to\infty} \|Ty_{2n}\| = 0.$$

However, this result contradicts our assumption, $\|Ty_{2n}\| > \epsilon$ for all $n \in \mathbb{N}$. Therefore the desirable $l \in \mathbb{N}$ exists. Consider the following elements

$$x_1 = y_1,\ x_2 = y_3,\ ...,\ x_l = y_{2l-1},\ x_{l+1} = y_{2l+1},\ x_{l+2} = y_{2l+2},\ x_{l+3} = z.$$

Then $x = \bigsqcup\limits_{i=1}^{n} x_i$ with $n = l+3$ is desirable decomposition of x. $\square$

Proposition 3.5.20. *Let E be an atomless C-complete vector lattice and V a finite-dimensional Banach space. Then every orthogonally additive horizontally-to-norm continuous operator $G : E \to V$ is narrow.*

Proof. By Proposition 3.5.19, for any $x \in E$ and $\epsilon > 0$, there exists a decomposition of $x = \bigsqcup\limits_{i=1}^{n} x_i$ such that $\|Gx_i\| < \frac{\epsilon}{dimV}$ for all $i \in \{1, 2, ..., n\}$. Considering the Proposition 3.5.16, for $\lambda_i = \frac{1}{2}$, there exists $\theta_i \in \{0, 1\}$ such that

$$2\left\|\sum_{i=1}^{n}(\frac{1}{2} - \theta_i)Gx_i\right\| \leq dimV \max\|Gx_i\| < \epsilon.$$

Put by definition, $I_0 = \{i \in \{1, 2, ..., n\} \colon \theta_i = 0\}$, $I_1 = \{i \in \{1, 2, ..., n\} \colon \theta_i = 1\}$, $y_0 = \bigsqcup\limits_{i\in I_0} x_i$ and $y_1 = \bigsqcup\limits_{i\in I_1} x_i$. One can observe that y_1 and y_2 are mutually complemented fragments of x. Therefore we have

$$\|Gy_1 - Gy_2\| = \left\| \sum_{i \in I_0 \cup I_1} (1 - 2\theta_i) Gx_i \right\| < \epsilon,$$

hence the operator G is narrow.

$\square$

Definition 3.5.21. Let E be a vector lattice and F a vector space. An orthogonally additive operator $T : E \to F$ is called of finite rank if the set $T(E)$ generates a finite-dimensional subspace in F.

Let us consider a Banach space X. We define a set $W := l_\infty(B_{X^*}) = \{q : B_{X^*} \to \mathbb{R}\colon \sup |q(f)| < \infty,\ f \in B_{X^*}\}$. Notice that W is a Banach space equipped with the supremum norm, and X can be considered as a closed subspace of W. Also, notice that if H is a relatively compact subset of the Banach space W and $\epsilon > 0$, then there exists a linear finite rank operator $R \in \mathcal{L}(W)$ such that $\|w - Rw\| \leq \epsilon$ for every $w \in H$. For a comprehensive and detailed discussion on this topic, please refer to [27, Lemma 10.25].

Finally, after these studies, we can now prove Theorem 3.5.15.

Proof. Fix an arbitrary $x \in E$ and $\epsilon > 0$. Since T is a C-compact operator, then the set $K = T(\mathcal{C}_x)$ is relatively compact in X and therefore in W. Thus there exists a finite rank operator $S \in \mathcal{L}(W)$ such that $\|y - Sy\| \leq \frac{\epsilon}{4}$ for all $y \in K$. We define the operator $G := SoT$. Notice that G is an orthogonally additive horizontally-to-norm continuous finite rank operator. By Proposition 3.5.20 G is narrow. Thus there exist mutually complemented fragments x_1, x_2 of x such that $\|Gx_1 - Gx_2\| < \frac{\epsilon}{2}$. It follows from $\|Tx_i - Gx_i\| < \frac{\epsilon}{4}$ for $i \in \{1, 2\}$ that

$$\begin{aligned}
\|Tx_1 - Tx_2\| &= \big\|(Tx_1 - Gx_1) + (Gx_2 - Tx_2) + (Gx_1 - Gx_2)\big\| \\
&\leq \frac{\epsilon}{4} + \frac{\epsilon}{4} + \frac{\epsilon}{2} \\
&= \epsilon.
\end{aligned}$$

Hence T is narrow.

□

Definition 3.5.22. Let E and F be vector lattices. An orthogonally additive operator $T : E \to F$ is called strictly narrow if there exist mutually complemented fragments u, w of x, such that $Tu = Tw$.

Proposition 3.5.23. *Let E be an atom-filled vector lattice, F be a vector lattice and $T : E \to F$ be an orthogonally additive operator. Then T is a strictly narrow operator if and only if $Tx = 0$ for all $x \in E$.*

Proof. Let $T : E \to F$ be a strictly narrow orthogonally additive operator. Take an element $x \in E$. By Theorem 3.5.7, the only mutually complemented fragments of x are only 0 and itself. Thus, we have

$$\begin{aligned} x = 0 \sqcup x \Rightarrow T0 &= Tx \\ \Rightarrow 0 &= Tx. \end{aligned}$$

Additionally, the operator $T = 0$ is always strictly narrow. This finishes the proof. □

3.6. Extensions of an OAO

Consider a vector lattice E, a vector space F, $A \subseteq E$ such that $x + y \in A$ for all disjoint elements $x, y \in A$. A map $T : A \to F$ is called an orthogonally additive map if $T(x + y) = Tx + Ty$ for all disjoint elements $x, y \in A$.

The extension problem can be considered for the orthogonally additive operators. That is, whether every orthogonally additive map on an arbitrary subset A of a vector lattice E has an extension to an orthogonally additive operator on E. So we get the following question. For what subsets A of E every orthogonally additive map $T : A \to F$ can be extended to an orthogonally additive operator $\tilde{T} : E \to F$? Therefore, we need the following definitions of the lateral ideal and lateral band.

3.6.1. Lateral Ideals and Lateral Bands

Definition 3.6.1. Let E be a vector lattice. A subset $\emptyset \neq \mathcal{I}$ of E is called a lateral ideal if the following hold:

(1) $x \sqcup y \in \mathcal{I}$ for every disjoint $x, y \in \mathcal{I}$;

(2) if $x \in \mathcal{I}$ then $y \in \mathcal{I}$ for all $y \in \mathcal{C}_x$.

Every order ideal of a vector lattice E is a lateral ideal of E. Indeed, let $\mathcal{I}$ be an order ideal of E. Take disjoint elements $x, y \in \mathcal{I}$. Then $x + y \in \mathcal{I}$ since $\mathcal{I}$ is a vector subspace of E. Also, take an element $y \in \mathcal{C}_x$. Therefore, one has $|y| \leq |x|$ and hence $y \in \mathcal{I}$.

A lateral ideal need not be an order ideal, generally. Additionally, a lateral ideal may not be a vector subspace of E. The following proposition explains these facts.

Proposition 3.6.2. *Let E be a vector lattice. Then the set $\mathcal{C}_x$ of all fragments of an element $x \in E_+$ is a lateral ideal of E.*

Proof. Let y, z be disjoint elements of $\mathcal{C}_x$

(1) First we remind that $y + z = y \wedge z + y \vee z$. Since $y, z \in \mathcal{C}_x$ and $x \in E_+$, we obtain that $y + z = y \vee z$. By Theorem 2.1.9, the set $\mathcal{C}_x$ is a Boolean algebra. It follows that $y \vee z \in \mathcal{C}_x$. Hence we obtain $y + z \in \mathcal{C}_x$.

(2) Let $y \in \mathcal{C}_x$. Given any $z \sqsubseteq y$, we have $z \sqsubseteq x$, by Proposition 2.4.2.

The proof is finished. □

Corollary 3.6.3. *Let E be a vector lattice. Then the set $\mathcal{C}_x$ of all fragments of an element $x \in E$ is a lateral ideal of E.*

Proof. It is well-known in the literature that $(y+z)^+ = y^+ + z^+$ and $(y+z)^- = y^- + z^-$ for all disjoint elements y, z of E, (see [21, 1.3.3]). Take disjoint $y, z \in \mathcal{C}_x$. Therefore, for the positive part of these elements, we have $y^+, z^+ \in \mathcal{C}_{x^+}$. Also, the disjointness of y and z imply the disjointness of y^+ and z^+. Thus, by Proposition 3.6.2 one has $(y+z)^+ = y^+ + z^+ \in \mathcal{C}_{x^+}$. The same is valid for the negative parts of these elements. So $(y+z)^- \in \mathcal{C}_{x^-}$. It directly follows that $(y+z) \in \mathcal{C}_x$, by Theorem 2.4.1. This finishes the proof. □

We note that the set of fragments need not generally be an ideal of any vector lattice E.

Example 3.6.4. Let E, F be vector lattices and $T : E \to F$ be a positive orthogonally additive operator. Then $Ker(T) = \{x \in E \colon Tx = 0\}$ is an example of a lateral ideal of E. Since T is an orthogonally additive operator, $0 \in Ker(T)$. Therefore the set $Ker(T) \neq \emptyset$. Let $x, y \in Ker(T)$ and $x \perp y$. It follows from $T(x+y) = Tx + Ty = 0$ that $x + y \in Ker(T)$. In the end, take an arbitrary $x \in Ker(T)$ and $y \in \mathcal{C}_x$. Since T is a positive operator, one has

$$Tx = T(x - y + y) = T(x - y) + T(y) \implies 0 \leq T(y) = T(x) - T(x - y) \leq Tx = 0,$$

and hence $y \in Ker(T)$.

Definition 3.6.5. Let E be a vector lattice and $(x_\alpha)_{\alpha \in \Delta}$ be a net in E. The net $(x_\alpha)_{\alpha \in \Delta}$ is called order fundamental if the net $(x_\alpha - x_\beta)_{(\alpha,\beta) \in \Delta \times \Delta}$ order converges to zero.

Definition 3.6.6. An order fundamental net $(x_\alpha)_{\alpha \in \Delta}$ in E is called horizontally fundamental if $x_\alpha \sqsubseteq x_\beta$ for all $\alpha, \beta \in \Delta$ with $\alpha \leq \beta$.

Definition 3.6.7. A subset D of the vector lattice E is called horizontally closed if every horizontally fundamental net $(x_\alpha)_{\alpha \in \Delta}$ in D order converges to some $x \in D$. Horizontally closed lateral ideal $\mathcal{B}$ is called lateral band of E.

Lemma 3.6.8. *Let E be a vector lattice, $x \in E$ and $y_\alpha \xrightarrow{h} y$, where $y \in E$ and $y_\alpha \sqsubseteq x$ for all $\alpha \geq \alpha_0$. Then $y \sqsubseteq x$.*

Proof. The proof follows from Lemma 2.4.3. □

Proposition 3.6.9. *Let E be a vector lattice and $(x_\alpha)_{\alpha\in\Delta}$ be a net in E. If the net $(x_\alpha)_{\alpha\in\Delta}$ horizontally converges, then it is horizontally fundamental.*

Proof. Let $(x_\alpha) \xrightarrow{h} x$. Then we have

$$|x_\alpha - x| \le \beta_\alpha \downarrow 0 \text{ and } x_\alpha \sqsubseteq x_{\alpha'} \text{ for all } \alpha, \alpha' \in \Delta \text{ and } \alpha \le \alpha'.$$

It follows that

$$\begin{aligned} |x_\alpha - x_{\alpha'}| &\le |x_\alpha - x| + |x - x_{\alpha'}| \\ &\le \beta_\alpha + \gamma_{\alpha'}. \end{aligned}$$

Now put by definition $\theta_{(\alpha,\alpha')} = \beta_\alpha + \gamma_{\alpha'}$. Clearly $\theta_{(\alpha,\alpha')} \downarrow 0$, and hence the net $(x_\alpha)_{\alpha\in\Delta}$ is horizontally fundamental. □

Example 3.6.10. Let E be a C-complete vector lattice. The set $\mathcal{C}_x$, which is a lateral ideal for all $x \in E$, is horizontally closed by Lemma 3.6.8. Hence $\mathcal{C}_x$ is a lateral band.

Example 3.6.11. Every band B of a Dedekind complete vector lattice E is a lateral band of E.

We note that Example 3.6.10 shows not every lateral band has to be a band.

Proposition 3.6.12. *Let E be a vector lattice and $(\mathcal{B}_i)_{i\in I}$ be a family of lateral bands of E. Then the set $\bigcap_{i\in I} \mathcal{B}_i$ is a lateral band of E.*

Proof. First, we claim that $\bigcap_{i\in I} \mathcal{B}_i \neq \emptyset$. Indeed, take an arbitrary $i \in I$. Since $\mathcal{B}_i$ is a lateral band, it is also a lateral ideal. Therefore $\mathcal{B}_i \neq \emptyset$. Given any element $x \in \mathcal{B}_i$, we obtain $\mathcal{C}_x \subseteq \mathcal{B}_i$ by the definition of lateral ideal. It directly follows that $0 \in \mathcal{B}_i$, and hence $\bigcap_{i\in I} \mathcal{B}_i \neq \emptyset$. We show that $\bigcap_{i\in I} \mathcal{B}_i$ is a lateral ideal of E. Indeed, let x, y be disjoint elements of $\bigcap_{i\in I} \mathcal{B}_i$. Considering the $\mathcal{B}_i$ is a lateral ideal for all $i \in I$, we have $x + y \in \mathcal{B}_i$, and hence $x + y \in \bigcap_{i\in I} \mathcal{B}_i$. Additionally, for an arbitrary element $x \in \bigcap_{i\in I} \mathcal{B}_i$, we have $x \in \mathcal{B}_i$ for all

$i \in I$. Therefore $\mathcal{C}_x \subseteq \mathcal{B}_i$ for all $i \in I$, and hence $\mathcal{C}_x \subseteq \bigcap_{i \in I} \mathcal{B}_i$. Finally, we show that the set $\bigcap_{i \in I} \mathcal{B}_i$ is horizontally closed. Take a horizontally fundamental net $(x_\alpha)_{\alpha \in \Delta}$ in $\bigcap_{i \in I} \mathcal{B}_i$. It follows that the net $(x_\alpha)_{\alpha \in \Delta}$ is a horizontally fundamental net in the lateral band $\mathcal{B}_i$ for all $i \in I$. Therefore $x_\alpha \xrightarrow{o} x \in \mathcal{B}_i$, which implies $x \in \bigcap_{i \in I} \mathcal{B}_i$. $\square$

Definition 3.6.13. Let H be a subset of a vector lattice E. We denote by $\mathcal{B}(H)$ the intersection of all lateral bands $(\mathcal{B}_\alpha)_{\alpha \in \Delta}$ with $H \subseteq \mathcal{B}_\alpha$, $\alpha \in \Delta$. We say that $\mathcal{B}(H)$ is the lateral band generated by H.

3.6.2. Extensions of an OAO on C-complete Vector Lattices

In this subsection, we construct the extensions of orthogonally additive operators. Some of the results presented in this section can be found in [4, 5]

Definition 3.6.14. Let E, F be vector lattices and $\mathcal{I}$ be a lateral ideal of E. A map $T : \mathcal{I} \to F$ is called orthogonally additive if $T(x + y) = Tx + Ty$ for all disjoint $x, y \in \mathcal{I}$. An orthogonally additive map $T : \mathcal{I} \to F$ is called:

(1) positive if $Tx \geq 0$ for every $x \in \mathcal{I}$;

(2) order bounded if it maps order bounded subsets of $\mathcal{I}$ to order bounded subsets of F.

Definition 3.6.15. Let E be a vector lattice, F be a Banach lattice and $\mathcal{B}$ be a lateral band of E. A positive orthogonally additive map $T : \mathcal{B} \to F$ is called:

(1) horizontally-to-order continuous if $(Tx_\alpha)_{\alpha \in \Delta}$ order converges to Tx whenever a net $(x_\alpha)_{\alpha \in \Delta}$ in $\mathcal{B}$ horizontally converges to x;

(2) narrow if for every $x \in \mathcal{B}$ and $\epsilon > 0$ there exists a disjoint decomposition $x = x_1 \sqcup x_2$ such that $\|Tx_1 - Tx_2\| < \epsilon$;

(3) strictly narrow if for every $x \in \mathcal{B}$ there exist a disjoint decomposition $x = x_1 \sqcup x_2$ such that $Tx_1 = Tx_2$.

(4) AM-compact if T maps order bounded subsets of $\mathcal{B}$ to relatively compact subsets of F;

(5) C-compact if $T(\mathcal{C}_x)$ is relatively compact subset of F for every $x \in \mathcal{B}$;

(6) disjointness preserving if $Tx \perp Ty$ for all disjoint $x, y \in \mathcal{B}$.

Theorem 3.6.16. *Let E be a vector lattice, F be a Dedekind complete vector lattice, $\mathcal{I}$ be a lateral ideal of E and $T : \mathcal{I} \to F^+$ be an order bounded positive orthogonally additive map. Then there exists a positive abstract Urysohn operator $\tilde{T}_{\mathcal{I}} : E \to F$, such that $Tx = \tilde{T}_{\mathcal{I}}x$ for every $x \in \mathcal{I}$. The operator $\tilde{T}_{\mathcal{I}}$ (or $\tilde{T}$ for brevity) is called the minimal extension (with respect to $\mathcal{I}$) of the order bounded positive orthogonally additive map $T : \mathcal{I} \to F$. Moreover,*

$$\tilde{T}x = \sup\{Ty \colon y \in \mathcal{I} \cap \mathcal{C}_x\}$$

for all $x \in E$.

Proof. We show that $\tilde{T}$ is an order bounded positive orthogonally additive operator. Take disjoint elements $x, y \in E$. For an arbitrary $z \in \mathcal{I} \cap \mathcal{C}_{x+y}$, by the decomposition property (see [20, Theorem 1.13]), there exist z_1, z_2 of E such that $z = z_1 + z_2$ and $|z_1| \leq |x|$ and $|z_2| \leq |y|$. By using the same idea in Theorem 3.2.3, we note that $z_1 \perp z_2$, $z_1 \sqsubseteq x$ and $z_2 \sqsubseteq y$. Additionally, we note that $\mathcal{C}_z \subseteq \mathcal{I}$ since $\mathcal{I}$ is a lateral ideal. It follows from $|z_1| \wedge |z - z_1| = |z_1| \wedge |z_2| = 0$ that $z_1 \in \mathcal{C}_z$, and hence $z_1 \in \mathcal{I}$. The same is true for z_2.

Thus, we have

$$\begin{aligned}
T(z) = T(z_1 + z_2) = Tz_1 + Tz_2 \\
&\le \sup\{Tw \colon w \in \mathcal{I} \cap \mathcal{C}_x\} + \sup\{Tw \colon w \in \mathcal{I} \cap \mathcal{C}_y\} \\
&= \tilde{T}x + \tilde{T}y.
\end{aligned}$$

Since z is an arbitrary element of $\mathcal{I} \cap \mathcal{C}_{x+y}$, one has

$$\tilde{T}(x+y) = \sup\{Tw \colon w \in \mathcal{I} \cap \mathcal{C}_{x+y}\} \le \tilde{T}x + \tilde{T}y.$$

We claim that if $z_1 \in \mathcal{I} \cap \mathcal{C}_x$ and $z_2 \in \mathcal{I} \cap \mathcal{C}_y$ then $z_1 + z_2 \in \mathcal{I} \cap \mathcal{C}_{x+y}$. Because

$$\begin{aligned}
|z_1 + z_2| \wedge |x - z_1 + y - z_2| &\le (|z_1| + |z_2|) \wedge (|x - z_1| + |y - z_2|) \\
&\le |z_1| \wedge |y - z_2| + |z_2| \wedge |x - z_1| \\
&\le |z_1| \wedge |y| + |z_1| \wedge |z_2| + |z_2| \wedge |x| + |z_2| \wedge |z_1| \\
&\le |x| \wedge |y| + |z_1| \wedge |z_2| + |y| \wedge |x| + |z_2| \wedge |z_1| \\
&= 0,
\end{aligned}$$

and hence $z_1 + z_2 \in C_{x+y}$. We proved the claim. It directly follows that

$$Tz_1 + Tz_2 = T(z_1 + z_2) \le \sup\{Tw \colon w \in \mathcal{I} \cap \mathcal{C}_{x+y}\} = \tilde{T}(x+y).$$

Now passing to the supremum over $z_1 \in \mathcal{I} \cap \mathcal{C}_x$ and $z_2 \in \mathcal{I} \cap \mathcal{C}_y$, we have

$$\tilde{T}x + \tilde{T}y \le \tilde{T}(x+y).$$

Therefore $\tilde{T}$ is an orthogonally additive operator. In addition, the positivity of T implies the positivity of $\tilde{T}$ by the definition of $\tilde{T}$. Let A be an order bounded subset of E. It follows that the set $\bigcup\limits_{x \in A} \mathcal{C}_x$ is an order bounded subset of E. Since F is Dedekind complete, the set $\tilde{T}(A)$ is order bounded, and hence $\tilde{T}$ is an order bounded operator. Finally, if $x \in \mathcal{I}$, then

$\mathcal{I} \cap \mathcal{C}_x = \mathcal{C}_x$. Take an element $y \in \mathcal{C}_x$. It follows from $y \perp (x-y)$ that

$$T(x) = T(y + (x-y)) = T(y) + T(x-y).$$

By the positivity of T, we have

$$T(y) = T(x) - T(x-y) \leq T(x) \ \text{ for all } y \in \mathcal{C}_x,$$

and hence $\tilde{T}x \leq Tx$. Also, clearly $Tx \leq \tilde{T}x$. Therefore we obtain $Tx = \tilde{T}x$. So, $\tilde{T}$ is an extension of T. This finished the proof. $\square$

Definition 3.6.17. Let E, F be vector lattices, where F is Dedekind complete, $\mathcal{I}$ is a lateral ideal in E and $T : \mathcal{I} \to F^+$ is an order bounded orthogonally additive mapping. A positive abstract Urysohn operator $S : E \to F$ is called the least extension of T if $Tx = Sx$ for every $x \in \mathcal{I}$ and $Sx \leq Rx$ for all $x \in E$ and for every positive abstract Urysohn operator $R : E \to F$ such that $Rx = Tx$ for all $x \in \mathcal{I}$.

Theorem 3.6.18. *Let $E, F, \mathcal{I}, T,$ and $\tilde{T}$ be as in Theorem 3.6.16. The operator $\tilde{T} : E \to F$ is the least extension of $T : \mathcal{I} \to F^+$.*

Proof. Take an arbitrary positive abstract Urysohn operator $R : E \to F$ such that $Tu = Ru$ for all $u \in \mathcal{I}$. Since T is a positive operator, given any $x \in E$ and $y \in \mathcal{I} \cap \mathcal{C}_x$, we have

$$Rx = R(y + (x-y)) = Ry + R(x-y) = Ty + R(x-y) \geq Ty.$$

It directly follows that $\tilde{T}x = \sup\{Ty\colon y \in \mathcal{I} \cap \mathcal{C}_x\} \leq Rx$. Since x is an arbitrary element of E, the proof is complete.

$\square$

Theorem 3.6.19. *Let E be a C-complete vector lattice, F be a Dedekind complete Banach lattice, $\mathcal{B}$ be a lateral band of E, $T : \mathcal{B} \to F$ be a positive orthogonally additive map and $\tilde{T} : E \to F$ be an extension of T defined by $\tilde{T}x = \sup\{Ty\colon y \in \mathcal{B} \cap \mathcal{C}_x\}$. Then*

(1) $\tilde{T}$ *is horizontally-to-order continuous if and only if* T *is;*

(2) $\tilde{T}$ *is strictly narrow if and only if* T *is;*

(3) $\tilde{T}$ *is narrow if and only if* T *is;*

(4) $\tilde{T}$ *is AM-compact if and only if* T *is;*

(5) $\tilde{T}$ *is C-compact if and only if* T *is;*

(6) $\tilde{T}$ *preserves disjointness if and only if* T *is.*

The proof of the theorem will be given after the subsequent studies.

Remark 3.6.20. *Let E be a vector lattice, $x \in E$ and $\mathcal{B}$ be a lateral band of E. We define the set $\mathcal{B}_x := \mathcal{B} \cap \mathcal{C}_x$. One can see that the set $\mathcal{B}_x$ is not empty because it contains at least* 0.

Proposition 3.6.21. *Let E be a C-complete vector lattice, $x \in E$ and $\mathcal{B}$ be a lateral band of E. Then $\mathcal{B}_x$ has a maximal element which we denote by $x^{\mathcal{B}}$.*

Proof. Since E is a C-complete vector lattice and $\mathcal{B}_x \subseteq \mathcal{C}_x$, $\sup \mathcal{B}_x$ exists and it is an element of $\mathcal{C}_x$. Let $v = \sup \mathcal{B}_x$, by definition. The set $\mathcal{B}_x$ is the upward directed set with respect the relation $\sqsubseteq$ because $\mathcal{B}_x \subseteq \mathcal{C}_x$. Thus, there exist a net $(v_\alpha)_{\alpha \in \Delta}$ such that $(v_\alpha) \xrightarrow{h} v$. We note that $\mathcal{C}_x$ is a lateral band for all $x \in E$ and $\mathcal{B}$ is a lateral band by the hypothesis; therefore we obtain $v \in \mathcal{B}$ and $v \in \mathcal{C}_x$, and hence $v \in \mathcal{B}_x$. It directly follows that $v = x^{\mathcal{B}}$. $\square$

Proposition 3.6.22. *Let E be a vector lattice, $\mathcal{B}$ be a lateral band of E, $x \in E$ and $x = y \sqcup z$. Then $x^{\mathcal{B}} = y^{\mathcal{B}} \sqcup z^{\mathcal{B}}$.*

Proof. We note that $x^{\mathcal{B}} \sqsubseteq x$ by Proposition 3.6.21 and there exists a decomposition of $x^{\mathcal{B}} = u + v$ for some $u, v \in E$. Using the method in Theorem 3.2.3, we can see that $u \sqsubseteq y$ and $v \sqsubseteq z$. Clearly, $u \perp v$. Therefore we have $x^{\mathcal{B}} = u \sqcup v$. We show that $u = y^{\mathcal{B}}$ and $v = z^{\mathcal{B}}$. It follows from $|u| \wedge |u + v - u| = |u| \wedge |v| = 0$ that $u \sqsubseteq u + v = x^{\mathcal{B}}$. Thus $u \sqsubseteq x^{\mathcal{B}}$. Clearly, $v \sqsubseteq x^{\mathcal{B}}$. Since $\mathcal{B}$ is a lateral band and $x^{\mathcal{B}} \in \mathcal{B}$, then $\mathcal{B}$ consists all fragments of $x^{\mathcal{B}}$.

So, we obtain $u \in \mathcal{B} \cap \mathcal{C}_y$ and $v \in \mathcal{B} \cap \mathcal{C}_z$. Therefore $u \sqsubseteq y^{\mathcal{B}}$ and $v \sqsubseteq z^{\mathcal{B}}$. Assume that either $u \neq y^{\mathcal{B}}$ or $v \neq z^{\mathcal{B}}$. Then we obtain

$$x^{\mathcal{B}} = u + v \sqsubseteq y^{\mathcal{B}} \sqcup z^{\mathcal{B}} \in \mathcal{B} \cap \mathcal{C}_x \quad \text{and} \quad x^{\mathcal{B}} \neq y^{\mathcal{B}} + z^{\mathcal{B}}.$$

Since $x^{\mathcal{B}}$ is the maximal element of $\mathcal{B} \cap \mathcal{C}_x$, it is a contradiction. $\square$

Lemma 3.6.23. *Let $E, F, \mathcal{B}, T$, and $\tilde{T}$ be as in Theorem 3.6.19. Then $\tilde{T}x = Tx^{\mathcal{B}}$.*

Proof. Take an element $y \in \mathcal{B} \cap \mathcal{C}_x$. It directly follows that

$$\begin{aligned} y \sqsubseteq x^{\mathcal{B}} &\Rightarrow Tx^{\mathcal{B}} = T(x^{\mathcal{B}} - y) + Ty \\ &\Rightarrow Ty \leq Tx^{\mathcal{B}} \qquad \text{(as } T \text{ is positive.)} \\ &\Rightarrow \tilde{T}x \leq Tx^{\mathcal{B}}. \end{aligned}$$

On the other hand, since $x^{\mathcal{B}} \sqsubseteq x$, we have

$$\begin{aligned} &\tilde{T}x^{\mathcal{B}} = \tilde{T}x - \tilde{T}(x - x^{\mathcal{B}}) \\ &\tilde{T}x^{\mathcal{B}} \leq \tilde{T}x \\ &Tx^{\mathcal{B}} \leq \tilde{T}x \qquad \text{(as } x^{\mathcal{B}} \in \mathcal{B} \cap \mathcal{C}_x). \end{aligned}$$

This ends the proof. $\square$

We can now give the proof of Theorem 3.6.19.

Proof. Since T is a restriction of $\tilde{T}$, the "only if" part holds for all items (1)-(6). Now we can give the proof for the other side.
(1) Take a net $(x_\alpha)_{\alpha \in \Delta}$ in E which horizontally converges to $x \in E$. Since $x = x_\alpha \sqcup (x - x_\alpha)$, one has $x^{\mathcal{B}} = x_\alpha^{\mathcal{B}} \sqcup (x - x_\alpha)^{\mathcal{B}}$ by Proposition 3.6.22. Besides, since $(x - x_\alpha)^{\mathcal{B}} \sqsubseteq (x - x_\alpha)$ and $(x - x_\alpha) \xrightarrow{o} 0$ is valid, we obtain $(x - x_\alpha)^{\mathcal{B}} \xrightarrow{o} 0$. It directly follows that $x_\alpha^{\mathcal{B}} \xrightarrow{o} x$. We note that for any $a, b \in E$, $a \sqsubseteq b$ implies $\mathcal{C}_a \subseteq \mathcal{C}_b$. It follows from $\mathcal{B} \cap \mathcal{C}_{x_\alpha} \subseteq \mathcal{B} \cap \mathcal{C}_{x_\beta}$ for all

$\alpha, \beta \in \Delta$ with $\alpha \leq \beta$ that $x_\alpha^{\mathcal{B}} \sqsubseteq x_\beta^{\mathcal{B}}$. Thus, we obtain the net $(x_\alpha^{\mathcal{B}})_{\alpha \in \Delta}$ horizontally converges to $x^{\mathcal{B}}$. Consequently, by Lemma 3.6.23, we have

$$o - \lim_{\alpha \in \Delta} \tilde{T} x_\alpha = o - \lim_{\alpha \in \Delta} T x_\alpha^{\mathcal{B}} = T x^{\mathcal{B}} = \tilde{T} x.$$

(2) Take an arbitrary $x \in E$. Since T is strictly narrow, then there exist mutually complemented fragments u, w of $x^{\mathcal{B}}$ such that $Tu = Tw$. We note that $x^{\mathcal{B}} \sqsubseteq x$ and $x = x^{\mathcal{B}} \sqcup (x - x^{\mathcal{B}}) = u \sqcup w \sqcup (x - x^{\mathcal{B}})$. Assign $\tilde{u} = u$ and $\tilde{w} = w + x - x^{\mathcal{B}}$. Clearly $\tilde{u} \sqsubseteq x$. Additionally, it follows from

$$\begin{aligned} |\tilde{w}| \wedge |x - \tilde{w}| &= \left| w + x - x^{\mathcal{B}} \right| \wedge \left| x - (w + x - x^{\mathcal{B}}) \right| \\ &= \left| x - (x^{\mathcal{B}} - w) \right| \wedge \left| x^{\mathcal{B}} - w \right| \\ &= |x - \tilde{u}| \wedge |\tilde{u}| \\ &= 0, \end{aligned}$$

and hence $\tilde{w} \sqsubseteq x$ and $\tilde{u}, \tilde{w}$ are mutually complemented fragments of x. We claim that $\tilde{T}\tilde{w} = \tilde{T}\tilde{u}$. Indeed, it follows from

$$\begin{aligned} \tilde{T}x = \tilde{T}(x - x^{\mathcal{B}}) + \tilde{T}x^{\mathcal{B}} &\Rightarrow Tx^{\mathcal{B}} = \tilde{T}(x - x^{\mathcal{B}}) + Tx^{\mathcal{B}} \quad (\text{as } x^{\mathcal{B}} \in \mathcal{B} \cap \mathcal{C}_x\ ) \\ &\Rightarrow \tilde{T}(x - x^{\mathcal{B}}) = 0 \end{aligned}$$

that

$$\begin{aligned} \tilde{T}\tilde{u} - \tilde{T}\tilde{w} &= \tilde{T}u - \tilde{T}(w + (x - x^{\mathcal{B}})) \quad (\text{as } w \perp (x - x^{\mathcal{B}})) \\ &= \tilde{T}u - \tilde{T}w - \tilde{T}(x - x^{\mathcal{B}}) \\ &= \tilde{T}u - \tilde{T}w \quad (\text{as } u, w \in \mathcal{B} \cap \mathcal{C}_x) \\ &= Tu - Tw \\ &= 0. \end{aligned}$$

The same steps can be applied to the claim (3).

(4) Take an order bounded subset G of E. Let us define the set $G^{\mathcal{B}} := \{x^{\mathcal{B}} \colon x \in G\}$. Since G is an order bounded set, $G^{\mathcal{B}}$ is also an order bounded subset of E. It follows from

$$\begin{aligned}\tilde{T}(G) &= \{\tilde{T}x \colon x \in G\} \\ &= \{Tx^{\mathcal{B}} \colon x^{\mathcal{B}} \in G^{\mathcal{B}}\} \\ &= T(G^{\mathcal{B}})\end{aligned}$$

that the set $\tilde{T}(G)$ is relatively compact.

The same steps can be applied to the claim (5).

(6) Take disjoint elements $x, y \in E$. We show that $\tilde{T}x \perp \tilde{T}y$. Indeed, since $|x^{\mathcal{B}}| \wedge |y^{\mathcal{B}}| \leq |x| \wedge |y| = 0$, one has $\tilde{T}x \wedge \tilde{T}y = Tx^{\mathcal{B}} \wedge Ty^{\mathcal{B}} = 0$. The proof is finished. $\square$

3.7. Projection Lateral Bands

In this section, our main purpose is to provide fundamental definitions and introduce the concept of projection bands within the context of orthogonally additive operators. Some of the results presented in this section can be found in [4].

Theorem 3.7.1. *Let E be a C-complete vector lattice and $\mathcal{B}$ be a lateral band of E. Then the map $\mathbf{p}_{\mathcal{B}} : E \to E$ defined by $\mathbf{p}_{\mathcal{B}}(x) = \sup\{y \colon y \in \mathcal{B} \cap \mathcal{C}_x\}$ satisfies the following properties:*

(1) $\mathbf{p}_{\mathcal{B}}$ is an orthogonally additive operator;

(2) $\mathbf{p}_{\mathcal{B}}$ is a horizontally-to-order continuous operator;

(3) $\mathbf{p}_{\mathcal{B}}$ is a non-expanding operator;

(4) $\mathbf{p}_{\mathcal{B}}$ is an projection from E onto $\mathcal{B}$.

Proof. First, we note that the supremum on $\mathbf{p}_{\mathcal{B}}$ is taken with respect to the partial order $\sqsubseteq$. Secondly, the map $\mathbf{p}_{\mathcal{B}}$ is well defined and $\mathbf{p}_{\mathcal{B}}(x) = x^{\mathcal{B}}$ according to Proposition 3.6.21.

(1) Take disjoint elements $y, z \in E$. Assign $x = y \sqcup z$. Therefore, by Proposition 3.6.22, we have

$$x^{\mathcal{B}} = y^{\mathcal{B}} + z^{\mathcal{B}} \iff \mathbf{p}_{\mathcal{B}}(x) = \mathbf{p}_{\mathcal{B}}(y \sqcup z) = \mathbf{p}_{\mathcal{B}}(y) + \mathbf{p}_{\mathcal{B}}(z).$$

(2) Take a net $(x_\alpha)_{\alpha \in \Delta}$, which horizontally converges to x. It was proved in Theorem 3.6.19 that $(x_\alpha^{\mathcal{B}})_{\alpha \in \Delta}$ horizontally converges to $x^{\mathcal{B}}$. This completes the proof.

(3) We show that $\mathbf{p}_{\mathcal{B}}(x) \in \{x\}^{dd}$. Take any $y \in \{x\}^d$. It follows from $x^{\mathcal{B}} \sqsubseteq x$ that $0 \leq |x^{\mathcal{B}}| \wedge |y| \leq |x| \wedge |y| = 0$, and hence $\mathbf{p}_{\mathcal{B}}(x) \in \{x\}^{dd}$.

(4) If $x \in \mathcal{B}$, then $\mathcal{C}_x \subseteq \mathcal{B}$. It directly follows that $\mathbf{p}_{\mathcal{B}}(x) = x$. Therefore one has $\mathbf{p}_{\mathcal{B}}(\mathbf{p}_{\mathcal{B}}(x)) = \mathbf{p}_{\mathcal{B}}(x)$ for any $x \in E$. Hence $\mathbf{p}_{\mathcal{B}}$ is a projection onto $\mathcal{B}$.

□

Definition 3.7.2. Let E be a C-complete vector lattice and $\mathcal{B}$ be a lateral band of E. The map $\mathbf{p}_{\mathcal{B}}$ is called lateral projection onto $\mathcal{B}$. The set of all projections of E is denoted by $\mathcal{OA}_{\mathbf{p}}(E)$. If a lateral band $\mathcal{B}$ coincide with $\mathcal{C}_y$ for some $y \in E$, then we denote the operator $\mathbf{p}_{\mathcal{C}_y}$ by $\mathbf{p}_y$.

Example 3.7.3. Let E be a Dedekind complete vector lattice and $\mathcal{B}$ be a band of E. Then the order projection $\pi_{\mathcal{B}}$ is a lateral projection.

Definition 3.7.4. Let E be a vector lattice and $x, y \in E$. We say that x is laterally disjoint to y and write $x \dagger y$ if $\{z \in \mathcal{C}_x \cap \mathcal{C}_y\} = \{0\}$. We say that two subsets G and H of E are laterally disjoint and use the notation $G \dagger H$ if $x \dagger y$ for every $x \in G$ and $y \in H$.

Proposition 3.7.5. *Let E be a vector lattice, $x, y \in E$ and $x \perp y$. Then $x \dagger y$.*

Proof. Take an element $z \in \mathcal{C}_x \cap \mathcal{C}_y$. It follows that $z \sqsubseteq x$ and $z \sqsubseteq y$. Thus one has $|z| \leq |x|$ and $|z| \leq |y|$. It directly follows that $|z| \leq |x| \wedge |y| = 0$, and hence $x \dagger y$. □

The converse side of Proposition 3.7.5 is invalid. Indeed, consider the vector lattice c, and its elements $\mathbf{1} = (1, 1, 1, ...)$ and $\mathbf{2} = (2, 2, 2, ...)$. Since $\mathcal{C}_{\mathbf{1}} \cap \mathcal{C}_{\mathbf{2}} = \{0\}$, one has $\mathbf{1} \dagger \mathbf{2}$. However, this does not imply the disjointness of $\mathbf{1}, \mathbf{2}$.

Proposition 3.7.6. *Let E be a vector lattice, $x, y \in E$, $x \dagger y$ and $x, y \in \mathcal{C}_v$ for some $v \in E$. Then $x \perp y$.*

For detailed proof, please refer to [4, Prop. 5.7].

Proposition 3.7.7. *Let E be an atom-filled vector lattice, and H, G be subsets of E. Then $H \dagger G$ if and only if $H \cap G = \{0\}$ or $H \cap G = \emptyset$.*

Proof. Let H, G be subsets of E. Given any $x \in H$ and $y \in G$ we have the following:

$$\begin{aligned} H \dagger G &\iff \mathcal{C}_x \cap \mathcal{C}_y = \{0\} \\ &\iff x \neq y \quad \text{or} \quad x = y = 0 \qquad (\text{as } x \text{ and } y \text{ are atom}) \\ &\iff H \cap G = \emptyset \quad \text{or} \quad H \cap G = \{0\}. \end{aligned}$$

□

The following example shows that Proposition 3.7.7 may not be accurate in the more general settings of vector lattices.

Example 3.7.8. Consider the vector lattice $\mathbb{R}^2$ with respect to pointwise order. We note that the vector lattice $\mathbb{R}^2$ is not atom-filled. Now take the subsets $H = \{(1, 1)\}$ and $G = \{(1, 0)\}$ of $\mathbb{R}^2$. Clearly $H \cap G = \emptyset$. However, considering the

$$\mathcal{C}_{(1,1)} = \{(0,0), (1,0), (0,1), (1,1)\} \quad \text{and} \quad \mathcal{C}_{(1,0)} = \{(0,0), (1,0)\},$$

we obtain $\mathcal{C}_{(1,1)} \cap \mathcal{C}_{(1,0)} = \{(0,0), (1,0)\} \neq \{(0,0)\}$, and hence H and G are not laterally disjoint.

Proposition 3.7.9. *Let E, F be vector lattices with F Dedekind complete, $T \in \mathcal{U}^+(E, F)$, $\mathcal{I}$ be a lateral ideal of E and ρ be an order projection on F. Then a map $\pi^{\mathcal{I}}T$ defined by*

$$\pi^{\mathcal{I}}Tx = \sup\{Ty \colon y \in \mathcal{I} \cap \mathcal{C}_x\}, \quad x \in E$$

is a positive abstract Urysohn operator and $\rho\pi^{\mathcal{I}}T \in \mathcal{C}_T$.

Proof. Take any disjoint $x, y \in E$. We show that $\pi^{\mathcal{I}}T(x+y) = \pi^{\mathcal{I}}Tx + \pi^{\mathcal{I}}Ty$. Let us take an arbitrary $z \in \mathcal{I} \cap \mathcal{C}_{x+y}$. By decomposition property, there exist $z_1, z_2 \in E$ such that $z = z_1 + z_2$ and $|z_1| \leq x$, $|z_2| \leq y$. By using the same idea in Theorem 3.2.3, we note that $z_1 \perp z_2$, $z_1 \sqsubseteq z$, $z_2 \sqsubseteq z$, $z_1 \sqsubseteq x$ and $z_2 \sqsubseteq y$. Thus $z_1, z_2 \in \mathcal{I}$. Therefore one has the following:

$$\begin{aligned}
Tz &= Tz_1 + Tz_2 \\
&\leq \sup\{Tu\colon u \in \mathcal{I} \cap \mathcal{C}_x\} + \sup\{Tw\colon w \in \mathcal{I} \cap \mathcal{C}_y\} \\
&= \pi^{\mathcal{I}}Tx + \pi^{\mathcal{I}}Ty.
\end{aligned}$$

Considering the z is an arbitrary element of $\mathcal{I} \cap \mathcal{C}_{x+y}$, one has $\pi^{\mathcal{I}}T(x+y) \leq \pi^{\mathcal{I}}Tx + \pi^{\mathcal{I}}Ty$. On the other hand, since $z_1 \in \mathcal{I} \cap \mathcal{C}_x$ and $z_2 \in \mathcal{I} \cap \mathcal{C}_y$, we get $z_1 + z_2 \in \mathcal{I} \cap \mathcal{C}_{x+y}$. It directly follows that

$$Tz_1 + Tz_2 = T(z_1 + z_2) \leq \pi^{\mathcal{I}}T(x+y).$$

Taking supremum over $z_1 \in \mathcal{I} \cap \mathcal{C}_x$ and $z_2 \in \mathcal{I} \cap \mathcal{C}_y$, we get

$$\pi^{\mathcal{I}}Tx + \pi^{\mathcal{I}}Ty \leq \pi^{\mathcal{I}}T(x+y),$$

and hence we obtain $\pi^{\mathcal{I}}Tx + \pi^{\mathcal{I}}Ty = \pi^{\mathcal{I}}T(x+y)$. Considering the idea in Theorem 3.6.16, we conclude that $\pi^{\mathcal{I}}Tx$ is a positive abstract Urysohn operator. Finally, we show that $\rho\pi^{\mathcal{I}}T \in \mathcal{C}_T$. Consider the operator $\Pi : \mathcal{U}^+(E,F) \to \mathcal{U}^+(E,F)$ defined by $\Pi(T) = \rho\pi^{\mathcal{I}}T$. We observe that the inequality $0 \leq \Pi(T) \leq T$ holds for all $T \in \mathcal{U}^+(E,F)$. By [20, Theorem 1.44] we deduce that Π is an order projection. Additionally, again corresponding to [20, Theorem 1.44], we have

$$\Pi(T) \perp S - \Pi(S) \quad \text{for all } T, S \in \mathcal{U}^+(E,F).$$

It directly follows that $\Pi(T) \perp T - \Pi(T)$, and hence $\rho\pi^{\mathcal{I}}T \perp (T - \rho\pi^{\mathcal{I}}T)$. The proof is finished. $\square$

Proposition 3.7.10. *Let E be a C-complete vector lattice, F be a Dedekind complete vector lattice, $x \in E$ and $T \in \mathcal{U}^+(E, F)$. Then $\pi^{\mathcal{C}_x}T = T\mathbf{p}_x$*

Proof. Let $v \in E$. Take any element $z \in \mathcal{C}_v \cap \mathcal{C}_x$. Consider the maximal element $\mathbf{p}_x$ of $\mathcal{C}_v \cap \mathcal{C}_x$. Since $z \sqsubseteq \mathbf{p}_x$, one has $Tz = T\mathbf{p}_x - T(\mathbf{p}_x - z) \leq T\mathbf{p}_x$. It directly follows that

$$\pi^{\mathcal{C}_x}Tv = \sup\{Ty\colon y \in \mathcal{C}_v \cap \mathcal{C}_x\} \leq T\mathbf{p}_x.$$

On the other hand, considering the $\mathbf{p}_x$ is an element of $\mathcal{C}_v \cap \mathcal{C}_x$ one has

$$T\mathbf{p}_x \leq \sup\{Ty\colon y \in \mathcal{C}_v \cap \mathcal{C}_x\} = \pi^{\mathcal{C}_x}Tv.$$

Hence $\pi^{\mathcal{C}_x}T = T\mathbf{p}_x$. $\square$

4. OAO on Lattice-Normed Spaces

In this chapter, our main purpose is to explore the concept of orthogonally additive operators within the context of lattice-normed spaces. Moreover, we introduce a new class of orthogonally additive operators known as dominated orthogonally additive operators. We note that further information about lattice-normed spaces can be found in Section 2.3.

4.1. Dominated OAO

Some of the results presented in this section can be found in [3].

Definition 4.1.1. Let (V, E) and (W, F) be lattice-normed spaces. A map $T : V \to W$ is said to be an orthogonally additive if $T(u+v) = Tu + Tv$ for any $u, v \in V$ with $u \perp v$.

Definition 4.1.2. An orthogonally additive map $T : V \to W$ is said to be a dominated Popov operator (or dominated $\mathcal{P}$-operator for brevity) if there exists a positive orthogonally additive operator $S : E \to F$ such that $\|Tv\| \leq S\|v\|$ for any $v \in V$. In this case, we say that S is a dominant for T. The set of all dominants of an operator T is denoted by $\mathcal{D}(T)$. If there exist the least element in $\mathcal{D}(T)$ with respect to the order induced by $\mathcal{P}_+(E, F)$, then it is called the least or the exact dominant of T, and it is denoted by $\|T\|$. The set of all dominated $\mathcal{P}$-operators from V to W is denoted by $\mathcal{DP}(V, W)$.

Example 4.1.3. Consider the vector space $\mathbb{R}^{\mathbb{R}}$, which is the space of all functions from $\mathbb{R}$ to $\mathbb{R}$. Let us define the set $\mathbb{R}_0^{\mathbb{R}} := \{f \in \mathbb{R}^{\mathbb{R}} \colon f(0) = 0\}$. We claim that

$$\mathcal{DP}(\mathbb{R}, \mathbb{R}) = \mathbb{R}_0^{\mathbb{R}}.$$

Take an element $f \in \mathbb{R}_0^{\mathbb{R}}$. Consider the function $S : \mathbb{R} \to \mathbb{R}$ defined by $S(x) = \left|f(x)\right| + \left|f(-x)\right|$. Clearly, $S(0) = 0$. Therefore S is a positive orthogonally additive operator. Additionally, the inequality $\left|f(x)\right| \leq \left|f(|x|)\right| + \left|f(-|x|)\right| = S\,|x|$ holds true for all $x \in \mathbb{R}$. Therefore S is a dominant for f and $f \in \mathcal{DP}(\mathbb{R}, \mathbb{R})$. Hence $\mathbb{R}_0^{\mathbb{R}} \subseteq \mathcal{DP}(\mathbb{R}, \mathbb{R})$. The other side follows from the definition of $\mathcal{DP}(\mathbb{R}, \mathbb{R})$.

Example 4.1.4. Let X and Y be normed spaces. Consider the lattice-normed spaces $(X, \mathbb{R})$ and $(Y, \mathbb{R})$. Then the map $T : X \to Y$ is an element of $\mathcal{DP}(X, Y)$ if and only if there exists a function $f : \mathbb{R} \to \mathbb{R}_+$ such that $f(0) = 0$, and the inequality $\|Tx\| \leq f(\|x\|)$ holds for every $x \in X$. Indeed, let $T \in \mathcal{DP}(X, Y)$. Then there exists a positive orthogonally additive operator $S : \mathbb{R} \to \mathbb{R}$ such that $\|Tx\| \leq S\|x\|$. Additionally, since S is an orthogonally additive operator from $\mathbb{R}$ to $\mathbb{R}$, then $S(0) = 0$. We prove now the "if" implication. Consider the function $S : \mathbb{R} \to \mathbb{R}$ defined by $S(x) = \big|f(x)\big| + \big|f(-x)\big|$. We note that S is a positive orthogonally additive operator and it satisfies the following inequality

$$\|Tx\| \leq f(\|x\|) \leq S\,|x|\,.$$

This finishes the proof.

Example 4.1.5. Let E and F be vector lattices with F Dedekind complete. Consider the lattice-normed spaces (E, E) and (F, F) where the lattice-valued norms coincide with the modules. We claim that $\mathcal{P}(E, F) = \mathcal{DP}(E, F)$. Take any C-bounded operator $T \in \mathcal{P}(E, F)$. Considering the F is Dedekind complete, T is dominated by its module $|T|$. Therefore $\mathcal{P}(E, F) \subseteq \mathcal{DP}(E, F)$. On the other hand, given any $T \in \mathcal{DP}(E, F)$ there exist a positive orthogonally additive operator $S : E \to F$ such that $|T| \leq S\,|x|$ for all $x \in E$. By Proposition 3.2.4, S is a C-bounded operator. This directly implies that T is a C-bounded operator. Hence $\mathcal{DP}(E, F) \subseteq \mathcal{P}(E, F)$.

Now we consider a decomposable lattice-normed space (V, E). We introduce an important set denoted by $\tilde{E}_+ = \{e \in E_+ \colon e = \bigsqcup_{i=1}^{n} \|v_i\|\ ;\ v_i \in V\ ;\ n \in \mathbb{N}\}$, which plays a significant role in the following lemmas and theorems. We note that the set $\tilde{E}_+$ is a lateral ideal of the vector lattice E. For detailed proof see [3].

Lemma 4.1.6. *Let E, F be vector lattices with F Dedekind complete, D be a lateral ideal in E, and $(T_\alpha)_{\alpha \in \Delta}$ be a downward directed set of positive orthogonally additive maps from D to F. For any $e \in D$, put*

$$Re = \inf\{T_\alpha e \colon \alpha \in \Delta\}.$$

Then R defines a positive orthogonally additive map from D to F.

Proof. Let us take an element $e \in D$. Consider the set $\mathcal{R}e := \{T_\alpha e \colon \alpha \in \Delta\}$. We note that since the net $(T_\alpha)_{\alpha\in\Delta}$ is downward directed, then the set $\mathcal{R}e$ is also a downward directed set in F. Considering the Dedekind completness of F, we conclude that R is well defined. Additionally, by definition of itself, R is positive. Now we show that R is an orthogonally additive map. Take any disjoint elements $e, f \in D$ and any $\alpha \in \Delta$. It follows from

$$T_\alpha(e+f) = T_\alpha e + T_\alpha f$$

that

$$R(e+f) \leq T_\alpha e + T_\alpha f.$$

Passing to the infimum in the right-hand side of the previous inequality over all $\alpha \in \Delta$, we obtain that $R(e+f) \leq Re + Rf$. For the other side, we have $Re + Rf \leq T_\alpha e + T_\alpha f = T_\alpha(e+f)$. Therefore the element $Re + Rf$ is a lower bound for the set $\{T_\alpha e \colon \alpha \in \Delta\}$. It directly follows that

$$Re + Rf \leq R(e+f).$$

This finishes the proof. $\square$

Definition 4.1.7. Let E, F be vector lattices with F Dedekind complete, and let D be a lateral ideal in E. With any positive orthogonally additive map $T : D \to F$, we can associate a map $\tilde{T}_D : E \to F$ defined by $\tilde{T}_D e = \sup\{Te_0 \colon e_0 \sqsubseteq e\ ;\ e_0 \in D\}$. The map $\tilde{T}_D$ is called the minimal extension (with respect to D) of T.

Lemma 4.1.8. *Let E, F be vector lattices with F Dedekind complete, D be a lateral ideal in E, and $T : D \to F$ be a positive orthogonally additive map. Then $\tilde{T}_D \in \mathcal{P}_+(E, F)$ and $\tilde{T}_D e = Te$ for any $e \in D$.*

For detailed proof, please refer to [16, Theorem 4.4].

Theorem 4.1.9. *Let (V, E) and (W, F) be lattice-normed spaces with V decomposable and F Dedekind complete. Then every dominated $\mathcal{P}$-operator $T : V \to W$ has an exact* *dominant* $\|T\|$ *. Moreover the exact dominant of a dominated* $\mathcal{P}$*-operator* $T : V \to W$ *can be calculated by the following formulas:*

(1) $\|T\|(e) = \sup\{\sum_{i=1}^{n} \|Tu_i\| : \bigsqcup_{i=1}^{n} \|u_i\| = e,\ n \in \mathbb{N}\} \quad (e \in \tilde{E}_+)$;

(2) $\|T\|(e) = \sup\{\|T\|(e_0) \colon e_0 \in \tilde{E}_+,\ e_0 \sqsubseteq e\} \quad (e \in E)$.

Proof. To complete the proof, we need to show that $\mathcal{D}(T)$ is the downward-directed set. Take any $S_1, S_2 \in \mathcal{D}(T)$. Let $x \in V$ and $e = \|x\|$. Consider disjoint pair $f, h \in E$ with $e = f \sqcup h$. Since (V, E) is a decomposable lattice-normed space, there exist $y, z \in V$ such that $x = y + z$ and $\|y\| = f, \|z\| = h$. It directly follows that

$$
\begin{aligned}
\|Tx\| = \big\|T(y+z)\big\| = \|Ty + Tz\| &\le \|Ty\| + \|Tz\| \\
&\le S_1\|y\| + S_2\|z\| \\
&= S_1 f + S_2 h.
\end{aligned}
$$

Passing to the infimum in the right-hand side of the above inequality over all $h, f \in E_+$, we have

$$\|Tx\| \le (S_1 \wedge S_2)\|x\| .$$

Take an any $S \in \mathcal{D}(T)$. We define $\overline{S}$ as the restriction of S to positive cone E_+. We note that $\tilde{S} : E_+ \to F$ is an orthogonally additive map. Consider the set $\mathcal{R} := \{\overline{S} \colon S \in \mathcal{D}(T)\}$. One can see that $\mathcal{R}$ is a downward directed set of positive orthogonally additive maps from E_+ to F. Therefore considering the Lemma 4.1.6, the operator $R : E_+ \to F$ defined by

$$Re = \inf\{\overline{S}e \colon S \in \mathcal{D}(T)\}$$

is a positive orthogonally additive map, and

$$\|Tx\| \le R\|x\| \le S\|x\|$$

is valid for all $x \in V$ and $S \in \mathcal{D}(T)$. Hence by Lemma 4.1.8, there exists the minimal extension of R. Let $\tilde{R}$ be the minimal extension of R. It follows that $\tilde{R} = \|T\|$. Take an element $e \in \tilde{E}_+$. We denote by Ge the right-hand side of the formula (1). For any decomposition $e = \bigsqcup_{i=1}^{n} \|v_i\|$, $v_1, ..., v_n \in V$, we have the following

$$\|Tv_i\| \leq \|T\|\|v_i\| \implies \sum_{i=1}^{n} \|Tv_i\| \leq \sum_{i=1}^{n} \|T\|\|v_i\| = \|T\| \, (\bigsqcup_{i=1}^{n} \|v_i\|) = \|T\| \, (e).$$

It follows that $\|T\| \, (e)$ is an upper bound for Ge. Considering the Dedekind completeness of F, we conclude that $Ge : \tilde{E}_+ \to F$ a is well defined positive map. Now we show that G is an orthogonally additive map. Take disjoint $f, h \in \tilde{E}_+$ with $f = \bigsqcup_{i=1}^{n} \|u_i\|$ and $h = \bigsqcup_{j=1}^{m} \|v_j\|$. It follows that

$$\begin{aligned} G(f+h) &= \sup\{\sum_{i=1}^{k} \|Tw_i\| : \bigsqcup_{i=1}^{k} \|w_i\| = f + h, \ k \in \mathbb{N}\} \\ &\geq \sum_{i=1}^{n} \|Tu_i\| + \sum_{j=1}^{m} \|Tv_j\| \end{aligned}$$

Passing to the supremum in the right-hand side of the above inequality over all $h, f \in \tilde{E}_+$, we have $Gf + Gh \leq G(f+h)$. On the other hand, let us consider a decomposition $f + h = \bigsqcup_{i=1}^{n} \|w_i\|$. Since the lattice normed space (V, E) is decomposable, there exist mutually disjoint elements $u_1, ..., u_n \in V$ and $v_1, ..., v_n \in V$, such that

$$f = \bigsqcup_{i=1}^{n} \|u_i\| \quad \text{and} \quad h = \bigsqcup_{i=1}^{n} \|v_i\|$$

with $w_i = u_i + v_i$ for all $i = 1, ..., n$. It directly follows that

$$\sum_{i=1}^{n} \|Tw_i\| \leq \sum_{i=1}^{n} \|Tu_i\| + \sum_{i=1}^{n} \|Tv_i\| \leq Ge + Gf.$$

Thus $G(e+f) \leq Ge + Gf$, and hence G is an orthogonally additive map from $\tilde{E}_+$ to F and $\|Tx\| \leq G\|x\|$ for all $x \in V$. By Lemma 4.1.8, it follows that $\tilde{G} \in \mathcal{P}_+(E, F)$. We claim that $\tilde{G}$ is the exact dominant of T. Take an arbitrary $S \in \mathcal{D}(T)$ and $e \in \tilde{E}_+$. For any decomposition $e = \bigsqcup_{i=1}^{n} \|u_i\|$, we have

$$\sum_{i=1}^{n} \|Tu_i\| \leq S(\bigsqcup_{i=1}^{n} \|u_i\|) = Se.$$

It follows that

$$\tilde{G}e = Ge \leq Se, \quad e \in \tilde{E}_+.$$

Take an arbitrary element $e \in E$. Then we have

$$\tilde{G}e_0 = Ge_0 \leq Se_0 \leq Se, \quad e_0 \sqsubseteq e, \quad e_0 \in \tilde{E}_+.$$

Passing to the supremum in the left-hand side of the above inequality over all fragments $e_0 \in \tilde{E}_+$ of an element e, we have $\tilde{G}_e \leq Se$ for all $e \in E$. □

Corollary 4.1.10. *Let (V, E) and (W, F) be the same as Theorem 4.1.9. Then an orthogonally additive operator $T : V \to W$ is dominated if and only if the set*

$$\mathcal{G}e = \{\sum_{i=1}^{n} \|Tu_i\| : \bigsqcup_{i=1}^{n} \|u_i\| = e, \ n \in \mathbb{N}\}$$

is order bounded for any $e \in \tilde{E}_+$.

Proof. The proof follows from Theorem 4.1.9. □

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