

On the domination number and the total domination number of Fibonacci cubes

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Abstract

Fibonacci cubes are special subgraphs of the hypercube graphs. Their domination numbers and total domination numbers are obtained for some small dimensions by integer linear programming. For larger dimensions upper and lower bounds on these numbers are given. In this paper, we present the up-down degree polynomials for Fibonacci cubes containing the degree information of all vertices in more detail. Using these polynomials we define optimization problems whose solutions give better lower bounds on the domination numbers and total domination numbers of Fibonacci cubes. Furthermore, we present better upper bounds on these numbers.

Keywords: Fibonacci cubes, domination number, total domination number, integer linear programming.

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1 Introduction

Let $G = (V(G), E(G))$ be a graph with vertex set $V(G)$ and edge set $E(G)$. $D \subseteq V(G)$ is a *dominating set* of G if every vertex in $V(G)$ either belongs to D or is adjacent to some vertex in D . The *domination number* $\gamma(G)$ is defined as the minimum cardinality of a dominating set of the graph G . Similarly, $D \subseteq V(G)$ is a *total dominating set* if every vertex in $V(G)$ is adjacent to some vertex in D and the *total domination number* $\gamma_t(G)$ is defined as the minimum cardinality of a total dominating set of G . Note that the total domination number is defined for isolate-free graphs and it is not defined for the graphs that contain isolated vertices. The domination number of the Fibonacci cubes Γ_n is first given

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in [14] and [2]. These results are extended in [8] by using integer linear programming for some cases. Total domination number of Γ_n is considered in [1], in which an upper bound and a lower bound on $\gamma_t(\Gamma_n)$ are obtained. The exact values of $\gamma(\Gamma_n)$ and $\gamma_t(\Gamma_n)$ are also considered by using integer programming in [1]. The upper bound on $\gamma_t(\Gamma_n)$ given in [1] is improved in [15]. We summarize these results in Section 2. The aim of this work is to improve some of the results given in [1] and [15].

The hypercube Q_n of dimension $n \geq 1$ is the graph with vertex set $V(Q_n) = \{0, 1\}^n$, in which two vertices are adjacent if they differ in one coordinate. For convenience $Q_0 = K_1$. All the vertices of Q_n are labeled by the binary strings of length n . The Fibonacci cubes Γ_n are special subgraphs of Q_n and they were introduced by Hsu [7] as a model of interconnection networks. In literature, many interesting properties of the Fibonacci cubes have been investigated, see survey [9] for details. In recent years results on disjoint hypercubes in Γ_n are presented in [5, 13, 16] and the cube enumerator polynomial of Γ_n is considered in [10, 11, 17] and many combinatorial results are given. The domination-type invariants of Γ_n are considered in [1, 2, 8, 14, 15] and some numerical results and bounds are presented.

It is known that Fibonacci strings of length n are the binary strings of length n that contain no consecutive ones. For this reason we can write

$$V(\Gamma_n) = \{b_1b_2 \cdots b_n \mid b_i \in \{0, 1\}, 1 \leq i \leq n, \text{ and } b_i \cdot b_{i+1} = 0 \text{ for } 1 \leq i < n\} \text{ and}$$

$$E(\Gamma_n) = \{(u, v) \mid u, v \in V(\Gamma_n), d_H(u, v) = 1\},$$

where $d_H(u, v)$ denotes the Hamming distance between u and v , that is, the number of different coordinates in u and v . The number of vertices of the Fibonacci cubes Γ_n is F_{n+2} , where F_n are the Fibonacci numbers defined as $F_0 = 0, F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. For $n \geq 2$ we will use the following formulation for the fundamental decomposition of Γ_n (see, [9]):

$$\Gamma_n = 0\Gamma_{n-1} + 10\Gamma_{n-2}. \tag{1.1}$$

Here note that $0\Gamma_{n-1}$ is the subgraph of Γ_n induced by the vertices that start with 0 and $10\Gamma_{n-2}$ is the subgraph of Γ_n induced by the vertices that start with 10. Furthermore, $0\Gamma_{n-1}$ has a subgraph isomorphic to $00\Gamma_{n-2}$, and there is a matching between $00\Gamma_{n-2}$ and $10\Gamma_{n-2}$ (see Figure 1).

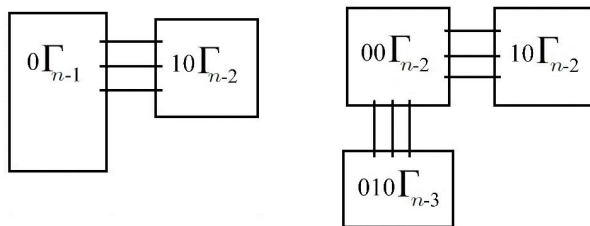


Figure 1: Fundamental decompositions of the Fibonacci cube $\Gamma_n, n \geq 3$.

In this paper, we present upper bounds on $\gamma(\Gamma_n)$ and $\gamma_t(\Gamma_n)$. Furthermore, we introduce the up-down degree polynomials for Γ_n containing the degree information of all vertices $V(\Gamma_n)$ in more detail. Using these polynomials we define optimization problems whose solutions give lower bound on $\gamma(\Gamma_n)$ and $\gamma_t(\Gamma_n)$.

Table 1: Known values of $\gamma(\Gamma_n)$ and $\gamma_t(\Gamma_n)$.

n	1	2	3	4	5	6	7	8	9	10	11	12	13
$ V(\Gamma_n) $	2	3	5	8	13	21	34	55	89	144	233	377	610
$\gamma(\Gamma_n)$	1	1	2	3	4	5	8	12	17	25	39	54–61	
$\gamma_t(\Gamma_n)$	2	2	2	3	5	7	10	13	20	30	44	65	97–101

Now we describe the integer linear programming used in [8] and [1]. Suppose each vertex $v \in V(\Gamma_n)$ is associated with a binary variable x_v . Let $N(v)$ be the set of vertices adjacent to v and $N[v] = N(v) \cup \{v\}$. The problems of determining $\gamma(\Gamma_n)$ and $\gamma_t(\Gamma_n)$ can be expressed as a problem of minimizing the objective function

$$\sum_{v \in V(\Gamma_n)} x_v \tag{2.1}$$

subject to the condition that for every $v \in V(\Gamma_n)$ we have

$$\sum_{a \in N[v]} x_a \geq 1 \text{ (for domination number),}$$

$$\sum_{a \in N(v)} x_a \geq 1 \text{ (for total domination number).}$$

The value of the objective function is then $\gamma(\Gamma_n)$ and $\gamma_t(\Gamma_n)$ respectively. Note that this problem has F_{n+2} variables and F_{n+2} constraints. In [1], it is stated that $\gamma(\Gamma_{12})$ and $\gamma_t(\Gamma_{13})$ were not computed in real time using the above optimization problem. They got the estimates

$$54 \leq \gamma(\Gamma_{12}) \leq 61 \quad \text{and} \quad 97 \leq \gamma_t(\Gamma_{13}) \leq 101.$$

Here, the main difficulty is the order of Γ_n which equals to the number of variables and the number of constraints.

By using the degree information of the vertices in Γ_n the following lower bound on $\gamma(\Gamma_n)$ is presented in [14].

Theorem 2.1 ([14]). *If $n \geq 9$, then*

$$\gamma(\Gamma_n) \geq \left\lceil \frac{F_{n+2} - 2}{n - 2} \right\rceil.$$

By using a similar technique the following lower bound on $\gamma_t(\Gamma_n)$ is obtained in [1].

Theorem 2.2 ([1]). *If $n \geq 9$, then*

$$\gamma_t(\Gamma_n) \geq \left\lceil \frac{F_{n+2} - 11}{n - 3} \right\rceil - 1.$$

In Section 3 we propose an optimization problem having less number of variables and constraints to estimate lower bounds on $\gamma(\Gamma_n)$ and $\gamma_t(\Gamma_n)$. Our results improve the lower

bounds given in Theorem 2.1 and Theorem 2.2 and we present some numerical values in Table 2 and Table 3.

By using the exact values in Table 1 and the fundamental decomposition (1.1) of Γ_n , the following upper bound on $\gamma_t(\Gamma_n)$ is obtained in [1].

Theorem 2.3 ([1]). *If $n \geq 11$, then $\gamma_t(\Gamma_n) \leq 21F_{n-8} + 2F_{n-10}$.*

In [1], using the computer result $\gamma_t(\Gamma_{13}) \leq 101$ the upper bound in Theorem 2.3 improved to

$$\gamma_t(\Gamma_n) \leq 601F_{n-1} - 371F_n, \quad n \geq 12.$$

These two upper bounds further improved in [15] by using the values of $\gamma(\Gamma_n)$ and the fundamental decomposition (1.1) of Γ_n more than once.

Theorem 2.4 ([15]). *If $n \geq 15$, then*

$$\begin{aligned} \gamma(\Gamma_n) \leq \gamma_t(\Gamma_n) &\leq 3\gamma(\Gamma_{n-3}) + 2\gamma(\Gamma_{n-4}) \\ &\leq 116F_n - 187F_{n-1} = 21F_{n-8} - (2F_{n-10} + F_{n-12}). \end{aligned}$$

Furthermore, $\gamma_t(\Gamma_{14}) \leq 166$.

We implemented the same integer linear programming problem (2.1) using CPLEX in NEOS Server [3, 4, 6] for $n = 13$ and obtain the estimates (takes approximately 2 hours)

$$78 \leq \gamma(\Gamma_{13}) \leq 93.$$

Using this result with $\gamma(\Gamma_{12}) \leq 61$ we obtain the following bound on the domination number of Γ_n .

Theorem 2.5. *If $n \geq 12$, then $\gamma(\Gamma_n) \leq 21F_{n-8} - (2F_{n-10} + 8F_{n-12})$.*

Proof. The proof mimics the proof of [1, Theorem 2.1]. By the fundamental decomposition (1.1) of Γ_n we have $\gamma(\Gamma_n) \leq \gamma(\Gamma_{n-1}) + \gamma(\Gamma_{n-2})$. We know that $\gamma(\Gamma_{12}) \leq 61$ and $\gamma(\Gamma_{13}) \leq 93$. For $n \geq 12$ define the sequence (b_n) with $b_n = b_{n-1} + b_{n-2}$ where $b_{12} = 61$ and $b_{13} = 93$. Then by induction we have $b_n = 21F_{n-8} - 2F_{n-10} - 8F_{n-12}$ for any $n \geq 12$. We complete the proof since $\gamma(\Gamma_n) \leq b_n$ for $n \geq 12$. \square

Combining the results in Theorem 2.5 and Theorem 2.4 we get the following result which improves Theorem 2.3 and Theorem 2.4.

Theorem 2.6. *If $n \geq 16$, then*

$$\gamma_t(\Gamma_n) \leq 21F_{n-8} - (2F_{n-10} + 8F_{n-12}).$$

3 Up-down degree enumerator polynomial

In this section we present the up-down degree enumerator polynomial for Γ_n . It contains the degree information of all vertices $V(\Gamma_n)$ in more detail. Using this polynomial we write optimization problems whose solutions are lower bounds on $\gamma(\Gamma_n)$ and $\gamma_t(\Gamma_n)$.

For each fixed $v \in V(\Gamma_n)$ we write a monomial $x^u y^d$ where $d = w(v)$ is the Hamming weight of v and u is $\deg(v) - d$ (that is, $\deg(v) = u + d$). Recall that, by the definition of Γ_n , $(v, v') \in E(\Gamma_n)$ if and only if $d_H(v, v') = 1$. Therefore, the number of neighbors of v

whose weight is one more than the weight of v (say up neighbors of v , $w(v') = w(v) + 1$) is u and the number of neighbors of v whose weight is one less than the weight of v (say down neighbors of v , $w(v') = w(v) - 1$) is d . For this reason we call the polynomial

$$P_n(x, y) = \sum_{v \in V(\Gamma_n)} x^{\deg(v)-w(v)} y^{w(v)} = \sum_{v \in V(\Gamma_n)} x^u y^d$$

as the up-down degree enumerator polynomial of Γ_n . By using the fundamental decomposition (1.1) of Γ_n we obtain the following recursive relation which will be useful to calculate $P_n(x, y)$.

Theorem 3.1. *Let $P_n(x, y)$ be the up-down degree enumerator polynomial of Γ_n . Then for $n \geq 3$ we have*

$$P_n(x, y) = xP_{n-1}(x, y) + yP_{n-2}(x, y) + yP_{n-3}(x, y) - xyP_{n-3}(x, y),$$

where

$$P_0(x, y) = 1, \quad P_1(x, y) = x + y \quad \text{and} \quad P_2(x, y) = x^2 + 2y.$$

Proof. P_0, P_1 and P_2 are clear from Figure 2. Assume that $n \geq 3$. We know that the up-down degree enumerator polynomials of $\Gamma_{n-1}, \Gamma_{n-2}$ and Γ_{n-3} are $P_{n-1}(x, y), P_{n-2}(x, y)$ and $P_{n-3}(x, y)$ respectively. By (1.1) we have

$$\Gamma_n = 0\Gamma_{n-1} + 10\Gamma_{n-2} \tag{3.1}$$

$$= (00\Gamma_{n-2} + 010\Gamma_{n-3}) + 10\Gamma_{n-2} \tag{3.2}$$

and there is a matching between $00\Gamma_{n-2}$ and $10\Gamma_{n-2}$ (see also Figure 1). From this decomposition we have the following three different cases:

1. Assume that $v \in 10\Gamma_{n-2}$. These vertices are the ones in Γ_{n-2} whose weights $d = w(v)$ increase by one in Γ_n . Furthermore, their degrees increase by one due to the matching between $00\Gamma_{n-2}$ and $10\Gamma_{n-2}$, which means that $u = \deg(v) - w(v)$ remains the same in Γ_n . Therefore, these vertices contribute $yP_{n-2}(x, y)$ to $P_n(x, y)$.
2. Assume that $v \in 010\Gamma_{n-3}$. These vertices are the ones in Γ_{n-3} whose weights $d = w(v)$ increase by one in Γ_n and their degrees increase by one due to the matching between $010\Gamma_{n-3}$ and $000\Gamma_{n-3} \subset 00\Gamma_{n-2}$, which means that $u = \deg(v) - w(v)$ remains the same in Γ_n . Therefore, these vertices contribute $yP_{n-3}(x, y)$ to $P_n(x, y)$.
3. Assume that $v \in 00\Gamma_{n-2}$. These vertices are the ones in $0\Gamma_{n-1}$ that are not in $010\Gamma_{n-3}$. In Γ_{n-1} the up-down degree enumerator polynomial of these vertices becomes $P_{n-1}(x, y) - yP_{n-3}(x, y)$. The weights $d = w(v)$ of all such vertices remain the same in Γ_n but their degrees increase by one due to the matching between $00\Gamma_{n-2}$ and $10\Gamma_{n-2}$, that is, $u = \deg(v) - w(v)$ increase by one in Γ_n . Therefore, these vertices contribute $x(P_{n-1}(x, y) - yP_{n-3}(x, y))$ to $P_n(x, y)$.

By adding all of the above contributions we get the desired result. □

Now we describe an optimization problem using the up-down degree enumerator polynomial $P_n(x, y)$. Let D_T be a total dominating set of Γ_n . Then by the definition of Fibonacci cubes for every vertex $v \in V(\Gamma_n)$ with weight $w(v)$ then there must exist a vertex

$v_D \in N(v) \cap D_T$ with $w(v_D) = w(v) \mp 1$. Furthermore, assume that for any fixed vertex $v_D \in D_T$ its corresponding monomial be $x^u y^d$ in the $P_n(x, y)$. This means that v_D dominates u distinct vertices $v \in V(\Gamma_n)$ with weight $w(v) = w(v_D) + 1$ and d distinct vertices $v \in V(\Gamma_n)$ with weight $w(v) = w(v_D) - 1$. Note that for all $v_D \in D_T$ some of the dominated vertices may coincide. Now assume that

$$P_n(x, y) = \sum_{v \in V(\Gamma_n)} x^u y^d = \sum c_d^u x^u y^d. \tag{3.3}$$

For each pair (u, d) in $P_n(x, y)$ we associate an integer variable z_d^u which counts the number of vertices in D_T with weight d and degree $u + d$, that is, the number of vertices in D_T having d down neighbors and u up neighbors. Clearly, we have the bounds $0 \leq z_d^u \leq c_d^u$. Our aim is to minimize $|D_T|$, that is, our objective function is to minimize

$$\sum_{u,d} z_d^u.$$

Then by the above observation to dominate all the vertices having a fixed weight d such that $1 \leq d \leq \lceil \frac{n}{2} \rceil - 1$ we must have

$$r_d: \sum_u (u \cdot z_{d-1}^u + (d + 1) \cdot z_{d+1}^u) \geq \sum_u c_d^u$$

since any vertex with weight $d - 1$ having u up neighbors can dominate u distinct vertices with weight d and any vertex with weight $d + 1$ (all have $d + 1$ down neighbors) can dominate $d + 1$ distinct vertices with weight d . By the same argument, for $d = 0$ we must have

$$r_0: \sum_u z_1^u \geq \sum_u c_0^u = 1$$

and for $d = \lceil \frac{n}{2} \rceil$ we must have

$$r_{\lceil \frac{n}{2} \rceil}: \sum_u u \cdot z_{\lceil \frac{n}{2} \rceil - 1}^u \geq \sum_u c_{\lceil \frac{n}{2} \rceil}^u = \begin{cases} 1 & \text{if } n \text{ is odd,} \\ \frac{n}{2} + 1 & \text{if } n \text{ is even.} \end{cases}$$

Now subject to these constraints $r_0, \dots, r_{\lceil \frac{n}{2} \rceil}$ the value of the objective function will be a lower bound on $\gamma_t(\Gamma_n)$. Similarly, to find a lower bound on $\gamma(\Gamma_n)$ we need to modify all of the constraints $r_d, 0 \leq d \leq \lceil \frac{n}{2} \rceil$. By the definition of the dominating set, for each fixed d we need to add all of the variables z_d^u to the left side of the constraint r_d .

Remark 3.2. We remark that using [12, Theorem 4.6] we can easily obtain the coefficients c_d^u in (3.3). By the definition of the up-down degree enumerator polynomial we know that c_d^u is the number of vertices in Γ_n whose number of up neighbors is u and weight is d . That is, c_d^u equals to the number of vertices of Γ_n having degree $u + d$ and weight d . Therefore, [12, Theorem 4.6] gives

$$c_d^u = \binom{d + 1}{n - 2d - u + 1} \binom{n - 2d}{u}.$$

Remark 3.3. We know that the number of vertices of Γ_n with weight d is equal to the right hand side of the above constraints r_d . By definition of Γ_n this number is equal to the number of Fibonacci strings of length n and weight d . Therefore we have

$$\sum_u c_d^u = \binom{n-d+1}{d}.$$

Remark 3.4. To find the number of variables z_d^u we need to find the number of monomials in $P_n(x, y)$. Assume that n is even. Then by the structure of the vertices in Fibonacci cubes (it can also be seen from the structure of Fibonacci strings) $n - 3d \leq u \leq n - 2d$. Therefore the number of variables z_d^u becomes

$$\sum_{d=0}^{\lfloor \frac{n}{3} \rfloor} (d+1) + \sum_{d=\lfloor \frac{n}{3} \rfloor + 1}^{\frac{n}{2}} (n-2d+1)$$

which is equal to

$$s^2 - 2sr + \frac{3r(r+1)}{2} + 1$$

where $r = \lfloor \frac{n}{3} \rfloor$ and $s = n/2$. Similarly, if n is odd we obtain that the number of variables z_d^u is

$$s^2 - s(2r+1) + \frac{r(3r+5)}{2} + 2$$

where $r = \lfloor \frac{n}{3} \rfloor$ and $s = \lceil \frac{n}{2} \rceil$.

Now we illustrate our optimization problem for $n = 14$. We have the following polynomial by Theorem 3.1.

$$\begin{aligned} P_{14}(x, y) = & 8y^7 + \\ & 7y^6x^2 + 42y^6x + 35y^6 + \\ & 6y^5x^4 + 60y^5x^3 + 120y^5x^2 + 60y^5x + 6y^5 + \\ & 5y^4x^6 + 60y^4x^5 + 150y^4x^4 + 100y^4x^3 + 15y^4x^2 + \\ & 4y^3x^8 + 48y^3x^7 + 112y^3x^6 + 56y^3x^5 + \\ & 3y^2x^{10} + 30y^2x^9 + 45y^2x^8 + \\ & 2yx^{12} + 12yx^{11} + \\ & x^{14} \end{aligned}$$

and this polynomial corresponds to the following optimization problem:

Objective function:

$$\begin{aligned} \min: & z_0^{14} + z_1^{12} + z_1^{11} + z_2^{10} + z_2^9 + z_2^8 + z_3^8 + z_3^7 + z_3^6 + z_3^5 + \\ & z_4^6 + z_4^5 + z_4^4 + z_4^3 + z_4^2 + z_5^4 + z_5^3 + z_5^2 + z_5^1 + z_5^0 + z_6^2 + z_6^1 + z_6^0 + z_7^0 \end{aligned}$$

For illustration we implemented the above integer linear programming problem using CPLEX in NEOS Server [3, 4, 6] for $13 < n \leq 26$ and immediately obtain the lower bounds presented in Table 2 and Table 3. Note that for $n = 26$, the number of variables in our optimization problem is 70 by Remark 3.4 and it is $F_{28} = 317811$ for the general optimization problem (2.1). In addition, the upper bounds in these tables are obtained by Theorem 2.5 for $n \geq 14$ and Theorem 2.6 for $n \geq 16$. Note that the first bounds in both tables are obtained in [1] and the upper bounds on $\gamma_t(\Gamma_{14})$ and $\gamma_t(\Gamma_{15})$ comes from Theorem 2.4.

Table 2: Current best bounds on $\gamma(\Gamma_n)$, $12 \leq n \leq 26$.

n	$\gamma(\Gamma_n)$	n	$\gamma(\Gamma_n)$	n	$\gamma(\Gamma_n)$
12	54–61	17	344–648	22	3060–7189
13	78–93	18	528–1049	23	4748–11632
14	98–154	19	819–1697	24	7381–18821
15	148–247	20	1270–2746	25	11472–30453
16	224–401	21	1970–4443	26	17912–49274

Table 3: Current best bounds on $\gamma_t(\Gamma_n)$, $13 \leq n \leq 26$.

n	$\gamma(\Gamma_n)$	n	$\gamma(\Gamma_n)$	n	$\gamma(\Gamma_n)$
13	97–101	18	578–1049	23	5075–11632
14	110–166	19	890–1697	24	7865–18821
15	164–261	20	1374–2746	25	12191–30453
16	246–401	21	2121–4443	26	19033–49274
17	376–648	22	3281–7189		

Remark 3.5. For $n = 12$, Theorem 2.1 gives $\gamma(\Gamma_{12}) \geq 38$ and Theorem 2.2 gives $\gamma_t(\Gamma_{12}) \geq 40$. The values of the objective function in our optimization problems having 19 variables and 7 constraints give lower bounds $\gamma(\Gamma_{12}) \geq 44$ and $\gamma_t(\Gamma_{12}) \geq 50$.

For the case $n = 13$, Theorem 2.1 gives $\gamma(\Gamma_{13}) \geq 56$ and Theorem 2.2 gives $\gamma_t(\Gamma_{13}) \geq 59$. The values of the objective function in our optimization problems having 22 variables and 8 constraints give lower bounds $\gamma(\Gamma_{13}) \geq 65$ and $\gamma_t(\Gamma_{13}) \geq 75$.

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