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LONG-TIME BEHAVIOUR OF DOUBLY NONLINEAR PARABOLIC EQUATIONS

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Abstract. We consider a doubly nonlinear parabolic equation in R^n. Under suitable hypotheses we prove that a semigroup generated by this equation possesses a global attractor.

1. Introduction.

We are interested in the study of the long-time behaviour (in terms of attractors) of a doubly nonlinear parabolic equation of the form

$$\alpha(u_t) - \Delta u + \lambda u + f(u) = g \tag{1.1}$$

in R^n.

In the case when $\alpha(x) \equiv x$, the equation (1.1) becomes a reaction-diffusion equation, whose attractors in bounded domains were studied in [1], [9], [19] and references therein. For unbounded domains, there are technical difficulties coming from the lack of compact embeddings of Sobolev spaces. To overcome these difficulties, some authors, as in [2] and [3], used weighted Sobolev spaces, while some authors, as in [15] and [18], used the cut-off function technique introduced in [20]. In [7], using the weighted energy method the authors studied the global attractors for the reaction-diffusion equations with more general source terms in three dimensional unbounded domains. The weighted energy method presented in [7] is widely applicable and in present paper we use this method to prove the uniform tail estimate (see proof of Lemma 4.3).

The long-time behaviour of the solutions of (1.1) in the bounded domain when $\alpha(\cdot)$ is sub-linear was studied in [17]. In the case that $\alpha(v)$ is like $|v|^p v$, the existence of a global attractor for (1.1) in a three dimensional bounded domain was established in [8] assuming that the force term g is a bounded function. As mentioned in that article, when the nonlinearity $\alpha(\cdot)$ grows sufficiently fast at infinity, unlike the case of usual reaction-diffusion equations, there is a principal difference between weak and strong solutions of doubly nonlinear equations of the form (1.1). Namely, in contrast to strong solutions, weak solutions may contain so-called "pathological" solutions which do not possess any smoothing properties for $t > 0$. In [8], the global attractors were studied for the solutions which are not "pathological". Recently, in [16], the long-time behaviour of the solutions of equation (1.1) with the bounded force term was studied in a three dimensional bounded domain. In that article also, the existence of the attractors was established for the strong solutions.

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We also note that there are several articles, such as [5], [6], [13], [14] devoted to the study of global attractors of doubly nonlinear parabolic equations of the form

$$\frac{\partial}{\partial t}\alpha(u) - \Delta u + f(u) = g.$$

In this paper, we study the long-time behaviour of the weak solutions of (1.1) in the whole space. The paper is organized as follows: In the next section we state our main result, in Section 3 we prove the well-posedness of the problem, in Section 4 we establish the asymptotic compactness property of solutions and then prove the existence of a global $(H^1(R^n), H^1(R^n))_{\mathfrak{B}}-$ attractor for the equation (1.1), and finally the proofs of some auxiliary lemmas are given in Appendix.

2. **Statement of the problem and main result.**

We consider the following Cauchy problem:

$$\begin{cases} \alpha(u_t) - \Delta u + \lambda u + f(u) = g(x), & (t,x) \in (0,\infty) \times R^n, \\ u(0,x) = u_0(x), & x \in R^n, \end{cases} \tag{2.1}$$

where $\lambda > 0$, $g \in L^2(R^n)$ and the nonlinear functions α, f satisfy the following conditions:

Assumption 2.1.

- $\alpha \in C^1(R), \qquad \alpha(0) = 0, \qquad \alpha$ is odd function, (2.2)
- $\alpha'(0) > 0$, $\alpha'(\cdot)$ is nondecreasing function on R_+, $\limsup\limits_{x\to\infty}\dfrac{\alpha(2x)}{\alpha(x)} < \infty$, (2.3)
- $f \in C^2(R)$, $\liminf\limits_{|v|\to\infty} f'(v) > 0$, $f(v)v \geq 0$, $|\, f''(v)| \leq c$ for every $v \in R$, (2.4)
- $|\, f'(v)| \leq c(1 + |v|^p)$ for every $v \in R$, where $0 \leq p \leq \min\{1, \dfrac{2}{(n-2)^+}\}$. (2.5)

Now to define a global $(H^1(R^n), H^1(R^n))_{\mathfrak{B}}-$attractor let us introduce the following family of sets:

$$\mathfrak{B} = \Big\{ B : B \text{ is a bounded subset of } H^1(R^n) \text{ and for any } \varepsilon > 0, \text{ there exists}$$

$$m = m(\varepsilon, B) > 0 \text{ such that } \sup_{u\in B} \int\limits_{\{x: x\in R^n,\ |u(x)|>m\}} |\nabla u(x)|^2\, dx \leq \varepsilon \Big\}.$$

Definition 2.1. Let $\{S(t)\}_{t\geq 0}$ be an operator semigroup on $H^1(R^n)$. We say that a set $\mathcal{A} \in \mathfrak{B}$ is a global $(H^1(R^n), H^1(R^n))_{\mathfrak{B}}-$attractor for the semigroup $\{S(t)\}_{t\geq 0}$ iff

- $\mathcal{A}$ is compact in $H^1(R^n)$;
- $\mathcal{A}$ is invariant, i.e. $S(t)\mathcal{A} = \mathcal{A}, \forall\, t \geq 0$;
- $\lim\limits_{t\to\infty} \sup\limits_{v\in B} \inf\limits_{u\in\mathcal{A}} \|S(t)v - u\|_{H^1(R^n)} = 0$ for each $B \in \mathfrak{B}$;

Our main result is:

Theorem 2.1. *Under Assumption 2.1, a semigroup generated by the problem (2.1) possesses a global $(H^1(R^n), H^1(R^n))_{\mathfrak{B}}-$attractor.*

Remark 2.1. By the definition it follows that a global $(H^1(R^n), H^1(R^n))_{\mathfrak{B}}$-attractor is maximal as an invariant set belonging to $\mathfrak{B}$ and minimal as a closed attractor attracting every element of $\mathfrak{B}$. Since every bounded subset of $H^1(R^n)\cap L^\infty(R^n)$ and $W^{1,\,2+\varepsilon}(R^n)$ belongs to $\mathfrak{B}$, a global $(H^1(R^n), H^1(R^n))_{\mathfrak{B}}-$attractor attracts each bounded subset of $H^1(R^n)\cap L^\infty(R^n)$ and $W^{1,\,2+\varepsilon}(R^n)$ in the topology of $H^1(R^n)$, where $\varepsilon > 0$.

Remark 2.2. We also note that Theorem 2.1 remains true if we assume

$$f \in C^1(R), \quad f(v)v \geq -\sigma \text{ for every } v \in R,$$

$$\liminf_{|v|\to\infty} f'(v) > -\lambda, \text{ and } f'(\cdot) \text{ satisfies the global Lipschitz condition,}$$

instead of (2.4), where $\sigma \in (0, \lambda)$.

3. **Well-posedness.**

Let us consider the following initial-boundary value problem:

$$\begin{cases} \alpha(v_t) - \Delta v + \lambda v + f(w) - f(w-v) = \widehat{g}(x), & (t,x) \in (0,T)\times B_\rho, \\ v(t,x) = 0, \quad (t,x) \in (0,T)\times \partial B_\rho \\ v(0,x) = v_0(x), \quad x \in B_\rho, \end{cases} \tag{3.1}$$

where $B_\rho = \{x : x \in R^n,\, |x| < \rho\}$.

To prove well-posedness of (2.1) we will use the following lemma:

Lemma 3.1. *Let Assumption 2.1 hold. Also assume that $w \in L^2(0,T; H^2(B_\rho) \cap H_0^1(B_\rho))$, $w_t \in L^2(0,T; H_0^1(B_\rho))$ and $\widehat{g} \in L^2(B_\rho)$. Then for every $v_0 \in H^2(B_\rho)\cap H_0^1(B_\rho)$ there exists a unique strong solution $v(t,x)$ of (3.1), that is $v \in W^{1,2}(0,T; H_0^1(B_\rho))\cap W^{2,2}_{loc}(0,T; L^2(B_\rho))\cap L^\infty(0,T; H^2(B_\rho))$ satisfies $(3.1)_1$ a.e. on $(0,T)\times B_\rho$ and $(3.1)_3$ a.e. on B_ρ.*

Proof. Uniqueness. Let $v^{(i)}(t,x) \in W^{1,2}(0,T; H_0^1(B_\rho)) \cap W^{2,2}_{loc}(0,T; L^2(B_\rho))\cap$ $\cap L^\infty(0,\infty; H^2(B_\rho))(i = 1,2)$ be solutions of (3.1). Then multiplying both sides of

$$\alpha(v_t^{(1)}) - \alpha(v_t^{(2)}) - \Delta(v^{(1)} - v^{(2)}) + \lambda(v^{(1)} - v^{(2)}) = f(w - v^{(1)}) - f(w - v^{(2)})$$

by $2(v_t^{(1)} - v_t^{(2)})$ and integrating over $(0,t)\times B_\rho$ we have

$$\left\|\nabla(v^{(1)}(t) - v^{(2)}(t))\right\|^2_{L^2(B_\rho)} + \lambda\left\|v^{(1)}(t) - v^{(2)}(t)\right\|^2_{L^2(B_\rho)}$$

$$+2\int_0^t\int_{B_\rho}(\alpha(v_t^{(1)}(s,x)) - \alpha(v_t^{(2)}(s,x)))(v_t^{(1)}(s,x) - v_t^{(2)}(s,x))dxds$$

$$= 2\int_0^t\int_{B_\rho}(f(w(s,x) - v^{(1)}(s,x)) - f(w(s,x) - v^{(2)}(s,x)))$$

$$\times(v_t^{(1)}(s,x) - v_t^{(2)}(s,x))dxds$$

and consequently

$$\left\|v^{(1)}(t) - v^{(2)}(t)\right\|^2_{H^1(B_\rho)} \leq C\int_0^t \left\|v^{(1)}(s) - v^{(2)}(s)\right\|^2_{H^1(B_\rho)} ds, \qquad \forall\, t \in [0,T].$$

Applying Gronwall's lemma to the last inequality we find $v^{(1)} \equiv v^{(2)}$.

Existence. Let $\{\varphi_i\}_{i=1}^{\infty}$ be eigenfunctions of $-\Delta$ in $H_0^1(B_\rho)$, i.e.

$$\begin{cases} -\Delta\varphi_i = \mu_i\varphi_i, \text{ in } B_\rho, \\ \varphi_i\mid_{B_\rho}= 0, \end{cases}, \quad i = 1, 2, \dots .$$

By standard elliptic theory we have $\varphi_i \in C^\infty(\overline{B_\rho})$, $i = 1, 2, \dots$. Set $v^m(t) = \sum_{j=1}^m a_{mj}(t)\varphi_j$ and consider the following system of ordinary differential equations:

$$\frac{1}{m}\frac{d^2}{dt^2}\langle v^m(t), \varphi_j\rangle + \langle\nabla v^m(t), \nabla\varphi_j\rangle + \left\langle \alpha(\frac{d}{dt}v^m(t)), \varphi_j \right\rangle$$
$$+ \lambda\langle v^m(t), \varphi_j\rangle + \langle f(w(t)) - f(w(t) - v^m(t)), \varphi_j\rangle = \langle g, \varphi_j\rangle, \quad j = \overline{1, m} \tag{3.2}$$

with initial conditions

$$v^m(0) = \sum_{j=1}^m b_{mj}\varphi_j, \qquad \frac{d}{dt}\, v^m(0) = 0, \tag{3.3}$$

where $\langle\psi, \varphi\rangle = \int\limits_{B_\rho}\psi(x)\varphi(x)dx$ and $\sum\limits_{j=1}^m b_{mj}\varphi_j \to v_0$ strongly in $H^2(B_\rho) \cap H_0^1(B_\rho)$ as $m \to \infty$. Existence theory of ordinary differential equations implies that there exists a solution of (3.2)-(3.3) on $[0, T_m)$. Multiplying both sides of (3.2) by $2\frac{d}{dt}a_{mj}(t)$, summing from 1 to m and integrating over $[0, t] \subset [0, T_m)$ we obtain

$$\frac{1}{m}\|v_t^m(t)\|^2_{L^2(B_\rho)} + \|\nabla v^m(t)\|^2_{L^2(B_\rho)} + \lambda\|v^m(t)\|^2_{L^2(B_\rho)}$$
$$+2\int_0^t\int_{B_\rho}\alpha(v_t^m(s, x))v_t^m(s, x)ds - 2\langle g, v^m(0)\rangle$$
$$+2\int_0^t\langle f(w(s)) - f(w(s) - v^m(s)), v_t^m(s)\rangle\, ds$$
$$= \|\nabla v^m(0)\|^2_{L^2(B_\rho)} + \lambda\|v^m(0)\|^2_{L^2(B_\rho)} + 2\langle g, v^m(t)\rangle, \qquad 0 \le t < T_m. \tag{3.4}$$

By condition (2.4)-(2.5) we have

$$\int_0^t\langle f(w(s)) - f(w(s) - v^m(s)), v_t^m(s)\rangle\, ds = \int_0^t\langle f(w(s)), v_t^m(s)\rangle\, ds$$
$$+\int_0^t\langle f(w(s) - v^m(s)), w_t(s) - v_t^m(s)\rangle\, ds - \int_0^t\langle f(w(s) - v^m(s)), w_t(s)\rangle\, ds$$
$$\ge -c\int_0^t\int_{B_\rho}(1 + |w(s, x)|^p)\,|w(s, x)|\,|v_t^m(s, x)|\, dxds$$
$$+\int_{B_\rho}F(w(t, x) - v^m(t, x))dx - \int_{B_\rho}F(w(0, x) - v^m(0, x))dx$$
$$-c\int_0^t\int_{B_\rho}(1 + |w(s, x)|^p + |v^m(s, x)|^p)(|w(s, x)| + |v^m(s, x)|)\,|w_t(s, x)|\, dxds, \tag{3.5}$$

where $F(u) = \int_0^u f(v)dv$. Taking into account (3.5) in (3.4) we find

$$\frac{1}{m}\|v_t^m(t)\|^2_{L^2(B_\rho)} + \|\nabla v^m(t)\|^2_{L^2(B_\rho)} + \lambda\|v^m(t)\|^2_{L^2(B_\rho)}$$
$$+\int_0^t\int_{B_\rho}\alpha(v_t^m(s,x))v_t^m(s,x)dxds$$
$$\le c_1(T,\|w\|_{C([0,T];H^1(B_\rho))})\|w\|_{C(0,T;H^1(B_\rho))} + c_2(\|v_0\|_{H^1(B_\rho)},\|g\|_{L^2(B_\rho)})$$
$$+c_3(T,\|w\|_{C([0,T];H^1(B_\rho))})\|w_t\|_{L^2(0,T;H^1(B_\rho))}\times(1+\int_0^t\|v^m\|^4_{H^1(B_\rho))})^{\frac{1}{2}}, \quad 0\le t<T_m,$$

and consequently

$$\|v^m(t)\|^4_{H^1(R^n))} \le c_4(T,\|w\|_{C([0,T];H^1(B_\rho))})(\|w\|^2_{C(0,T;H^1(B_\rho))} + \|w_t\|^2_{L^2(0,T;H^1(B_\rho))})$$
$$\times\left(1+\int_0^t\|v^m\|^4_{H^1(B_\rho))}\right) + 2c_2^2(\|v_0\|_{H^1(B_\rho)},\|g\|_{L^2(B_\rho)}),\ 0\le t<T_m,$$

where $c_i : R_+\times R_+\to R_+$ $(i=\overline{1,\ 4})$ are nondecreasing functions with respect to each variable. Applying Gronwall's lemma we obtain

$$\|v^m(t)\|_{H^1(B_\rho))} \le 1 + c_2(\|v_0\|_{H^1(B_\rho)},\|g\|_{L^2(B_\rho)})$$
$$+c_5(T,\|w\|_{C([0,T];H^1(B_\rho))},\|w_t\|_{L^2(0,T;H^1(B_\rho))}),\ 0\le t<T_m,$$

where $c_5 : R_+\times R_+\times R_+\to R_+$ is a nondecreasing function with respect to each variable and $c_5(\cdot,0,0)=0$. Hence $v^m(t,\cdot)$ can be extended to an interval $[0,T]$ and

$$\frac{1}{m}\|v_t^m(t)\|^2_{L^2(B_\rho)} + \|\nabla v^m(t)\|^2_{L^2(B_\rho)} + \lambda\|v^m(t)\|^2_{L^2(B_\rho)} + \int_0^t\int_{B_\rho}\mathcal{N}(\alpha(v_t^m(s,x)))dxds$$
$$\le c_6(\|v_0\|_{H^1(B_\rho)},\|g\|_{L^2(B_\rho)}) + c_7(T,\|w\|_{C([0,T];H^1(B_\rho))},\|w_t\|_{L^2(0,T;H^1(B_\rho))}),$$
$$0\le t\le T, \tag{3.6}$$

where $\mathcal{N}(x) = \int_0^x \alpha^{-1}(y)dy$ and $c_6 : R_+\times R_+\to R_+$, $c_7 : R_+\times R_+\times R_+\to R_+$ are nondecreasing functions with respect to each variable and $c_7(\cdot,0,0)=0$.

Multiplying both sides of (3.2) by $2\mu_j\frac{d}{dt}a_{mj}(t)$, summing from 1 to m and integrating over $[0,t]$ we obtain

$$\frac{1}{m}\|\nabla v_t^m(t)\|^2_{L^2(B_\rho)} + \|\Delta v^m(t)\|^2_{L^2(B_\rho)} + \lambda\|v^m(t)\|^2_{L^2(B_\rho)}$$
$$+2\alpha'(0)\int_0^t\|\nabla v_t^m(s)\|^2_{L^2(B_\rho)}ds + 2\int_0^t\langle f(w(s)-v^m(s))-f(w(s)),\Delta v_t^m(s)\rangle ds$$
$$\le\|\Delta v^m(0)\|^2_{L^2(B_\rho)} + \lambda\|v^m(0)\|^2_{L^2(B_\rho)} - 2\langle g,\Delta v^m(t)\rangle + 2\langle g,\Delta v^m(0)\rangle,$$
$$0\le t\le T. \tag{3.7}$$

By condition (2.4)-(2.5) we find

$$\int_0^t \langle f(w(s)-v^m(s))-f(w(s)),\Delta v_t^m(s)\rangle\, ds$$

$$=\int_0^t \langle f'(w(s))\nabla w(s),\nabla v_t^m(s)\rangle\, ds$$

$$-\int_0^t \langle f'(w(s)-v^m(s))\nabla (w(s)-v^m(s)),\nabla v_t^m(s)\rangle\, ds$$

$$\geq -c\int_0^t\int_{B_\rho}(1+|w(s,x)|^p)\,|\nabla w(s,x)|\,|\nabla v_t^m(s,x)|\,dxds$$

$$-c\int_0^t\int_{B_\rho}(1+|w(s,x)|^p+|v^m(s,x)|^p)\,|\nabla v^m(s,x)|\,|\nabla v_t^m(s,x)|\,dxds$$

$$-c\int_0^t\int_{B_\rho}(1+|w(s,x)|^p+|v^m(s,x)|^p)\,|\nabla w(s,x)|\,|\nabla v_t^m(s,x)|\,dxds. \tag{3.8}$$

Taking into account (3.6) and (3.8) in (3.7) we obtain

$$\frac{1}{m}\left\|v_t^m(t)\right\|^2_{H^1(B_\rho)}+\left\|v^m(t)\right\|^2_{H^2(B_\rho)}+\int_0^t \left\|\nabla v_t^m(s)\right\|^2_{L^2(B_\rho)} ds$$

$$\leq c_8(\left\|v_0\right\|_{H^2(B_\rho)},\left\|g\right\|_{L^2(B_\rho)})+c_9(T,\left\|w\right\|_{L^2([0,T];H^2(B_\rho))},\left\|w_t\right\|_{L^2(0,T;H^1(B_\rho))}),$$

$$0\leq t\leq T, \tag{3.9}$$

where $c_8: R_+\times R_+\to R_+$, $c_9: R_+\times R_+\times R_+\to R_+$ are nondecreasing functions with respect to each variable and $c_9(\cdot,0,0)=0$.

Now multiplying both sides of (3.2) by $\frac{d^2}{dt^2}a_{mj}(t)$, summing from 1 to m and integrating over $[0,t]$ we find

$$\frac{1}{m}\int_0^t \left\|v_{tt}^m(s)\right\|^2_{L^2(B_\rho)} ds+\int_{B_\rho}\widehat{\alpha}(v_t^m(t,x))dx-\langle \Delta v^m(t),v_t^m(t)\rangle+\lambda\langle v^m(t),v_t^m(t)\rangle$$

$$=\langle g,v_t^m(t)\rangle+\int_0^t \left\|\nabla v_t^m(s)\right\|^2_{L^2(B_\rho)} ds+\lambda\int_0^t \left\|v_t^m(s)\right\|^2_{L^2(B_\rho)} ds$$

$$-\langle f(w(t))-f(w(t)-v^m(t)),v_t^m(t)\rangle+\int_0^t\langle f'(w(s))w_t(s),v_t^m(s)\rangle\, ds$$

$$-\int_0^t\langle f'(w(s)-v^m(s))(w_t(s)-v_t^m(s)),v_t^m(s)\rangle\, ds,\qquad 0\leq t\leq T, \tag{3.10}$$

where $\widehat{\alpha}(v) = \int_0^v \alpha(v)dv$.

Differentiating both sides of (3.2), multiplying by $\frac{d^2}{dt^2}a_{mj}(t)$, summing from 1 to m and integrating over $[s,t]$ we have

$$\frac{1}{2m}\left\|v_{tt}^m(t)\right\|_{L^2(B_\rho)}^2 - \frac{1}{2m}\left\|v_{tt}^m(s)\right\|_{L^2(B_\rho)}^2 + \int_s^t\int_{B_\rho}\alpha'(v_t^m(\tau,x))\left|v_{tt}^m(\tau,x)\right|^2 dxd\tau$$
$$+\frac{1}{2}\left\|\nabla v_t^m(t)\right\|_{L^2(B_\rho)}^2 - \frac{1}{2}\left\|\nabla v_t^m(s)\right\|_{L^2(B_\rho)}^2 + \frac{\lambda}{2}\left\|v_t^m(t)\right\|_{L^2(B_\rho)}^2$$
$$-\frac{\lambda}{2}\left\|v_t^m(s)\right\|_{L^2(B_\rho)}^2 + \int_s^t \left\langle f'(w(\tau))w_t(\tau), v_{tt}^m(\tau)\right\rangle d\tau$$
$$-\int_s^t \left\langle f'(w(\tau)-v^m(\tau))(w_t(\tau)-v_t^m(\tau)), v_{tt}^m(\tau)\right\rangle d\tau = 0, \quad 0 < s < t \le T.$$

Integrating the last equality with respect to s from 0 to t we find

$$\frac{1}{2m}t\left\|v_{tt}^m(t)\right\|_{L^2(B_\rho)}^2 - \frac{1}{2m}\int_0^t \left\|v_{tt}^m(s)\right\|_{L^2(B_\rho)}^2 ds$$
$$+\int_0^t\int_s^t\int_{B_\rho}\alpha'(v_t^m(\tau,x))\left|v_{tt}^m(\tau,x)\right|^2 dxd\tau ds + \frac{1}{2}t\left\|\nabla v_t^m(t)\right\|_{L^2(B_\rho)}^2$$
$$-\frac{1}{2}\int_0^t \left\|\nabla v_t^m(s)\right\|_{L^2(B_\rho)}^2 ds + \frac{\lambda}{2}t\left\|v_t^m(t)\right\|_{L^2(B_\rho)}^2$$
$$-\frac{\lambda}{2}\int_0^t \left\|v_t^m(s)\right\|_{L^2(B_\rho)}^2 ds + \int_0^t\int_s^t \left\langle f'(w(\tau))w_t(\tau), v_{tt}^m(\tau)\right\rangle d\tau ds$$
$$-\int_0^t\int_s^t \left\langle f'(w(\tau)-v^m(\tau))(w_t(\tau)-v_t^m(\tau)), v_{tt}^m(\tau)\right\rangle d\tau ds = 0, \quad 0 \le t \le T. \quad (3.11)$$

By (2.3), (2.4), (2.5), (3.9), (3.10) and (3.11) we have

$$\int_t^T \left\|v_{tt}^m(s)\right\|_{L^2(B_\rho)}^2 ds \le \frac{1+T}{t}c_{10}(\left\|v_0\right\|_{H^2(B_\rho)}, \left\|g\right\|_{L^2(B_\rho)})$$
$$+\frac{1}{t}c_{11}(T, \left\|w\right\|_{L^2([0,T];H^2(B_\rho))}, \left\|w_t\right\|_{L^2(0,T;H^1(B_\rho))}), \quad 0 \le t \le T, \quad (3.12)$$

where $c_{10} : R_+ \times R_+ \to R_+$, $c_{11} : R_+ \times R_+ \times R_+ \to R_+$ are nondecreasing functions with respect to each variable and $c_{11}(\cdot, 0, 0) = 0$. Taking into account (3.6), (3.9), (3.12) and applying [11, Theorem 14.4, p. 131] we can say that there exists a subsequence $\{m_k\}$ such that

$$\begin{cases} v^{m_k} \to v \quad \text{weakly star in } L^\infty(0,T;H^2(B_\rho)\cap H_0^1(B_\rho)) \\ v_t^{m_k} \to v_t \quad \text{weakly in } L^2(0,T;H_0^1(B_\rho)) \\ v_{tt}^{m_k} \to v_{tt} \quad \text{weakly in } L^2_{loc}(0,T;L^2(B_\rho)) \\ \int_0^T\int_{B_\rho} \alpha(v_t^m)\psi dxds \to \int_0^T\int_{B_\rho} \alpha(v_t)\psi dxds, \quad \forall\, \psi\in L^\infty((0,T)\times B_\rho) \end{cases} \tag{3.13}$$

Now taking into account (3.13) and passing to limit in (3.2)-(3.3) we obtain

$$\begin{aligned} &\langle \alpha(v_t(t)),\varphi_j\rangle - \langle \Delta v(t),\varphi_j\rangle + \lambda\langle v(t),\varphi_j\rangle + \langle f(w(t)),\varphi_j\rangle \\ &- \langle f(w(t)-v(t)),\varphi_j\rangle = \langle g,\varphi_j\rangle, \quad \text{a.e. on } (0,T), \quad j=1,2... \end{aligned}$$

and

$$v(0)=v_0$$

from which we find that $\alpha(v_t)\in L^\infty(0,T;L^2(B_\rho))$ and $v\in W^{1,2}(0,T;H_0^1(B_\rho))\cap W^{2,2}_{loc}(0,T;L^2(R^n))\cap L^\infty(0,T;H^2(B_\rho))$ satisfies (3.1). □

Now let us prove the existence and uniqueness of the strong solution of (2.1).

Theorem 3.1. *Let Assumption (2.1) hold and $u_0\in H^2(R^n)$. Then for every $T>0$, the problem (2.1) has a unique strong solution $u(t,x)$ on $[0,T[\times R^n$, that is $u\in W^{1,2}(0,\,T;\,L^2(R^n))\cap W^{2,2}_{loc}(0,T;L^2(R^n))\cap L^\infty(0,\,T;\,H^2(R^n))$ satisfies $(2.1)_1$ a.e. on $(0,T)\times R^n$ and $(2.1)_2$ a.e. on R^n.*

Proof. Since proof of the uniqueness is trivial we prove the existence of the strong solution. Since the function $-f(-x)$ satisfies conditions (2.4)-(2.5) choosing $w(t,x)\equiv 0$, taking $-f(-x)$ instead of $f(x)$ and applying Lemma 3.1 we obtain that there exists a function $u_m\in W^{1,2}(0,\,T;\,H_0^1(B_m))\cap W^{2,2}_{loc}(0,T;L^2(B_m))\cap L^\infty(0,\,T;\,H^2(B_m)\cap H_0^1(B_m))$ which satisfies $(2.1)_1$ a.e. on $(0,T)\times B_m$ and $(2.1)_2$ a.e. on B_m. Also by (3.9), (3.11) and (3.12) we have

$$\begin{aligned} &\|\alpha(u_{mt}(t))\|^2_{L^2(B_m)} + \|u_m(t)\|^2_{H^2(B_m)} \\ &+\frac{\tau}{1+t}\int_\tau^t \|u_{mtt}(s)\|^2_{L^2(B_m)}\,ds + \int_0^t \|\nabla u_{mt}(s)\|^2_{L^2(B_m)}\,ds \\ &\le c(\|u_0\|_{H^2(R^n)},\|g\|_{L^2(R^n)}), \quad 0<\tau\le t\le T, \quad \forall\, m\in\mathbb{N}, \end{aligned} \tag{3.14}$$

where $c: R_+\times R_+\to R_+$ is a nondecreasing function with respect to each variable. Setting $\widetilde{u}_m(t,x)=\begin{cases} u_m(t,x), & x\in B_m \\ 0, & x\in R^n\backslash B_m \end{cases}$ by (3.14) we can say that there exists a subsequence $\{m_k\}\subset\{m\}$ such that

$$\begin{cases} \widetilde{u}_{m_k}\to u \quad \text{weakly star in } L^\infty(0,T;H^1(R^n)) \\ u_{m_k}\to u \quad \text{weakly star in } L^\infty(0,T;H^2(B_\rho)) \\ u_{m_kt}\to u_t \text{ weakly in } L^2(0,T;H^1(B_\rho)) \\ u_{m_ktt}\to u_{tt} \text{ weakly in } L^2_{loc}(0,T;L^2(B_\rho)) \\ \alpha(u_{m_kt})\to\alpha(u_t) \quad \text{weakly in } L^2(0,T;L^2(B_\rho)) \end{cases}$$

and consequently

$$\|\alpha(u_t(t))\|^2_{L^2(B_\rho)} + \|u(t)\|^2_{H^2(B_\rho)} + \frac{\tau}{1+t}\int_\tau^t \|u_{tt}(s)\|^2_{L^2(B_\rho)}\,ds$$

$$+\int_0^t \|\nabla u_t(s)\|^2_{L^2(B_\rho)}\, ds \le c(\|u_0\|_{H^2(R^n)}, \|g\|_{L^2(R^n)}), \quad 0 < \tau \le t \le T, \quad \forall \rho > 0.$$

Hence $u(t,x)$ is the strong solution of (2.1). □

Now let us define a weak solution.

Definition 3.1. A function $u \in C([0,T]; H^1(R^n))$ possessing the property $u(0,\cdot) = u_0$ is said to be a weak solution to problem (2.1) on $[0,\ T[\times R^n$, iff there exists a sequence of strong solutions $\{u^m(t,x)\}$ to problem (2.1) with initial data u_0^m instead of u_0 such that

$$\lim_{n\to\infty} \|u - u^n\|_{C([0,T];\ H^1(R^n))} = 0.$$

Remark 3.1. It is easy to see that, for sub-linear $\alpha(\cdot)$ and non-decreasing $f(\cdot)$, the weak solution defined here coincides with the solution studied in [4].

Using Theorem 3.1 and also density argument we have the following existence theorem:

Theorem 3.2. *Let Assumption 2.1 hold. Then for every $T > 0$ and $u_0 \in H^1(R^n)$, the problem (2.1) has the unique weak solution $u(t,x)$ on $[0,T[\times R^n$, which satisfies the following inequality*

$$E(u(t)) + \int_{R^n} F(u(t,x))dx - \int_{R^n} g(x)u(t,x)dx + \int_\tau^t \int_{R^n} \alpha(u_t(t,x))u_t(t,x)dxdt$$

$$\le\ E(u(\tau)) + \int_{R^n} F(u(\tau,x))dx - \int_{R^n} g(x)u(\tau,x)dx, \qquad 0 \le \tau \le t \le T. \tag{3.15}$$

Moreover if $v(t,x)$ is a weak solution to (2.1) on $[0,T] \times R^n$ with initial data v_0 and $\max\left\{\|u_0\|_{H^1(R^n)}, \|v_0\|_{H^1(R^n)}\right\} \le R$, then there exists $c = c(T,R) > 0$ such that

$$E(u(t) - v(t)) \le cE(u_0 - v_0), \qquad \forall\, t \in [0,T],$$

where $E(u) = \frac{1}{2}(\|\nabla u(t)\|^2_{L^2(R^n)} + \lambda \|u(t)\|^2_{L^2(R^n)})$.

Thus, under Assumption 2.1, problem (2.1) generates a continuous semigroup $\{S(t)\}_{t\ge 0}$ in $H^1(R^n)$ by the formula $S(t)u_0 = u(t,)$, where $u(t,x)$ is a weak solution with initial data u_0.

4. **Asymptotic compactness and global attractors.** Let $u(t,x)$ be a solution of (2.1). We decompose $u(t,x)$ as a sum $v(t,x) + w(t,x)$, where

$$\begin{cases} \alpha(v_t) - \Delta v + \lambda v + f(u) - f(u-v) = g_0(x),\ (t,x) \in (0,\infty) \times R^n, \\ v(0,x) = 0, \quad x \in R^n, \end{cases} \tag{4.1}$$

$$\begin{cases} \alpha(v_t + w_t) - \alpha(v_t) - \Delta w + \lambda w + f(w) \\ = g(x) - g_0(x), \quad (t,x) \in (0,\infty) \times R^n, w(0,x) = u_0, \qquad x \in R^n, \end{cases} \tag{4.2}$$

and $g_0 \in L^2(R^n) \cap L^\infty(R^n)$.

To prove the asymptotic compactness of the solutions of (2.1) we will prove the compactness of the solutions of (4.1) in $H^1(R^n)$ (for fixed t and g_0) and then show that the solutions of (4.2) are sufficiently small in the norm of $H^1(R^n)$ for large t and for $g_0 \in L^2(R^n) \cap L^\infty(R^n)$ which is sufficiently close to g in $L^2(R^n)$.

Let us first prove the regularity of the solutions of (4.1). For this we will use the following maximum principle:

Lemma 4.1. *Let Assumption 2.1 hold. Also assume that* $w \in L^2(0,T;H^2(B_\rho)\cap H_0^1(B_\rho))$, $w_t \in L^2(0,T;H_0^1(B_\rho))$, $\widehat{g} \in L^\infty(B_\rho)$ *and* $v_0 = 0$. *Then the strong solution* $v(t,x)$ *of (3.1) satisfies the following inequality*

$$\|v\|_{L^\infty((0,T)\times B_\rho)} \leq \frac{\mu_0}{\lambda} + \frac{1}{\lambda}\|\widehat{g}\|_{L^\infty(B_\rho)}, \tag{4.3}$$

where the positive constant μ_0 *depends only on* $f(\cdot)$.

Proof. For the proof, see Appendix. □

Now using Lemma 4.1 let us prove the following lemma:

Lemma 4.2. *Assume that Assumption 2.1 holds . Then for every* $u_0 \in H^2(R^n)$ *and* $T > 0$ *there exists a unique strong solution* $v \in W^{1,2}(0,T;H^1(R^n))\cap$ $\cap W^{2,2}_{loc}(0,T;L^2(R^n)) \cap L^\infty(0,T;H^2(R^n))$ *of (4.1) on* $[0,T[\times R^n$ *such that*

$$\|v(t)\|_{H^2(R^n)} \leq c(\|u_0\|_{H^1(R^n)}, \|g\|_{L^2(R^n)}, \|g_0\|_{L^2(R^n)\cap L^\infty(R^n)}), \quad \forall\, t \geq 0, \tag{4.4}$$

where $c : R_+ \times R_+ \times R_+ \to R_+$ *is a nondecreasing function with respect to each variable.*

Proof. From Lemma 3.1, Theorem 3.1 and Lemma 4.1 it follows that there exists a unique strong solution $v_m \in W^{1,2}(0,T;H_0^1(B_m)) \cap W^{2,2}_{loc}(0,T;L^2(B_m)) \cap L^\infty(0,T;H^2(B_m))$ of the problem

$$\begin{cases} \alpha(v_{mt}) - \Delta v_m + \lambda v_m + f(u) - f(u - v_m) = g_0(x), \ (t,x) \in (0,T)\times B_m, \\ v_m(t,x) = 0, \quad (t,x) \in (0,T)\times \partial B_m, \\ v_m(0,x) = 0, \quad x \in B_m, \end{cases}$$

which satisfies

$$\begin{aligned} &\|\alpha(v_{mt}(t)\|^2_{L^2(B_m)} + \|v_m(t)\|^2_{H^2(B_m)} \\ &+\frac{\tau}{1+t}\int_\tau^t \|v_{mtt}(s)\|^2_{L^2(B_m)}\,ds + \int_0^t \|\nabla v_{mt}(s)\|^2_{L^2(B_m)}\,ds \\ &\leq c_1(T, \|u_0\|_{H^2(R^n)}, \|g_0\|_{L^2(R^n)}), \quad 0 < \tau \leq t \leq T, \quad \forall\, m \in \mathbb{N}, \end{aligned} \tag{4.5}$$

and

$$\|v_m\|_{L^\infty((0,T)\times B_m)} \leq \frac{\mu_0}{\lambda} + \frac{1}{\lambda}\|g_0\|_{L^\infty(B_m)}, \quad \forall\, m \in \mathbb{N}, \tag{4.6}$$

where $c_1 : R_+ \times R_+ \times R_+ \to R_+$ is a nondecreasing function with respect to each variable. Setting $\widetilde{v}_m(t,x) = \begin{cases} v_m(t,x), & x \in B_m \\ 0, & x \in R^n \backslash B_m \end{cases}$ by (4.5) and (4.6) we can say that there exists a subsequence $\{m_k\} \subset \{m\}$ such that

$$\begin{cases} \widetilde{v}_{m_k} \to v \ \text{ weakly star in } L^\infty(0,T;H^1(R^n)) \\ v_{m_k} \to v \ \text{ weakly star in } L^\infty(0,T;H^2(B_\rho)) \\ v_{m_k t} \to v_t \text{ weakly in } L^2(0,T;H^1(B_\rho)) \\ v_{m_k tt} \to v_{tt} \text{ weakly in } L^2_{loc}(0,T;L^2(B_\rho)) \\ \alpha(v_{m_k t}) \to \alpha(v_t) \ \text{ weakly in } L^2(0,T;L^2(B_\rho)) \\ \widetilde{v}_{m_k} \to v \ \text{ weakly star in } L^\infty((0,T)\times R^n) \end{cases} \tag{4.7}$$

for every $\rho > 0$. So by $(4.7)_1$-$(4.7)_5$ and (4.5) we have $v \in W^{1,2}(0,T;H^1(R^n)) \cap W^{2,2}_{loc}(0,T;L^2(R^n)) \cap L^\infty(0,T;H^2(R^n))$ is the strong solution of (4.1) on $[0,T[\times R^n$. Also from (4.6) and $(4.7)_6$ it follows that

$$\|v\|_{L^\infty((0,T)\times R^n)} \leq \frac{\mu_0}{\lambda} + \frac{1}{\lambda}\|g_0\|_{L^\infty(R^n)}.$$

Set $\overline{\alpha}_k(s)=\begin{cases}\alpha'(k), & |s|>k\\ \alpha'(s), & |s|\le k\end{cases}$ and $\alpha_k(s)=\int_0^s\overline{\alpha}_k(t)dt$ for $k\in\mathbb{N}$. Since $\alpha_k(\cdot)$ also satisfies conditions (2.2)-(2.3) for any $u_0\in H^2(R^n)$ and $T>0$ there exists a unique strong solution $v_k\in W^{1,2}(0,T;H^1(R^n))\cap W^{2,2}_{loc}(0,T;L^2(R^n))\cap L^\infty(0,T;H^2(R^n))$ of the problem

$$\begin{cases}\alpha_k(v_{kt})-\Delta v_k+\lambda v_k+f(u)-f(u-v_k)=g_0(x), & (t,x)\in(0,T)\times R^n,\\ v_k(0,x)=0, \quad x\in R^n,\end{cases}, \tag{4.8}$$

which also satisfies

$$\|v_k\|_{L^\infty((0,T)\times R^n)}\le\frac{\mu_0}{\lambda}+\frac{1}{\lambda}\|g_0\|_{L^\infty(R^n)}, \quad \forall\, k\in\mathbb{N}. \tag{4.9}$$

By $(4.8)_1$ we have $\frac{\partial}{\partial t}\alpha_k(v_{kt})\in L^2(0,T;H^{-1}(R^n))$, which together with the inclusion $\alpha_k(v_{kt})\in L^2(0,T;H^1(R^n))$ implies that $\alpha_k(v_{kt})\in C([0,T];L^2(R^n))$. Now differentiating $(4.8)_1$ with respect to t and testing obtained equation by $\alpha_k(v_{kt})$ we find

$$\begin{aligned}&\frac{1}{2}\|\alpha_k(v_{kt}(t))\|^2_{L^2(R^n)}-\frac{1}{2}\|g_0\|^2_{L^2(R^n)}\\ &\le\int_0^t\int_{R^n}(f'(u(s,x)-v_k(s,x))-f'(u(s,x)))u_t(s,x)\alpha_k(v_{kt}(s,x))dxds\\ &\quad+c\int_0^t\int_{R^n}v_{kt}(s,x)\alpha_k(v_{kt}(s,x))dxds, \quad \forall\, t\ge0,\end{aligned} \tag{4.10}$$

where the constant $c>0$ depends only on $f(\cdot)$. By (2.4) and (4.9) we have

$$\begin{aligned}&\left|\int_0^t\int_{R^n}(f'(u(s,x)-v_k(s,x))-f'(u(s,x)))u_t(s,x)\alpha_k(v_{kt}(s,x))dxds\right|\\ &\le\Big(\frac{c\mu_0}{\lambda}+\frac{c}{\lambda}\|g_0\|_{L^\infty(R^n)}\Big)\int_0^t\int_{R^n}|u_t(s,x)|\,|\alpha_k(v_{kt}(s,x))|\,dxds, \quad \forall\, t\ge0.\end{aligned} \tag{4.11}$$

Applying Young inequality (see for example [11]) to the integral on right side of (4.11) and taking into account (3.15) we obtain

$$\begin{aligned}&\int_0^t\int_{R^n}|u_t(s,x)|\,|\alpha_k(v_{kt}(s,x))|\,dxds\\ &\le\int_0^t\int_{R^n}u_t(s,x)\alpha_k(u_t(s,x))dxds+\int_0^t\int_{R^n}v_{kt}(s,x)\alpha_k(v_{kt}(s,x))dxds\\ &\le\int_0^t\int_{R^n}u_t(s,x)\alpha(u_t(s,x))dxds+\int_0^t\int_{R^n}v_{kt}(s,x)\alpha_k(v_{kt}(s,x))dxds\\ &\le c_2(\|u_0\|_{H^1(R^n)},\|g\|_{L^2(R^n)})+\int_0^t\int_{R^n}v_{kt}(s,x)\alpha_k(v_{kt}(s,x))dxds, \quad \forall\, t\ge0,\end{aligned} \tag{4.12}$$

where $c_2 : R_+ \times R_+ \to R_+$ is a nondecreasing function with respect to each variable. Taking into account (4.11) and (4.12) in (4.10) we have

$$\|\alpha_k(v_{kt}(t))\|^2_{L^2(R^n)} \le c_3(\|u_0\|_{H^1(R^n)}, \|g\|_{L^2(R^n)}, \|g_0\|_{L^2(R^n)\cap L^\infty(R^n)})$$

$$\left(c + \frac{c\mu_0}{\lambda} + \frac{c}{\lambda}\|g_0\|_{L^\infty(R^n)}\right)\int_0^t\int_{R^n} v_{kt}(s,x)\alpha_k(v_{kt}(s,x))dxds, \quad \forall t \ge 0, \tag{4.13}$$

where $c_3 : R_+ \times R_+ \times R_+ \to R_+$ is a nondecreasing function with respect to each variable.

On the other hand subtracting $(4.8)_1$ from $(4.1)_1$ and testing the obtained equation by $(v_t - v_{kt})$ we find

$$\frac{1}{3}\alpha'(0)\int_0^t \|v_t(s) - v_{kt}(s)\|^2_{L^2(R^n)} ds + \frac{1}{2}\|\nabla(v(t) - v_k(t))\|^2_{L^2(R^n)}$$

$$+\frac{\lambda}{2}\|v(t) - v_k(t)\|^2_{L^2(R^n)} \le \frac{3}{4\alpha'(0)}\int_0^t \|\alpha(v_t(s)) - \alpha_k(v_t(s))\|^2_{L^2(R^n)} ds$$

$$+ c_4(t, \|u_0\|_{H^1(R^n)}, \|g\|_{L^2(R^n)})\int_0^t \|v(s) - v_k(s)\|^2_{H^1(R^n)} ds, \forall t \ge 0. \tag{4.14}$$

From definition of $\alpha_k(\cdot)$ it follows that

$$\int_0^T \|\alpha(v_t(s)) - \alpha_k(v_t(s))\|^2_{L^2(R^n)} ds \le \int_0^T \int_{\{x: x\in R^n,\ |v_t(s,x)|>k\}} |\alpha(v_t(s,x))|^2 dxds. \tag{4.15}$$

Since $\alpha(v_t) \in L^2(0,T;L^2(R^n))$ (thanks to (4.5) and (4.7)), by (4.15) we have

$$\alpha_k(v_t) \to \alpha(v_t) \text{ strongly in } L^2(0,T;L^2(R^n))$$

for every $T > 0$. Then applying Gronwall's lemma to (4.14) we obtain

$$\begin{cases} v_k \to v \text{ strongly in } L^\infty(0,T;H^1(R^n)) \\ v_{kt} \to v_t \text{ strongly in } L^2(0,T;L^2(R^n)) \end{cases}$$

So passing to limit in (4.13) we find

$$\|\alpha(v_t(t))\|^2_{L^2(R^n)} \le c_3(\|u_0\|_{H^1(R^n)}, \|g\|_{L^2(R^n)}, \|g_0\|_{L^2(R^n)\cap L^\infty(R^n)})$$

$$+\left(c + \frac{c\mu_0}{\lambda} + \frac{c}{\lambda}\|g_0\|_{L^\infty(R^n)}\right)\int_0^t\int_{R^n} v_t(s,x)\alpha(v_t(s,x))dxds, \quad \forall t \ge 0. \tag{4.16}$$

Now let us estimate the second term on the right side of (4.16). Multiplying both sides of $(4.2)_1$ by w_t and integrating over $(s,T)\times R^n$ we obtain

$$E(w(T)) + \int_{R^n} F(w(T,x))dx - \int_{R^n}(g(x) - g_0(x))w(T,x)dx$$

$$+\int_s^T\int_{R^n} w_t(t,x)(\alpha(v_t(t,x) + w_t(t,x)) - \alpha(v_t(t,x)))dxdt$$

$$\leq E(w(s)) + \int_{R^n} F(w(s,x))dx - \int_{R^n} (g(x) - g_0(x))w(s,x)dx, \qquad \forall\, T \geq s \geq 0. \quad (4.17)$$

By (2.2)-(2.3), we have

$$(\alpha(x) - \alpha(y))(x - y) \geq \widehat{c}\alpha(x - y)\ (x - y), \quad \forall\, x, y \in R,$$

for some $\widehat{c} > 0$. By the last two inequalities we find

$$\|w(T)\|^2_{H^1(R^n)} + \int_0^T \int_{R^n} \alpha(w_t(s,x))w_t(s,x)dxds$$

$$\leq c_5(\|u_0\|_{H^1(R^n)}, \|g - g_0\|_{L^2(R^n)}), \qquad \forall\, T \geq 0, \tag{4.18}$$

and using Young inequality we have

$$\widehat{c}\int_0^T \int_{R^n} \alpha(v_t(s,x))v_t(s,x)dxds$$

$$\leq \int_0^T \int_{R^n} (\alpha(u_t(s,x)) - \alpha(w_t(s,x)))(u_t(s,x) - w_t(s,x))dxds$$

$$\leq 2\int_0^T \int_{R^n} \alpha(u_t(s,x))u_t(s,x)dxds + 2\int_0^T \int_{R^n} \alpha(w_t(s,x))w_t(s,x)dxds$$

$$\leq c_6(\|u_0\|_{H^1(R^n)}, \|g\|_{L^2(R^n)}, \|g_0\|_{L^2(R^n)}), \qquad \forall\, T \geq 0, \tag{4.19}$$

where $c_5 : R_+ \times R_+ \to R_+$ and $c_6 : R_+ \times R_+ \times R_+ \to R_+$ are nondecreasing functions with respect to each variable. The last inequality together with (4.16) yields

$$\|\alpha(v_t(t))\|^2_{L^2(R^n)} \leq c_7(\|u_0\|_{H^1(R^n)}, \|g\|_{L^2(R^n)}, \|g_0\|_{L^2(R^n)\cap L^\infty(R^n)}), \ \ \forall\, t \geq 0,$$

where $c_7 : R_+ \times R_+ \times R_+ \to R_+$ is a nondecreasing function with respect to each variable. Thus taking into account (3.15), (4.18) and the last inequality in $(4.1)_1$ we obtain (4.4). □

Now let us prove the uniform tail estimate for the solutions of (4.1):

Lemma 4.3. *Assume that Assumption 2.1 holds and $u_0 \in H^2(R^n)$. Then for any $\varepsilon > 0$ and $T > 0$ there exists $r = r(\varepsilon, T, \|u_0\|_{H^1(R^n)}) > 0$ such that*

$$\int_{\{x: x\in R^n,\ |x|\geq r\}} \left(|\nabla v(T,x)|^2 + |v(T,x)|^2\right) dx \leq \varepsilon, \tag{4.20}$$

where $r : R_+ \times R_+ \times R_+ \to R_+$ is a nondecreasing function with respect to the third variable.

Proof. We use techniques of [7]. Multiplying equation (4.1) by $v_t e^{-|x-x_0|}$, integrating over $(0,T) \times R^n$ and applying Gronwall's inequality we find

$$\int_{R^n} \left(|\nabla v(T,x)|^2 + \lambda\, |v(T,x)|^2\right) e^{-|x-x_0|}dx \leq \frac{4}{\lambda} e^{C_1(\|u_0\|_{H^1(R^n)})T} \int_{R^n} |g_0|^2\, e^{-|x-x_0|}dx,$$

where $C_1 : R_+ \to R_+$ is a nondecreasing function.

Integrating the last inequality with respect to x_0 over $\{x_0 : x_0 \in R^n,\ |x_0| \geq r\}$ we obtain

$$\int\limits_{\{x_0:x_0\in R^n,\ |x_0|\geq r\}}\int\limits_{R^n}\left(|\nabla v(T,x)|^2+\lambda\,|v(T,x)|^2\right)e^{-|x-x_0|}dxdx_0$$

$$\leq \frac{4}{\lambda}e^{C_1T}\int\limits_{\{x_0:x_0\in R^n,\ |x_0|\geq r\}}\int\limits_{R^n}|g_0|^2\,e^{-|x-x_0|}dxdx_0. \tag{4.21}$$

Let $\varphi \in L^2(R^n)$. Then we have

$$\int\limits_{\{x_0:x_0\in R^n,\ |x_0|\geq r\}}\int\limits_{R^n}|\varphi(x)|^2\,e^{-|x-x_0|}dxdx_0$$

$$=\int\limits_{\{x_0:x_0\in R^n,\ |x_0|\geq r\}}\int\limits_{\left\{x:x\in R^n,\ |x|\geq\frac{r}{2}\right\}}|\varphi(x)|^2\,e^{-|x-x_0|}dxdx_0$$

$$+\int\limits_{\{x_0:x_0\in R^n,\ |x_0|\geq r\}}\int\limits_{\left\{x:x\in R^n,\ |x|<\frac{r}{2}\right\}}|\varphi(x)|^2\,e^{-|x-x_0|}dxdx_0$$

$$\leq\left(\int\limits_{\left\{x:x\in R^n,\ |x|\geq\frac{r}{2}\right\}}|\varphi(x)|^2\,dx\right)\left(\int\limits_{R^n}e^{-|y|}dy\right)$$

$$+e^{-\frac{r}{4}}\int\limits_{\{x_0:x_0\in R^n,\ |x_0|\geq r\}}\int\limits_{\left\{x:x\in R^n,\ |x|<\frac{r}{2}\right\}}|\varphi(x)|^2\,e^{-\frac{1}{4}|x_0|}dxdx_0$$

$$\leq C_2\int\limits_{\left\{x:x\in R^n,\ |x|\geq\frac{r}{2}\right\}}|\varphi(x)|^2\,dx+C_3e^{-\frac{r}{4}}\int\limits_{R^n}|\varphi(x)|^2\,dx \tag{4.22}$$

and

$$\int\limits_{\{x_0:x_0\in R^n,\ |x_0|\geq r\}}\int\limits_{R^n}|\varphi(x)|^2\,e^{-|x-x_0|}dxdx_0$$

$$=\int\limits_{R^n}\int\limits_{R^n}|\varphi(x)|^2\,e^{-|x-x_0|}dxdx_0-\int\limits_{\{x_0:x_0\in R^n,\ |x_0|<r\}}\int\limits_{R^n}|\varphi(x)|^2\,e^{-|x-x_0|}dxdx_0$$

$$=C_2\int\limits_{R^n}|\varphi(x)|^2\,dx-\int\limits_{\{x_0:x_0\in R^n,\ |x_0|<r\}}\int\limits_{\{x:x\in R^n,\ |x|<2r\}}|\varphi(x)|^2\,e^{-|x-x_0|}dxdx_0$$

$$-\int\limits_{\{x_0:x_0\in R^n,\ |x_0|<r\}}\int\limits_{\{x:x\in R^n,\ |x|\geq 2r\}}|\varphi(x)|^2\,e^{-|x-x_0|}dxdx_0$$

$$\geq C_2\int\limits_{\{x:x\in R^n,\ |x|\geq 2r\}}|\varphi(x)|^2\,dx-e^{-r}\int\limits_{\{x_0:x_0\in R^n,\ |x_0|<r\}}\int\limits_{\{x:x\in R^n,\ |x|\geq 2r\}}|\varphi(x)|^2\,dxdx_0$$

$$=\left(C_2-C_4r^ne^{-r}\right)\int\limits_{\{x:x\in R^n,\ |x|\geq 2r\}}|\varphi(x)|^2\,dx. \tag{4.23}$$

Taking into account (4.22)-(4.23) in (4.21) we find (4.20). □

Now denote by $\mathcal{R}(t)$ a solution operator of (4.1), i.e. $v(t) = \mathcal{R}(t)(u_0, g_0)$, where $u_0 \in H^2(R^n)$, $g_0 \in L^2(R^n) \cap L^\infty(R^n)$ and $v(t,)$ is the solution of (4.1) determined by Lemma 4.2. By (4.1), it is easy to see that if the sequence $\{u_{0n}\}_{n=1}^\infty \subset H^2(R^n)$ converges in $H^1(R^n)$, then the sequence $\{\mathcal{R}(t)(u_{0n}, g_0)\}_{n=1}^\infty$ also converges in $H^1(R^n)$. Hence using density argument, the operator $\mathcal{R}(t)(\cdot, g_0)$ can be extended to $H^1(R^n)$, and so by Lemma 4.2 and Lemma 4.3 we immediately have the following corollary.

Corollary 4.1. *Assume that Assumption 2.1 holds. Then the operator* $\mathcal{R}(t)(\cdot, g_0) : H^1(R^n) \to H^1(R^n)$, $t \geq 0$, *is compact.*

Now let us denote $g_k(x) = \begin{cases} g(x), & |g(x)| \leq k \\ 0, & |g(x)| > k \end{cases}$.

Lemma 4.4. *Assume that Assumption 2.1 holds and B is a bounded subset of $H^1(R^n)$. Then for any $\varepsilon > 0$ and $m > 0$ there exist $k_0 = k_0(\varepsilon, B) \in \mathbb{N}$, $M_0 = M_0(\varepsilon, B) > 0$ and $T_0 = T_0(\varepsilon, B, m) > 0$ such that*

$$\|S(T)u_0 - \mathcal{R}(T)(u_0, g_k)\|^2_{H^1(R^n)} \leq c \int\limits_{\{x : x \in R^n,\ |u_0(x)| > m\}} |\nabla u_0(x)|^2 \, dx + \varepsilon,$$

$$\forall\, u_0 \in B,\ \forall\, T \geq T_0,\ \forall\, k \geq k_0,\ \forall\, m \geq M_0, \tag{4.24}$$

where the positive constant c depends only λ and $f(\cdot)$.

Proof. We apply the techniques used in [10]. Since $g \in L^2(R^n)$, we have $g_k \in L^2(R^n) \cap L^\infty(R^n)$ and $g_k \to g$ strongly in $L^2(R^n)$ as $k \to \infty$. Let $u_0 \in H^2(R^n)$. Denote $v_k(t) = \mathcal{R}(T)(u_0, g_k)$ and $w_k = u(t) - v_k(t)$, where $u(t) = S(t)u_0$. Then the function $w_k \in W^{1,2}(0, T; H^1(R^n)) \cap W^{2,2}_{loc}(0, T; L^2(R^n)) \cap L^\infty(0, T; H^2(R^n))$ satisfies $(4.2)_1$ (with force term $g(x) - g_k(x)$ instead of $g(x) - g_0(x)$) a.e. on $(0, T) \times R^n$ for every $T > 0$ and $w_k(0) = u_0$ a.e. on R^n. Denote $u_{0m}(x) = \begin{cases} u_0(x) + m, & u_0(x) < -m \\ 0, & |u_0(x)| \leq m \\ u_0(x) - m, & u_0(x) > m \end{cases}$. Putting g_k, v_k and w_k instead of g_0, v and w in (4.2) respectively, multiplying obtained equation by $\frac{1}{t+1}(w_k(t, x) - u_{0m}(x))$ and integrating over $(0, T) \times R^n$ we have

$$\int_0^T \frac{1}{t+1} \|\nabla w_k(t)\|^2_{L^2(R^n)} \, dt - \int_0^T \int_{R^n} \frac{1}{t+1} \sum_{i=1}^n w_{kx_i}(t, x) u_{0mx_i}(x) dx dt$$

$$+ \lambda \int_0^T \frac{1}{t+1} \|w_k(t)\|^2_{L^2(R^n)} \, dt - \lambda \int_0^T \int_{R^n} \frac{1}{t+1} w_k(t, x) u_{0m}(x) dx dt$$

$$+ \int_0^T \int_{R^n} \frac{1}{t+1} f(w_k(t, x)) w_k(t, x) dx dt - \int_0^T \int_{R^n} \frac{1}{t+1} f(w_k(t, x)) u_{0m}(x) dx dt$$

$$\leq \int_0^T \int_{R^n} \alpha(v_{kt}(t,x)) \frac{1}{t+1} \int_0^t w_{kt}(s,x) ds dx dt - \int_0^T \int_{R^n} \alpha(u_t(t,x)) \frac{1}{t+1} \int_0^t w_{kt}(s,x) ds dx dt$$

$$+ \int_0^T \int_{R^n} \alpha(v_{kt}(t,x)) \frac{1}{t+1} (u_0(x) - u_{0m}(x)) dx dt$$

$$- \int_0^T \int_{R^n} \alpha(u_t(t,x)) \frac{1}{t+1} (u_0(x) - u_{0m}(x)) dx dt$$

$$+ \int_0^T \int_{R^n} \frac{1}{t+1} (w_k(t,x) - u_{0m}(x))(g(x) - g_k(x)) dx dt. \tag{4.25}$$

Now let us estimate first four terms on the right side of (4.25).

Using Young and Jensen inequalities (see [11]) we find

$$\int_0^T \int_{R^n} |\alpha(v_{kt}(t,x))| \frac{1}{t+1} \int_0^t |w_{kt}(s,x)| ds dx dt$$

$$\leq \int_0^T \int_{R^n} \mathcal{N}(\mu |\alpha(v_{kt}(t,x))|) dx dt + \int_0^T \int_{R^n} \mathcal{M} \left(\frac{1}{\mu(t+1)} \int_0^t |w_{kt}(s,x)| ds \right) dx dt$$

$$\leq \int_0^T \int_{R^n} \mathcal{N}(\mu \alpha(v_{kt}(t,x))) dx dt + \int_0^T \int_{R^n} \frac{t}{\mu(t+1)} \mathcal{M} \left(\frac{1}{t} \int_0^t |w_{kt}(s,x)| ds \right) dx dt$$

$$\leq \int_0^T \int_{R^n} \mathcal{N}(\mu \alpha(v_{kt}(t,x))) dx dt + \int_0^T \int_{R^n} \frac{1}{\mu(t+1)} \int_0^t \mathcal{M}(w_{kt}(s,x)) ds dx dt, \forall \mu > 1, \tag{4.26}$$

where $\mathcal{M}(z) = \int_0^z \alpha(x) dx$. By (2.3), since α is odd function and $\alpha'(\cdot)$ is nondecreasing on R_+, we have

$$\alpha(\mu x) \geq \mu \alpha(x), \ \ \forall x \in R_+, \ \forall \mu > 1,$$

and consequently

$$\alpha^{-1}(\mu \alpha(x)) \leq \mu x, \ \ \ \forall x \in R_+, \ \forall \mu > 1.$$

The last inequality together with (2.3) yields that

$$\int_0^T \int_{R^n} \mathcal{N}(\mu \alpha(v_{kt}(t,x))) dx dt \leq \mu \int_0^T \int_{R^n} \alpha(v_{kt}(t,x)) \alpha^{-1}(\mu \alpha(v_{kt}(t,x))) dx dt$$

$$\leq \mu^2 \int_0^T \int_{R^n} \alpha(v_{kt}(t,x)) v_{kt}(t,x) dx dt. \tag{4.27}$$

By (4.26)-(4.27) we obtain

$$\int_0^T\int_{R^n} |\alpha(v_{kt}(t,x))| \frac{1}{t+1}\int_0^t |w_{kt}(s,x)|\, dsdxdt$$
$$\leq \mu^2 \int_0^T\int_{R^n} \alpha(v_{kt}(t,x))v_{kt}(t,x)dxdt + \frac{\ln(T+1)}{\mu}\int_0^T\int_{R^n} \alpha(w_{kt}(t,x))w_{kt}(t,x)dxdt,$$

which together with (4.18)-(4.19) implies

$$\int_0^T\int_{R^n} |\alpha(v_{kt}(t,x))| \frac{1}{t+1}\int_0^t |w_{kt}(s,x)|\, dsdxdt$$
$$\leq \left(\mu^2 + \frac{\ln(T+1)}{\mu}\right)\widehat{c}_1(\|u_0\|_{H^1(R^n)}, \|g\|_{L^2(R^n)}), \quad \forall\, T\geq 0,\ \forall\, \mu>1, \tag{4.28}$$

where $\widehat{c}_1 : R_+ \times R_+ \to R_+$ is a nondecreasing functions with respect to each variable. By the same way we find

$$\int_0^T\int_{R^n} |\alpha(u_t(t,x))| \frac{1}{t+1}\int_0^t |w_{kt}(s,x)|\, dsdxdt$$
$$\leq \left(\mu^2 + \frac{\ln(T+1)}{\mu}\right)\widehat{c}_2(\|u_0\|_{H^1(R^n)}, \|g\|_{L^2(R^n)}), \quad \forall\, T\geq 0,\ \forall\, \mu>1, \tag{4.29}$$

where $\widehat{c}_2 : R_+ \times R_+ \times R_+ \to R_+$ is a nondecreasing function with respect to each variable. By definition of $u_{0m}(x)$, we have

$$\int_0^T\int_{R^n} \frac{1}{t+1}|\alpha(v_{kt}(t,x))|\,|u_0(x)-u_{0m}(x)|\, dxdt$$
$$= \int_0^T \int_{\{x:x\in R^n,\ |u_0(x)|\leq m\}} |\alpha(v_{kt}(t,x))| \frac{1}{t+1}|u_0(x)|\, dxdt$$
$$+ m\int_0^T \int_{\{x:x\in R^n,\ |u_0(x)|> m\}} \frac{1}{t+1}|\alpha(v_{kt}(t,x))|\, dxdt$$
$$\leq \int_0^T\int_{R^n} \mathcal{N}(\alpha(v_{kt}(t,x)))dxdt + \int_0^T \int_{\{x:x\in R^n,\ |u_0(x)|\leq m\}} \mathcal{M}(\frac{1}{t+1}u_0(x))dxdt$$
$$+ m\int_0^T \int_{\{x:x\in R^n,\ |u_0(x)|> m,\ |v_{kt}(t,x)|\leq 1\}} \frac{1}{t+1}|\alpha(v_{kt}(t,x))|\, dxdt$$
$$+ m\int_0^T \int_{\{x:x\in R^n,\ |u_0(x)|> m,\ |v_{kt}(t,x)|> 1\}} \frac{1}{t+1}|\alpha(v_{kt}(t,x))|\, dxdt$$

$$
\begin{aligned}
&\leq \int_0^T \int_{R^n} \alpha(v_{kt}(t,x)) v_{kt}(t,x)\,dxdt \\
&\quad + \int_0^T \int_{\{x: x\in R^n,\ |u_0(x)|\leq m\}} \alpha\Big(\frac{1}{t+1}u_0(x)\Big)\frac{1}{t+1}u_0(x)\,dxdt \\
&\quad + m\alpha(1)\ln(T+1)\operatorname{mes}\{x : x\in R^n,\ \ |u_0(x)| > m\} \\
&\quad + m\int_0^T \int_{R^n} \frac{1}{t+1}\alpha(v_{kt}(t,x))v_{kt}(t,x)\,dxdt \\
&\leq (m+1)\int_0^T \int_{R^n} \alpha(v_{kt}(t,x))v_{kt}(t,x)\,dxdt \\
&\quad + \int_0^T \int_{\{x: x\in R^n,\ |u_0(x)|\leq m\}} \frac{1}{(t+1)^2}\alpha(u_0(x))u_0(x)\,dxdt \\
&\quad + \frac{1}{m}\alpha(1)\ln(T+1)\left\|u_0\right\|^2_{L^2(R^n)} \leq (m+1)\int_0^T \int_{R^n} \alpha(v_{kt}(t,x))v_{kt}(t,x)\,dxdt \\
&\quad + \alpha'(m)\left\|u_0\right\|^2_{L^2(R^n)} + \frac{1}{m}\alpha(1)\ln(T+1)\left\|u_0\right\|^2_{L^2(R^n)} \\
&\leq \left(\frac{1}{m}\alpha(1)\ln(T+1) + \alpha'(m)\right)\left\|u_0\right\|^2_{L^2(R^n)} \\
&\quad + (m+1)\widehat{c}_1(\left\|u_0\right\|_{H^1(R^n)}, \left\|g\right\|_{L^2(R^n)}) \quad \forall\, T\geq 0,\ \ \forall\, m>0. \qquad (4.30)
\end{aligned}
$$

Similarly we have

$$
\begin{aligned}
&\int_0^T \int_{R^n} |\alpha(u_t(t,x))|\,\frac{1}{t+1}\,|u_0(x) - u_{0m}(x)|\,dxdt \\
&\leq \left(\frac{1}{m}\alpha(1)\ln(T+1) + \alpha'(m)\right)\left\|u_0\right\|^2_{L^2(R^n)} \\
&\quad + (m+1)\widehat{c}_2(\left\|u_0\right\|_{H^1(R^n)}, \left\|g\right\|_{L^2(R^n)}) \quad \forall\, T\geq 0,\ \ \forall\, m>0. \qquad (4.31)
\end{aligned}
$$

Taking into account (4.28)-(4.31) in (4.25) we find

$$
\begin{aligned}
&\int_0^T \frac{1}{t+1}E(w_k(t))dt + \int_0^T \int_{R^n} \frac{1}{t+1}f((w_k(t,x))w_k(t,x)\,dxdt \\
&\leq \ln(T+1)E(u_{0m}) + 2\left(\frac{1}{m}\alpha(1)\ln(T+1) + \alpha'(m)\right)\left\|u_0\right\|^2_{L^2(R^n)} \\
&\quad + (m+1)\widehat{c}_3(\left\|u_0\right\|_{H^1(R^n)}, \left\|g\right\|_{L^2(R^n)}) \\
&\quad + \left(\mu^2 + \frac{\ln(T+1)}{\mu}\right)\widehat{c}_3(\left\|u_0\right\|_{H^1(R^n)}, \left\|g\right\|_{L^2(R^n)})
\end{aligned}
$$

$$+\int_0^T\int_{R^n}\frac{1}{t+1}(w_k(t,x)-u_{0m}(x))(g(x)-g_k(x))dxdt$$
$$+\int_0^T\int_{R^n}\frac{1}{t+1}f(w_k(t,x))u_{0m}(x)dxdt,\ \ \forall T>0,\ \ \forall m>0,\ \ \forall \mu>1,$$

where $\widehat{c}_3=\widehat{c}_1+\widehat{c}_2$. Taking into account (2.4), (2.5) and (4.18) in the last inequality we obtain

$$\begin{aligned}
&\int_0^T\frac{1}{t+1}E(w_k(t))dt+\int_0^T\int_{R^n}\frac{1}{t+1}F((w_k(t,x))dxdt\\
&-\int_0^T\int_{R^n}\frac{1}{t+1}(g(x)-g_k(x))w_k(t,x)dxdt\\
&\le\widehat{c}\ln(T+1)E(u_{0m})+2\widehat{c}\left(\frac{1}{m}\alpha(1)\ln(T+1)+\alpha'(m)\right)\|u_0\|^2_{L^2(R^n)}\\
&\quad+\widehat{c}(m+1)\widehat{c}_3(\|u_0\|_{H^1(R^n)},\|g\|_{L^2(R^n)})\\
&\quad+\widehat{c}\left(\mu^2+\frac{\ln(T+1)}{\mu}\right)\widehat{c}_3(\|u_0\|_{H^1(R^n)},\|g\|_{L^2(R^n)})\\
&\quad+\widehat{c}_4(\|u_0\|_{H^1(R^n)},\|g-g_k\|_{L^2(R^n)})\ln(T+1)\|u_{0m}\|_{L^2(R^n)}\\
&\quad+\widehat{c}_4(\|u_0\|_{H^1(R^n)},\|g-g_k\|_{L^2(R^n)})\|g-g_k\|_{L^2(R^n)}\ln(T+1)\\
&\quad+\widehat{c}\ln(T+1)\|u_{0m}\|_{L^2(R^n)}\|g-g_k\|_{L^2(R^n)},\quad \forall T>0,\ \ \forall m>0,\ \ \forall \mu>1,
\end{aligned}$$

where $\widehat{c}_4:R_+\times R_+\to R_+$ is a nondecreasing function with respect to each variable and the positive constant $\widehat{c}$ depends only on λ and $f(\cdot)$.

Now putting g_k, v_k and w_k instead of g_0, v and w in (4.17) respectively, multiplying both sides of obtained inequality by $\frac{1}{1+s}$ and integrating with respect to s from 0 to T we have

$$\begin{aligned}
&\ln(T+1)E(w_k(T))+\ln(T+1)\int_{R^n}F(w_k(T,x))dx\\
&\le\ln(T+1)\int_{R^n}(g(x)-g_k(x))w_k(T,x)dx+\int_0^T\frac{1}{s+1}E(w_k(s))ds\\
&\quad+\int_0^T\int_{R^n}\frac{1}{s+1}F(w_k(s,x))dxds-\int_0^T\int_{R^n}\frac{1}{s+1}(g(x)-g_k(x))w_k(s,x)dxds.
\end{aligned}$$

By the last two inequalities, for any $\varepsilon>0$ there exists $M_0=M_0(\varepsilon,\|u_0\|_{H^1(R^n)})>0$ such that

$$\begin{aligned}
\frac{1}{2}E(w_k(T))\le&\frac{\varepsilon}{4}+\widehat{c}\int_{\{x:x\in R^n,\ |u_0(x)|>m\}}|\nabla u_0(x)|^2\,dx\\
&+\frac{2\widehat{c}\alpha'(m)}{\ln(T+1)}\|u_0\|^2_{L^2(R^n)}+\frac{\widehat{c}(m+1)}{\ln(T+1)}\widehat{c}_3(\|u_0\|_{H^1(R^n)},\|g\|_{L^2(R^n)})
\end{aligned}$$

$$+\widehat{c}\left(\frac{\mu^2}{\ln(T+1)}+\frac{1}{\mu}\right)\widehat{c}_3(\|u_0\|_{H^1(R^n)},\|g\|_{L^2(R^n)})$$
$$+\widehat{c}_4(\|u_0\|_{H^1(R^n)},\|g-g_k\|_{L^2(R^n)})\|g-g_k\|_{L^2(R^n)}$$
$$+\frac{1}{\lambda}\|g-g_k\|^2_{L^2(R^n)},\ \ \forall m\geq M_0.$$

Thus choosing $\mu=\ln^{\frac{1}{4}}(T+1)$, we obtain (4.24) for large T and k. □

Lemma 4.5. *Assume that Assumption 2.1 holds and $B\in\mathfrak{B}$. Then for any $\varepsilon>0$ there exist $\delta_0=\delta_0(\varepsilon)>0$, $T_0=T_0(\varepsilon,B)>0$ and $M_0=M_0(\varepsilon,B)>0$ such that*

$$\int\limits_{\{x:x\in R^n,\ |u(T,x)|>M_0\}}|\nabla u(T,x)|^2\,dx\leq\varepsilon,\quad\forall\,T\geq T_0\ ,\ \forall\,u_0\in O_{\delta_0}(B),\tag{4.32}$$

where $u(T,)=S(T)u_0$ and $O_\delta(B)$ is $\delta-$neihbourhood of B in $H^1(R^n)$.

Proof. We present the proof for $n\geq 3$. By Lemma 4.4 for any $\varepsilon>0$ there exist $k_0=k_0(\varepsilon,B)\in\mathbb{N}$, $\delta_0=\delta_0(\varepsilon)>0$ and $T_0=T_0(\varepsilon,B)>0$ such that

$$\|S(T)u_0-\mathcal{R}(T)(u_0,g_{k_0})\|_{H^1(R^n)}\leq\frac{\varepsilon}{2},\quad\forall\,T\geq T_0\ ,\ \forall\,u_0\in O_{\delta_0}(B).\tag{4.33}$$

Also by Theorem 3.2 and Lemma 4.2 we have

$$\int\limits_{\{x:x\in R^n,\ |u(T,x)|>M\}}\left(|\nabla v(T,x)|^2+|v(T,x)|^2\right)dx$$
$$\leq\text{mes}^{\frac{2}{n}}\{x:x\in R^n,\ \ |u(T,x)|>M\}$$
$$\times\,c_1(\|u_0\|_{H^1(R^n)},\|g\|_{L^2(R^n)},\|g_{k_0}\|_{L^\infty(R^n)})$$
$$\leq M^{-\frac{4}{n-2}}c_2(\|u_0\|_{H^1(R^n)},\|g\|_{L^2(R^n)},\|g_{k_0}\|_{L^\infty(R^n)}),\tag{4.34}$$

where $c_i:R_+\times R_+\times R_+\to R_+$ $(i=1,2)$ are nondecreasing functions with respect to each variable and $v(T,)=\mathcal{R}(T)(u_0,g_{k_0})$. By (4.33)-(4.34) we obtain (4.32). □

We are now in a position to prove the asymptotic compactness of solution of (2.1), which is included in the following theorem:

Theorem 4.1. *Assume that Assumption 2.1 holds and $B\in\mathfrak{B}$. Then any sequence of the form $\{S(t_m)u_{0m}\}_{m=1}^\infty$, $t_m\to\infty$, $u_{0m}\in O_{\delta_m}(B)$, $\delta_m\searrow 0$, has a convergent subsequence in $H^1(R^n)$.*

Proof. Denote by $K_{H^1(R^n)}(A)$ the Kuratowski measure of non-compactness of the set A in $H^1(R^n)$, i.e.

$$K_{H^1(R^n)}(A):=\inf\{\varepsilon\ |A\text{ has a finite open cover of sets whose diameters are less than }\varepsilon\}.$$

By Lemma 4.5, for any $\varepsilon>0$ and $B\in\mathfrak{B}$ there exist $\delta_0=\delta_0(\varepsilon)>0$, $T_0=T_0(\varepsilon,B)>0$ and $M_0=M_0(\varepsilon,B)>0$ such that

$$\int\limits_{\{x:x\in R^n,\ |\varphi(x)|>M_0\}}|\nabla\varphi(x)|^2\,dx\leq\frac{\varepsilon}{2c},\ \ \forall\,\varphi\in\bigcup_{t\geq T_0}S(t)O_\delta(B),\ \ \forall\,\delta\in(0,\delta_0).$$

Then by Lemma 4.4, there exist $k_0 = k_0(\varepsilon, B) \in \mathbb{N}$ and $T_1 = T_1(\varepsilon, B, M_0) > 0$ such that

$$\|S(T_1)\varphi - \mathcal{R}(T_1)(\varphi, g_{k_0})\|_{H^1(R^n)} \leq \sqrt{\varepsilon}, \quad \forall\, \varphi \in \bigcup_{t\geq T_0} S(t)O_\delta(B), \quad \forall\, \delta \in (0, \delta_0). \tag{4.35}$$

Taking into account (4.35) and Corollary 4.1 we obtain

$$K_{H^1(R^n)}\left(S(T_1)\left(\bigcup_{t\geq T_0} S(t)O_\delta(B)\right)\right) \leq 4\sqrt{\varepsilon}, \ \forall\, \delta \in (0, \delta_0)$$

or

$$K_{H^1(R^n)}\left(\bigcup_{t\geq T_0+T_1} S(t)O_\delta(B)\right) \leq 4\sqrt{\varepsilon}, \ \forall\, \delta \in (0, \delta_0).$$

Now if $t_m \to \infty$, $u_{0m} \in O_{\delta_m}(B)$ and $\delta_m \searrow 0$, then from the last inequality it follows that

$$K_{H^1(R^n)}\left(\{S(t_m)u_{0m}\}_{m=1}^{\infty}\right) = 0,$$

which completes the proof. □

From this theorem immediately the following corollary follows.

Corollary 4.2. *Under Assumption 2.1 for every $B \in \mathfrak{B}$, the sets $\omega(B) = \bigcap_{t\geq 0}\overline{\bigcup_{\tau\geq t} S(\tau)B}$ and $\widehat{\omega}(B) = \bigcap_{\delta>0}\bigcap_{t\geq 0}\overline{\bigcup_{\tau\geq t} S(\tau)O_\delta(B)}$ are nonempty strictly invariant compacts which attract B.*

Now we can prove the main result.

Proof of Theorem 2.1. Set

$$Z = \left\{\varphi : \varphi \in H^1(R^n), \ -\Delta\varphi + \lambda\varphi + f(\varphi) = g\right\}.$$

It is easy to see that under conditions (2.4)-(2.5) the set Z is a bounded subset of $H^2(R^n)$ and consequently $Z \in \mathfrak{B}$. Then by Corollary 4.2 the set $\widehat{\omega}(Z)$ is invariant and compact in $H^1(R^n)$. We will show that $\widehat{\omega}(Z)$ is the global $(H^1(R^n), H^1(R^n))_{\mathfrak{B}}-$attractor for $\{S(t)\}_{t\geq 0}$. To this end it is sufficient to show that

$$\omega(B) \subset \widehat{\omega}(Z), \qquad \forall\, B \in \mathfrak{B}. \tag{4.36}$$

As shown in [1, p.159-161], since $\omega(B), (B \in \mathfrak{B})$ is a compact strictly invariant set and the problem (2.1) admits the Lyapunov function $L(u(t)) := E(u(t)) + \int_{R^n} F(u(t,x))dx - \int_{R^n} u(t,x)g(x)dx$ (thanks to (3.15)), for every $v \in \omega(B)$ there exists a complete trajectory $\gamma = \{u(t), t \in R\} \subset \omega(B)$ such that

$$u(0) = v \ \text{ and } \ \lim_{t\to-\infty}\inf_{\varphi\in Z}\|u(t) - \varphi\|_{H^1(R^n)} = 0. \tag{4.37}$$

Taking into account (4.37) and the equality $u(t + \tau) = S(t)u(\tau), t \geq 0, \tau \in R$, we find that $v \in \widehat{\omega}(Z)$. Since v and B are the arbitrary element of $\omega(B)$ and $\mathfrak{B}$ respectively, by the last conclusion we obtain (4.36).

Remark 4.1. If $g \in L^2(R^n) \cap L^\infty(R^n)$, then by the proof of Lemma 4.2 one can see that

$$\|\mathcal{R}(t)(u_0, g)\|_{L^\infty(R^n)} \leq c, \ \forall\, t \geq 0, \ \forall\, u_0 \in H^2(R^n), \tag{i}$$

where the positive constant c depends on λ, $f(\cdot)$ and $\|g\|_{L^\infty(R^n)}$. Also by Lemma 4.4 for any $B \in \mathfrak{B}$ we have

$$\|S(t)u_0 - \mathcal{R}(t)(u_0, g)\|_{H^1(R^n)} \to 0 \ \text{ as } t \to \infty \tag{ii}$$

uniformly with respect to all $u_0 \in B$. Since a global $(H^1(R^n), H^1(R^n))_{\mathfrak{B}}-$attractor is invariant, from (i)-(ii) it follows that a global $(H^1(R^n), H^1(R^n))_{\mathfrak{B}}-$attractor is a bounded subset of $L^\infty(R^n)$.

Remark 4.2. Let $\Omega \subset R^3$ be a bounded domain with smooth boundary and $g \in L^\infty(\Omega)$. Using the method of this paper it is easy to see that under Assumption 2.1 a semigroup generated by

$$\begin{cases} \alpha(u_t) - \Delta u + f(u) = g(x), & (t,x) \in (0,\infty) \times \Omega, \\ u(t,x) = 0, & (t,x) \in (0,\infty) \times \partial\Omega \\ u(0,x) = u_0(x), & x \in \Omega, \end{cases} \tag{*}$$

possesses a global $(H_0^1(\Omega), H_0^1(\Omega))_{\mathfrak{B}}-$attractor $\mathcal{A}_{\mathfrak{B}}$, which is also a bounded subset of $L^\infty(\Omega)$, as mentioned in Remark 5.1. If, in addition to (2.2)-(2.3), the function $\alpha(\cdot)$ satisfies also the conditions imposed in [8], then as shown in [8] the semigroup generated by (*) possesses also a global $(H_0^1(\Omega) \cap L^\infty(\Omega), H_0^1(\Omega) \cap L^\infty(\Omega))-$attractor $\mathcal{A}_\infty$. Since $\mathcal{A}_{\mathfrak{B}}$ is an invariant, bounded subset of $H_0^1(\Omega) \cap L^\infty(\Omega)$, we have $\mathcal{A}_{\mathfrak{B}} \subset \mathcal{A}_\infty$. On the other hand, since $\mathcal{A}_\infty$ is an invariant element of $\mathfrak{B}$, as mentioned in Remark 2.1, we have $\mathcal{A}_\infty \subset \mathcal{A}_{\mathfrak{B}}$. Thus under additional conditions the attractor constructed here coincides with the attractor constructed in [8].

5. **Appendix.** To prove Lemma 4.1 we need the following lemma:

Lemma 5.1. *Let (2.2) and (2.4) hold. Also assume that* $w \in C([0,T] \times \overline{B_\rho})$, $\widehat{g} \in C(\overline{B_\rho})$, $v_0 = 0$ *and* $v \in C^2([0,T] \times \overline{B_\rho})$ *is a classical solution of (3.1). Then*

$$\|v\|_{C([0,T]\times\overline{B_\rho})} \le \frac{\mu_0}{\lambda} + \frac{1}{\lambda} \|\widehat{g}\|_{C(\overline{B_\rho})}, \tag{5.1}$$

where the positive constant μ_0 *depends only on* $f(\cdot)$.

Proof. By (2.4) it follows that there exists $M > 0$ such that

$$\inf_{|x|>M} f'(x) > 0 \tag{5.2}$$

Let $\mu_0 = \max\limits_{x,y\in[-M,M]} (f(x) - f(y))$. Let us show that

$$(f(x) - f(y)) \operatorname{sgn}(x-y) \ge -\mu_0, \quad \forall\, x,y \in R. \tag{5.3}$$

If $x,y \in [-M,M]$ then (5.3) is trivial. If $x,y > M$ or $x,y < -M$ then (5.3) follows from (5.2). If $x > M$ and $y < M$ $(y > M$ and $x < M)$ then by (2.4) and (5.2) we have

$$(f(x) - f(y)) \operatorname{sgn}(x-y) > f(M) - f(y) \ge -\mu_0$$
$$((f(x) - f(y)) \operatorname{sgn}(x-y) > f(M) - f(x) \ge -\mu_0 \;).$$

If $x < -M$ and $y > -M$ $(x > -M$ and $y < -M)$ then

$$(f(x) - f(y)) \operatorname{sgn}(x-y) > f(y) - f(-M) \ge -\mu_0$$
$$(\;(f(x) - f(y)) \operatorname{sgn}(x-y) > f(x) - f(-M) \ge -\mu_0 \;).$$

Now let $v(t_0,x_0) = \max\limits_{[0,T]\times\overline{B_\rho}} v(t,x)$. Since $v(0,x) \equiv 0$ we have $v(t_0,x_0) \ge 0$. If $(t_0,x_0) \in (0,T] \times B_\rho$ then from $(3.1)_1$ we obtain

$$\lambda v(t_0,x_0) + f(w(t_0,x_0)) - f(w(t_0,x_0) - v(t_0,x_0)) \le \widehat{g}(x_0)$$

which together with (5.3) yields

$$v(t_0,x_0) \le \frac{\mu_0}{\lambda} + \frac{1}{\lambda} \|\widehat{g}\|_{C(\overline{B_\rho})}$$

If $t_0 = 0$ or $x_0 \in \partial B_\rho$ then by the initial-boundary conditions it follows that

$$v(t_0, x_0) = 0$$

So we have

$$v(t,x) \le \frac{\mu_0}{\lambda} + \frac{1}{\lambda} \|\widehat{g}\|_{C(\overline{B_\rho})}, \qquad \forall (t,x) \in [0,T] \times \overline{B_\rho}. \tag{5.4}$$

Similarly one can show that if $v(t_1, x_1) = \min\limits_{[0,T]\times\overline{B_\rho}} v(t,x)$, then

$$v(t_1, x_1) \ge -\frac{\mu_0}{\lambda} - \frac{1}{\lambda} \|\widehat{g}\|_{C(\overline{B_\rho})}$$

and consequently

$$v(t,x) \ge -\frac{\mu_0}{\lambda} - \frac{1}{\lambda} \|\widehat{g}\|_{C(\overline{B_\rho})}, \qquad \forall (t,x) \in [0,T] \times \overline{B_\rho}.$$

The last inequality and (5.4) imply (5.1). □

Proof of Lemma 4.1. *Step1.* We first prove lemma for $w \in C^2([0,T] \times \overline{B_\rho})$ and $\widehat{g} \in C_0^3(\overline{B_\rho})$. Since $\alpha_m(\cdot)$ (for definition see proof of Lemma 4.2) satisfies (2.2)-(2.3) and $\alpha_m'(0) = \alpha'(0)$, by Lemma (3.1) we can say there exists a unique strong solution $v_m \in W^{1,2}(0,T; H_0^1(B_\rho)) \cap W_{loc}^{2,2}(0,T; L^2(B_\rho)) \cap L^\infty(0,T; H^2(B_\rho))$ of the following initial-boundary value problem:

$$\begin{cases} \alpha_m(v_{mt}) - \Delta v_m + \lambda v_m + f(w) - f(w - v_m) = \widehat{g}(x), (t,x) \in (0,T) \times B_\rho, \\ v_m(t,x) = 0, (t,x) \in (0,T) \times \partial B_\rho, \\ v_m(0,x) = 0, x \in B_\rho, \end{cases} \tag{5.6}$$

which satisfies the following inequality

$$\begin{aligned} &\|v_m(t)\|_{H^2(B_\rho)}^2 + \int_0^t \|\nabla v_{mt}(s)\|_{L^2(B_\rho)}^2 ds + \frac{\tau}{1+t} \int_\tau^t \|v_{mtt}(s)\|_{L^2(B_\rho)}^2 ds \\ &\le c_1(T, \|w\|_{L^2([0,T]; H^2(B_\rho))}, \|w_t\|_{L^2(0,T; H^1(B_\rho))}, \|\widehat{g}\|_{L^2(B_\rho)}), \\ &\qquad \forall\, m \in \mathbb{N}, \quad 0 < \tau \le t \le T, \end{aligned} \tag{5.7}$$

where $c_1 : R_+ \times R_+ \times R_+ \times R_+ \to R_+$ is a nondecreasing function with respect to each variable. From (5.6) and (5.7) it follows that $\alpha_m(v_{mt}) \in L^2(0,T; H_0^1(B_\rho))$ and $\frac{\partial}{\partial t}\alpha_m(v_{mt}) \in L^2(0,T; H^{-1}(B_\rho))$. So we have $\alpha_m(v_{mt}) \in C(0,T; L^2(B_\rho))$ and consequently $v_{mt} \in C(0,T; L^2(B_\rho))$. Denoting $h_{q,k}(s) = \begin{cases} k^q s, & |s| > k \\ |s|^q s, & |s| \le k \end{cases}$, we obtain that $h_{q,k}(v_{mt}) \in L^2(0,T; H_0^1(B_\rho)) \cap C(0,T; L^2(B_\rho))$, where $q > 0$. Differentiating both sides of $(5.6)_1$ with respect to t and testing by $h_{q,k}(v_{mt})$ we obtain

$$\begin{aligned} &c\int_0^t \left\| h_{\frac{q}{2},k}(v_{mt}(s)) \right\|_{H^1(B_\rho)}^2 ds + \int_0^t \left\langle \frac{\partial}{\partial t}\alpha_m(v_{mt}(s)), h_{q,k}(v_{mt}) \right\rangle ds \\ &\quad + \int_0^t \left\langle \frac{\partial}{\partial t}(f(w(s)) - f(w(s) - v_m(s))), h_{q,k}(v_{mt}) \right\rangle ds \le 0 \end{aligned} \tag{5.8}$$

Now for $m > \left\|\alpha^{-1}(\widehat{g})\right\|_{L^\infty(B_\rho)}$ let us estimate the second and third terms in (5.8):

$$\begin{aligned}\int_0^t \left\langle \frac{\partial}{\partial t}\alpha_m(v_{mt}(s)), h_{q,k}(v_{mt})\right\rangle ds &= \lim_{\tau\to 0^+}\int_\tau^t \left\langle \frac{\partial}{\partial t}\alpha_m(v_{mt}(s)), h_{q,k}(v_{mt})\right\rangle ds \\ &= \lim_{\tau\to 0^+}\int_\tau^t \left\langle \overline{\alpha}_m(v_{mt}(s))v_{mtt}(s), h_{q,k}(v_{mt})\right\rangle ds \\ &= \left\langle \widehat{\alpha}_m(v_{mt}(t)), 1\right\rangle - \lim_{\tau\to 0^+}\left\langle \widehat{\alpha}_m(v_{mt}(\tau)), 1\right\rangle,\end{aligned}$$

where $\widehat{\alpha}_{mk}(s) = \int_0^s \overline{\alpha}_m(\tau)h_{q,k}(\tau)d\tau$. Since $v_{mt} \in C(0,T;L^2(B_\rho))$ we have $\widehat{\alpha}_{mk}(v_{mt}) \in C(0,T;L^2(B_\rho))$ and taking into account it in the above equality we obtain

$$\begin{aligned}&\int_0^t \left\langle \frac{\partial}{\partial t}\alpha_m(v_{mt}(s)), h_{q,k}(v_{mt})\right\rangle ds \\ &\geq \alpha'(0)\frac{1}{q+2}\left\|h_{\frac{q}{2},k}(v_{mt}(t))\right\|^2_{L^2(B_\rho)} - \left\langle \widehat{\alpha}_{mk}(v_{mt}(0)), 1\right\rangle \\ &= \alpha'(0)\frac{1}{q+2}\left\|h_{\frac{q}{2},k}(v_{mt}(t))\right\|^2_{L^2(B_\rho)} - \left\langle \widehat{\alpha}_{mk}(\alpha_m^{-1}(\widehat{g})), 1\right\rangle \\ &\geq \alpha'(0)\frac{1}{q+2}\left\|h_{\frac{q}{2},k}(v_{mt}(t))\right\|^2_{L^2(B_\rho)} - \frac{1}{q+2}\int_{B_\rho}\widehat{g}(x)\left|\alpha^{-1}(\widehat{g}(x))\right|^{q+1}dx \\ &\geq \alpha'(0)\frac{1}{q+2}\left\|h_{\frac{q}{2},k}(v_{mt}(t))\right\|^2_{L^2(B_\rho)} \\ &\quad - \frac{1}{(q+2)\alpha'(0)}\left\|\alpha^{-1}(\widehat{g})\right\|^q_{L^\infty(B_\rho)}\left\|\widehat{g}\right\|^2_{L^2(B_\rho)}, \quad \forall\, t\in[0,T]. \qquad (5.9)\end{aligned}$$

$$\begin{aligned}&\int_0^t \left\langle \frac{\partial}{\partial t}f(w(s)-v_m(s)) - \frac{\partial}{\partial t}f(w(s)), h_{q,k}(v_{mt})\right\rangle ds \\ &= \int_0^t \left\langle (f'(w(s)-v_m(s)) - f'(w(s)))w_t(s), h_{q,k}(v_{mt})\right\rangle \\ &\quad - \int_0^t \left\langle f'(w(s)-v_m(s))v_{mt}, h_{q,k}(v_{mt})\right\rangle ds \\ &\leq c_2(T,\rho,\|w\|_{C^2([0,T]\times\overline{B_\rho})}, \|\widehat{g}\|_{L^2(B_\rho)})\left(\int_0^t \left\|h_{\frac{q}{2},k}(v_{mt}(s))\right\|^2_{H^1(B_\rho)}ds\right)^{\frac{q+1}{q+2}} \\ &\quad + c\int_0^t \left\|h_{\frac{q}{2},k}(v_{mt}(s))\right\|^2_{L^2(B_\rho)}ds, \quad \forall\, t\in[0,T], \qquad (5.10)\end{aligned}$$

where $c_2 : R_+ \times R_+ \times R_+ \times R_+ \to R_+$ is a nondecreasing function with respect to each variable. Taking into account (5.9)-(5.10) in (5.8) we obtain

$$\alpha'(0)\frac{1}{q+2}\left\|h_{\frac{q}{2},k}(v_{mt}(t))\right\|^2_{L^2(B_\rho)} \le \frac{1}{(q+2)\alpha'(0)}\left\|\alpha^{-1}(\widehat{g})\right\|^{q+1}_{L^\infty(B_\rho)}\left\|\widehat{g}\right\|^2_{L^2(B_\rho)}$$

$$+\frac{1}{c^{q+1}}\frac{1}{q+2}c_2^{q+2}(T,\rho,\|w\|_{C^2([0,T]\times\overline{B_\rho})},\|\widehat{g}\|_{L^2(B_\rho)})$$

$$+c\int_0^t \left\|h_{\frac{q}{2},k}(v_{mt}(s))\right\|^2_{L^2(B_\rho)} ds, \quad \forall\, t\in[0,T].$$

Applying Gronwall's lemma to the last inequality we find

$$\left\|h_{\frac{q}{2},k}(v_{mt}(t))\right\|^2_{L^2(B_\rho)} \le d_1 e^{d_2 t}, \quad \forall\, t\in[0,T], \tag{5.11}$$

where $d_1 = \frac{1}{(\alpha'(0))^2}\left\|\alpha^{-1}(\widehat{g})\right\|^{q+1}_{L^\infty(B_\rho)}\|\widehat{g}\|^2_{L^2(B_\rho)} +$
$+\frac{1}{c^{q+1}\alpha'(0)}c_2^{q+2}(T,\rho,\|w\|_{C^2([0,T]\times\overline{B_\rho})},\|\widehat{g}\|_{L^2(B_\rho)})$ and $d_2 = \frac{c(q+2)}{\alpha'(0)}$. Passing to the limit in (5.11) with respect to k we have

$$\|v_{mt}(t)\|_{L^{q+2}(B_\rho)} \le (d_1)^{\frac{1}{q+2}} e^{\frac{d_2}{q+2}t}, \quad \forall\, t\in[0,T].$$

Now passing to limit in the last inequality as $q \to \infty$ we obtain

$$\|v_{mt}(t)\|_{L^\infty((0,T)\times B_\rho)} \le M_0(T,\rho,\|w\|_{C^2([0,T]\times\overline{B_\rho})},\|\widehat{g}\|_{L^\infty(B_\rho)}), \tag{5.12}$$

where $M_0 : R_+ \times R_+ \times R_+ \times R_+ \to R_+$ is a nondecreasing function with respect to each variable. By (5.7) and (5.12) it is easy to see that

$$\begin{cases} v_m \to v \ \text{ weakly star in } L^\infty(0,T;H^2(B_\rho)) \\ v_{mt} \to v_t \text{ weakly in } L^2(0,T;H^1(B_\rho)) \\ v_{mt} \to v_t \ \text{ weakly star in } L^\infty((0,T)\times B_\rho) \\ v_{mtt} \to v_{tt} \text{ weakly in } L^2_{loc}(0,T;L^2(B_\rho)) \\ \alpha_m(v_{mt}) \to \alpha(v_t) \ \text{ weakly in } L^2(0,T;L^2(B_\rho)) \end{cases}$$

where $v(t,x)$ is the solution of (3.1) with initial data $v_0 = 0$. It is also clear that $v(t,x)$ satisfies (5.7) and (5.12).

Now let us consider the following initial-boundary value problem:

$$\begin{cases} \alpha_\varepsilon(v_t^\varepsilon) - \Delta v^\varepsilon + \lambda v^\varepsilon + f(w) - f(w - v^\varepsilon) = \widehat{g}(x), \quad (t,x)\in(0,T)\times B_\rho, \\ v^\varepsilon(t,x) = 0, \quad (t,x)\in(0,T)\times\partial B_\rho, \\ v^\varepsilon(0,x) = 0, \quad x\in B_\rho, \end{cases} \tag{5.13}$$

where $\alpha_\varepsilon \in C^3(R)$, $\alpha_\varepsilon \to \alpha$ strongly in $C^1[-M_0, M_0]$ (M_0 is the same as in (5.12)) as $\varepsilon \to 0^+$, and $\alpha'_\varepsilon(x) \ge \alpha'(0)$ for every $x \in R$. By Lemma 3.1 and the argument done above we can say that there exists a unique strong solution of (5.13) which satisfies (5.7) and (5.12). Moreover

$$\begin{cases} v^\varepsilon \to v \ \text{ weakly star in } L^\infty(0,T;H^2(B_\rho)) \\ v_t^\varepsilon \to v_t \text{ weakly in } L^2(0,T;H^1(B_\rho)) \\ v_t^\varepsilon \to v_t \ \text{ weakly star in } L^\infty((0,T)\times B_\rho) \\ v_{tt}^\varepsilon \to v_{tt} \text{ weakly in } L^2_{loc}(0,T;L^2(B_\rho)) \\ \alpha_\varepsilon(v_t^\varepsilon) \to \alpha(v_t) \ \text{ weakly in } L^2(0,T;L^2(B_\rho)) \end{cases} \tag{5.14}$$

Since $v^\varepsilon(t,x)$ satisfies (5.12) by $(5.13)_1$ we have

$$-\Delta v^\varepsilon + \lambda v^\varepsilon + f(w) - f(w - v^\varepsilon) = \widehat{g}(x) - \alpha_\varepsilon(v_t^\varepsilon) \in L^\infty((0,T)\times B_\rho) \tag{5.15}$$

Since $w \in C^2([0,T] \times \overline{B_\rho})$ from condition (2.4) it follows that there exists $M_1 = M_1(\|w\|_{C([0,T]\times\overline{B_\rho})}) > 0$ such that

$$(f(w(t,x)) - f(w(t,x)-v))v > 0, \qquad \forall\, (t,x) \in [0,T] \times \overline{B_\rho}$$

for $|v| \geq M_1$. Setting $v^\varepsilon_M(t,x) = \begin{cases} v^\varepsilon(t,x) - M, & v^\varepsilon(t,x) > M \\ 0, & |v^\varepsilon(t,x)| \leq M \\ v^\varepsilon(t,x) + M, & v^\varepsilon(t,x) < -M \end{cases}$ and testing (5.15) by $v^\varepsilon_M(t,x)$ we obtain

$$\lambda M \|v^\varepsilon_M(t,x)\|_{L^1((0,T)\times B_\rho)} \leq \|\widehat{g}(x) - \alpha_\varepsilon(v^\varepsilon_t)\|_{L^\infty((0,T)\times B_\rho)} \|v^\varepsilon_M(t,x)\|_{L^1((0,T)\times B_\rho)}$$

and consequently

$$\|v^\varepsilon_M(t,x)\|_{L^1((0,T)\times B_\rho)} = 0,$$

for every $M > \max\left\{M_1, \frac{1}{\lambda}\|\widehat{g}(x) - \alpha_\varepsilon(v^\varepsilon_t)\|_{L^\infty((0,T)\times B_\rho)}\right\}$. The last equality means that $v^\varepsilon \in L^\infty((0,T)\times B_\rho)$, which together with (5.15) yields

$$v^\varepsilon \in L^\infty(0,T; W^{2,\infty}(B_\rho)) \tag{5.16}$$

Differentiating both sides of (5.13) with respect to t we have

$$\begin{cases} \varphi_t - \Delta\alpha_\varepsilon^{-1}(\varphi) + \lambda\alpha_\varepsilon^{-1}(\varphi) + f_1(t,x)\alpha_\varepsilon^{-1}(\varphi) + f_2(t,x) = 0, & (t,x) \in (0,T)\times B_\rho, \\ \varphi(t,x) = 0, \quad (t,x) \in (0,T)\times\partial B_\rho, \\ \varphi(0,x) = \widehat{g}(x), \quad x \in B_\rho, \end{cases} \tag{5.17}$$

where $\varphi(t,x) = \alpha_\varepsilon(v^\varepsilon_t(t,x))$, $f_1(t,x) = f'(w(t,x) - v^\varepsilon(t,x))$ and $f_2(t,x) = (f'(w(t,x) - f'(w(t,x) - v^\varepsilon(t,x)))w_t(t,x)$. Since $v^\varepsilon(t,x)$ satisfies (5.12) and (5.16), applying [12, Theorem 6.1, p.513] to (5.17) we find that $\varphi \in H^{2+\beta,1+\frac{\beta}{2}}([0,T]\times\overline{B_\rho})$ and consequently $v^\varepsilon(t,x) = \int_0^t \alpha_\varepsilon^{-1}(\varphi(s,x))ds \in C^2([0,T]\times\overline{B_\rho})$. Now we can apply Lemma 5.1 to (5.13) which gives us the following estimate:

$$\|v^\varepsilon(t,x)\|_{L^\infty((0,T)\times B_\rho)} \leq \frac{\mu_0}{\lambda} + \frac{1}{\lambda}\|\widehat{g}\|_{L^\infty(B_\rho)}.$$

The last inequality together with (5.14) yields (4.2).

Step2. Let $w \in L^2(0,T; H^2(B_\rho) \cap H^1_0(B_\rho))$, $w_t \in L^2(0,T; H^1_0(B_\rho))$, $\widehat{g} \in L^\infty(B_\rho)$ and $v_0 = 0$. Then by Lemma 3.1, the problem (3.1) has a unique strong solution $v \in W^{1,2}(0,T; H^1_0(B_\rho)) \cap W^{2,2}_{loc}(0,T; L^2(B_\rho)) \cap L^\infty(0,T; H^2(B_\rho))$. By the density there are $\{w_k\}_{k=1}^\infty \subset C^2([0,T]\times\overline{B_\rho})$ and $\{\widehat{g}_k\}_{k=1}^\infty \subset C^3(\overline{B_\rho})$ such that

$$\begin{cases} w_k \to w \text{ strongly in } L^2(0,T; H^2(B_\rho)) \\ w_{kt} \to w_t \text{ strongly in } L^2(0,T; H^1(B_\rho)) \\ \widehat{g}_k \to \widehat{g} \text{ strongly in } L^2(B_\rho) \\ \sup\limits_k \|\widehat{g}_k\|_{L^\infty(B_\rho)} \leq \|\widehat{g}\|_{L^\infty(B_\rho)} \end{cases} \tag{5.18}$$

Put $w_k(t,x)$ instead of $w(t,x)$ and $\widehat{g}_k(x)$ instead of $\widehat{g}(x)$ in $(3.1)_1$. Then by the arguments done in Step 1, we can say that there exists a unique strong solution $v_k(t,x)$ of

$$\begin{cases} \alpha(v_{kt}) - \Delta v_k + \lambda v_k + f(w_k) - f(w_k - v_k) = \widehat{g}_k(x), & (t,x) \in (0,T)\times B_\rho, \\ v_k(t,x) = 0, \quad (t,x) \in (0,T)\times\partial B_\rho, \\ v_k(0,x) = 0, \quad x \in B_\rho, \end{cases}$$

which (thanks to $(5.18)_4$) satisfies (4.2). On the other hand multiplying both sides of

$$\begin{aligned}&\alpha(v_t)-\alpha(v_{kt})-\Delta(v-v_k)+\lambda(v-v_k)\\=&f(w_k)-f(w)+f(w-v)-f(w_k-v_k)+\widehat{g}(x)-\widehat{g}_k(x),\end{aligned}$$

by $(v_t - v_{kt})$ and integrating over $(0,t)\times B_\rho$ we have

$$\|v(t)-v_k(t)\|^2_{H^1(B_\rho)}+\|v_t(t)-v_{kt}(t)\|^2_{L^2(B_\rho)}\le c\int_0^T\|w_k(s)-w(s)\|^2_{L^2(B_\rho)}\,ds$$

$$+cT\,\|\widehat{g}-\widehat{g}_k\|^2_{L^2(B_\rho)}+c\int_0^t\|v(t)-v_k(t)\|^2_{H^1(B_\rho)}\,ds,\qquad \forall\, t\in[0,T].$$

Taking into account $(5.18)_1$-$(5.18)_3$ in the last inequality we obtain

$$\begin{cases} v_k\to v \text{ strongly in } L^\infty(0,T;H^1(B_\rho))\\ v_{kt}\to v_t \text{ strongly in } L^2(0,T;L^2(B_\rho))\end{cases}$$

Thus since $v_k(t,x)$ satisfies (4.2), it yields that $v(t,x)$ also satisfies (4.2).

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