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A note on *ADS** modules

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Abstract We study the ADS^* modules which are the dualizations of ADS modules studied by Alahmadi et al. (J Algebra 352:215–222, 2012). Mainly we prove that an amply supplemented module M is ADS^* if and only if M_1 and M_2 are mutually projective whenever $M = M_1 \oplus M_2$ if and only if for any direct summand S_1 and a submodule S_2 with $M = S_1 + S_2$, the epimorphism $\alpha_i : M \longrightarrow S_i/(S_1 \cap S_2)$ with $\mathrm{Ker}(\alpha_i) = S_j (i \neq j = 1, 2)$ can be lifted to an idempotent endomorphism β_i of M with $\beta_i(M) \subseteq S_i$.

Keywords Supplement submodule · Amply supplemented module · π-Projective modules · ADS^* module

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1 Introduction

Throughout every module will be a right module. All rings have identity and all modules are unital.

Fuchs [4], calls a right module M ADS if for every decomposition $M = S \oplus T$ of M and every complement T' of S we have $M = S \oplus T'$. In 2012, Alahmadi, Jain and Leroy [1] studied the class of ADS rings and modules. In this note we dualize the concept of ADS modules as ADS^*. A right module $M ADS^*$ if for every decomposition $M = S \oplus T$ of M and every supplement T' of S we have $M = S \oplus T'$. Note that for

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the submodules A and B of M, A is called a *supplement* of B in M if $M = A + B$ and $A \cap B$ is small in A. We will write $X \ll M$ if X is a small submodule of M. Any module M is called *amply supplemented* if B contains a supplement of A in M whenever $M = A + B$ and M is called *supplemented* if every submodule of M has a supplement in M. Thus every amply supplemented module is supplemented.

Let M be any module. M is called π*-projective* if whenever $M = A + B$ then there exists a homomorphism $f : M \longrightarrow M$ such that $f(M) \subseteq A$ and $(1 - f)(M) \subseteq B$. If any module M is π-projective, then it is ADS^* by [6, 41.14(2)]. Since every ring R is π-projective, R_R and ${}_RR$ are ADS^*. If any module M is both supplemented and π-projective then M is called *quasi-discrete*. Now we have that

$$\text{quasi-discrete} \Rightarrow \pi\text{-projective} \Rightarrow ADS^*.$$

Note that the $\mathbb{Z}$-module $\mathbb{Q}_{\mathbb{Z}}$ is ADS^* but it is not π-projective and so not quasi-discrete. On the other hand, if M is ADS^* and lifting, then it is quasi-discrete by [6, 41.15(b)]. Note that any module M is called *lifting* if it is amply supplemented and every supplement submodule of M is a direct summand. We should note that there is an amply supplemented ADS^* module which is not π-projective (equivalently not quasi-discrete). For example; let R be a local artinian ring with radical W such that $W^2 = 0$, $Q = R/W$ is commutative, $dim({}_QW) = 2$ and $dim(W_Q) = 1$ (see, [3]). Consider the indecomposable injective right R-module $U = [(R \oplus R)/D]_R$ with $D = \{(ur, -vr) \mid r \in R\}$ in [3, Proposition 2(r)], where $W = Ru + Rv$. Note that U is 2-generated, ADS^* and amply supplemented. On the other and, U cannot be lifting since it is not cyclic. Hence U is not quasi-discrete and so not π-projective.

In this note we provide equivalent conditions for a module to be ADS^*. An amply supplemented module M is ADS^* if and only if any decomposition of M into two modules, the summands are mutually projective (Theorem 2.1). We give an example showing that "amply supplemented" condition in Theorem 2.1 is necessary (Example 2.2). We call a module M *completely* ADS^* if each of its subfactors is ADS^*. Let $M = \oplus_{i \in I} M_i$ be an indecomposable decomposition of M. Suppose that M is amply supplemented completely ADS^*. Then

(i) For every $i \neq j \in I$, M_i is M_j-projective.
(ii) If $i \neq j \in I$ are such that $Hom(M_i, M_j) \neq 0$, then M_i is simple (Theorem 2.7). We state that "amply supplemented" condition in Theorem 2.7 is necessary. Finally, we prove that a right semiartinian ring R is a right V-ring if and only if every right R-module is ADS^* if and only if every 2-generated right R-module is ADS^* (Theorem 2.8).

For the undefined notions we refer to [2,5,6].

2 Results

Theorem 2.1 *For an amply supplemented module M the following are equivalent:*

(i) *M is ADS^*.*

(ii) *For any direct summand S_1 and a submodule S_2 with $M = S_1 + S_2$, the epimorphism $\alpha_i : M \longrightarrow S_i/(S_1 \cap S_2)$ with $Ker(\alpha_i) = S_j (i \neq j = 1, 2)$ can be lifted to an idempotent endomorphism β_i of M with $\beta_i(M) \subseteq S_i$.*

(iii) *If $M = A \oplus B$, then A and B are mutually projective.*

Proof (i) $\Rightarrow$ (ii): Let S_1 be a direct summand of M and S_2 any submodule of M with $M = S_1 + S_2$. Let T be a supplement of S_1 in M contained in S_2. By (i), $M = S_1 \oplus T$. Let $\alpha_1 : M \longrightarrow S_1/(S_1 \cap S_2)$ be the epimorphism with $\text{Ker}(\alpha_1) = S_2$ and $\beta_1 : M \longrightarrow S_1$ be the projection map while $\pi_1 : S_1 \longrightarrow S_1/(S_1 \cap S_2)$ is the natural epimorphism. Clearly, $\pi_1\beta_1 = \alpha_1$. Therefore β_1 lifts α_1. Similarly, the projection map $\beta_2 : M \longrightarrow T$ lifts the epimorphism $\alpha_2 : M \longrightarrow S_2/(S_1 \cap S_2)$ with $\text{Ker}(\alpha_2) = S_1$.

(ii) $\Rightarrow$ (i): Let $M = A \oplus B = A + C$ and $A \cap C \ll C$. We want to show that $M = A \oplus C$. By hypothesis, there exists an idempotent endomorphism $f : M \longrightarrow C$ such that $\pi_2 f = \alpha_2$ where $\pi_2 : C \longrightarrow C/(A \cap C)$ is the natural epimorphism and $\alpha_2 : M \longrightarrow C/(A \cap C)$ is the epimorphism with $\text{Ker}(\alpha_2) = A$. Note that $M = f(M) \oplus \text{Ker}(f) = C + \text{Ker}(f)$. Clearly, $\text{Ker}(f) \subseteq A$. Since $A \cap C \ll C$, $[(A\cap C)+\text{Ker}(f)]/\text{Ker}(f) \ll M/\text{Ker}(f)$. $\text{Ker}(f)+(A\cap C) = A\cap(\text{Ker}(f)+C) = A$ implies that $A/\text{Ker}(f) \ll M/\text{Ker}(f)$. But A is coclosed in M, hence $A = \text{Ker}(f)$. Therefore $M = f(M) \oplus A$. Since C is a supplement of A in M, $f(M) = C$. Thus $M = A \oplus C$.

(i) $\Rightarrow$ (iii): Suppose that $M = A \oplus B$ is ADS^*. We will prove that A is B-projective. Let C be a submodule of M with $M = C + B$. Since M is amply supplemented, there exists a submodule C' of M with $M = C' + B$, $C' \cap B \ll C' \subseteq C$. Since M is ADS^*, $M = C' \oplus B$. Hence A is B-projective by [6, 41.14 (3)$\Leftrightarrow$ (4)].

(iii) $\Rightarrow$ (i): Let $M = A\oplus B = B+C$ and $B\cap C \ll C$. Since A is B-projective, there exists a submodule C' of M contained in C such that $M = B \oplus C'$. But $B \cap C \ll C$. Then $C = C'$. Hence M is ADS^*. □

Note that there exists an ADS^* module which is not amply supplemented with a decomposition $M = A \oplus B$ such that A is not B-projective:

Example 2.2 Let R be a right V-domain which is not a division ring and S be any simple right R-module which is not R-projective. Put $M = R \oplus S$. Since R is a right V-ring, every right R-module has a zero radical so M is (completely) ADS^*. Note that M is not amply supplemented since R is not a division ring.

Proposition 2.3 *Let A and B two submodules of an amply supplemented ADS^* module M such that A is a direct summand of M and B is coclosed in M and $M = A + B$. Then $A \cap B$ is coclosed in M.*

Proof Let C be a supplement of A contained in B. By hypothesis, $M = A \oplus C$. Then $B = C \oplus (A \cap B)$. By [2, 3.7(6)], $A \cap B$ is coclosed in M. □

Lemma 2.4 *Let $M = B \oplus C$ be a decomposition of M with projections $\beta : M \longrightarrow B, \gamma : M \longrightarrow C$. Then $M = B \oplus C_1$ if and only if there exists $\theta : M \longrightarrow M$ such that $C_1 = (\gamma - \beta\theta\gamma)(M)$.*

Proof It is Lemma 3.6 in [1]. □

With the same notations as in Lemma 2.4 we give an interesting property of an ADS^* module.

Theorem 2.5 *A module M is ADS^* if and only if for any decomposition $M = B \oplus C$ the supplements of B in M are all of the form $(\gamma - \beta\theta\gamma)(M)$ for some $\theta : M \longrightarrow M$.*

Lemma 2.6 *Let M be an ADS^* module. Then any direct summand of M is ADS^*.*

Proof Let $M = A \oplus B$ be ADS^*. Let $A = A_1 \oplus A_2 = A_1 + K$ and $A_1 \cap K \ll K$. Then $M = A_1 \oplus (A_2 \oplus B) = A_1 + (K + B)$. $A_1 \cap (K + B) \subseteq K \cap (A_1 + B) = K \cap A_1 \ll K + B$ implies that $M = A_1 \oplus (K + B)$. Hence $A = A_1 \oplus (A \cap (K + B)) = A_1 \oplus K$. Thus A is ADS^*. □

Theorem 2.7 *Let $M = \oplus_{i \in I} M_i$ be an indecomposable decomposition of M. Suppose that M is amply supplemented completely ADS^*. Then*

(i) *For every $i \neq j \in I$, M_i is M_j-projective.*
(ii) *If $i \neq j \in I$ are such that $Hom(M_i, M_j) \neq 0$, then M_i is simple.*

Proof (i) Assume M is ADS^* (not completely). By Theorem 2.1 and Lemma 2.6 it is satisfied.

(ii) Assume that $i = 1$, $j = 2$ and let $\sigma : M_1 \longrightarrow M_2$ be nonzero. Since $M_1 \oplus \sigma(M_1) \oplus \cdots$ is ADS^*, $\sigma(M_1) \cong M_1/\mathrm{Ker}(\sigma)$ is M_1-projective. Thus $\mathrm{Ker}(\sigma)$ is a direct summand of M_1. But M_1 is indecomposable, hence $\mathrm{Ker}(\sigma) = 0$. Let $m_1 \in M_1$ be nonzero. Note that $M_1/m_1R \cong \sigma(M_1)/\sigma(m_1R)$. Thus

$$\begin{aligned}(M_1/m_1R) \oplus M_1 &\cong [\sigma(M_1)/\sigma(m_1R)] \oplus M_1 \subseteq [M_2/\sigma(m_1R)] \oplus M_1 \\ &\cong (M_2 \oplus M_1)/\sigma(m_1R) \subseteq M/\sigma(m_1R).\end{aligned}$$

Since M is completely ADS^*, M_1/m_1R is M_1-projective. Hence m_1R is a direct summand of M_1 and so $m_1R = M_1$. This means that M_1 is simple. □

Note that Theorem 2.7 is not correct if M is not amply supplemented: If we consider Example 2.2, then $Hom(R_R, S) \neq 0$ but R is not simple. Therefore Theorem 2.7(i) and (ii) are not satisfied.

Finally, we investigate that when a semiartinian ring is a right V-ring in terms of ADS^* modules:

Theorem 2.8 *Let R be a right semiartinian ring. Then the following are equivalent:*

(i) *R is a right V-ring.*
(ii) *Every right R-module is ADS^*.*
(iii) *Every 2-generated right R-module is ADS^*.*

Proof (i) ⇒ (ii) ⇒ (iii) are clear.

(iii) ⇒ (i): Let S be a simple right R-module, and E be the injective hull of S. Assume $S \neq E$. Then by semiartinian assumption, we can find a cyclic submodule T of E containing S as a maximal submodule. Put $X = T \oplus (T/S)$ and $A = \{(t, t + S) \mid t \in T\}$. Clearly, A is a supplement of $T \oplus 0$ in X but $A \cap (T \oplus 0) \neq 0$, contradicting the ADS^* assumption. So R is a right V-ring. □

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