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Lebesgue and co-Lebesgue di-uniform texture spaces

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ABSTRACT

The author introduces the notions of Lebesgue di-uniformity and co Lebesgue di-uniformity and investigates the relationship between a Lebesgue quasi uniformity on X and the corresponding Lebesgue di-uniformity on the discrete texture $(X, \mathcal{P}(X))$. Finally a notion of a dual dicovering Lebesgue quasi di-uniform texture space is introduced and several properties are discussed.

1. Introduction

Textures and ditopological texture spaces were first introduced by L.M. Brown as a point-based setting for the study of fuzzy topology. However, the development of the theory has proceeded largely independently of this context. In particular this is true of the work on di-uniformities [8] which gives the foundations for a theory of uniformities in a textural setting, and provides a more unified setting for the study of quasi uniformities and uniformities than does the classical approach.

In [10] the authors investigated the effect of a complementation on a direlational uniformity and showed that although a direlational uniformity on a discrete texture corresponds to a quasi uniformity, a complemented di-uniformity corresponds to a uniformity.

Since covers cannot be used to define a quasi uniformity, T.E. Garnter and R.G. Steinlage [6] introduced the notion of *pairs of covers* having a common index. In the meantime L.M. Brown [1] introduced independently the notion of *dual cover*, and S. Romaguera and J. Marin [12] the closely related notion of a *pair open cover* of a quasi uniform space.

The notion of quasi di-uniformity was introduced by the author in [11] by removing the symmetry condition in the definition of a direlational uniformity, and instead of dicovers, which are the textural analogue of dual covers, dual dicovers were used to characterize quasi di-uniformities. In [7] J. Marin and S. Romaguera introduce a notion of Lebesgue quasi uniformity in terms of pair open covers.

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In this paper we will introduce the notions of Lebesgue di-uniformity and co-Lebesgue di-uniformity, and since di-uniformities on discrete textures correspond to quasi uniformities we will investigate the relationship between a Lebesgue quasi uniformity on X and the corresponding Lebesgue di-uniformity on the discrete texture $(X, \mathcal{P}(X))$.

We conclude this paper by defining dual dicovering Lebesgue quasi di-uniformities, which are the textural analogue of pair Lebesgue quasi uniformities in the sense of J. Marin and S. Romaguera.

This section concludes with some basic definitions from the theory, and the reader is referred to [3,4,8–10] for more background material.

Texture. ([3]) Let S be a nonempty set. We recall that a *texturing* on $\mathcal{S}$ is a point-separating, complete, completely distributive lattice $\mathcal{S}$ of subsets of S with respect to inclusion, which contains S and $\emptyset$, and for which arbitrary meet $\bigwedge$ coincides with intersection $\bigcap$ and finite joins $\bigvee$ with unions $\bigcup$. The pair $(S, \mathcal{S})$ is called a *texture*. For $s \in S$ the sets

$$P_s = \bigcap\{A \in \mathcal{S} \mid s \in A\} \quad \text{and} \quad Q_s = \bigvee\{A \in \mathcal{S} \mid s \notin A\} = \bigvee\{P_u \mid u \in S,\ s \notin P_u\}$$

are called respectively, the p-sets and q-sets of $(S, \mathcal{S})$. These sets are used in the definition of many textural concepts. We note in particular that $S^\flat = \{s \in S \mid S \not\subseteq Q_s\}$ is called the *core* of S. In general, $S^\flat \notin \mathcal{S}$.

In general, a texturing of S need not be closed under set complementation, but sometimes we have a notion of complementation.

Complementation. ([3]) A mapping $\sigma : \mathcal{S} \to \mathcal{S}$ satisfying $\sigma(\sigma(A)) = A$, $\forall A \in \mathcal{S}$ and $A \subseteq B \Rightarrow \sigma(B) \subseteq \sigma(A)$, $\forall A, B \in \mathcal{S}$ is called a *complementation* on $(S, \mathcal{S})$ and $(S, \mathcal{S}, \sigma)$ is then said to be a *complemented texture*.

Example 1.1.

(1) For any set X, $(X, \mathcal{P}(X), \pi_X)$, $\pi_X(Y) = X \setminus Y$ for $Y \subseteq X$, is the complemented *discrete texture* representing the usual set structure of X. Clearly, $P_x = \{x\}$ and $Q_x = X \setminus \{x\}$ for all $x \in X$.
(2) For $\mathbb{I} = [0, 1]$ define $\mathcal{I} = \{[0, t] \mid t \in [0, 1]\} \cup \{[0, t) \mid t \in [0, 1]\}$, $\iota([0, t]) = [0, 1 - t)$ and $\iota([0, t)) = [0, 1 - t]$, $t \in [0, 1]$. Then $(\mathbb{I}, \mathcal{I}, \iota)$ is a complemented texture, which we will refer to as the *unit interval texture*. Here $P_t = [0, t]$ and $Q_t = [0, t)$ for all $t \in I$.

Ditopology. A *dichotomous topology* on $(S, \mathcal{S})$ or *ditopology* for short, is a pair (τ, κ) of subsets of $\mathcal{S}$, where the set of *open sets* τ satisfies

(1) $S, \emptyset \in \tau$,
(2) $G_1, G_2 \in \tau \Rightarrow G_1 \cap G_2 \in \tau$ and
(3) $G_i \in \tau,\ i \in I \Rightarrow \bigvee_i G_i \in \tau$,

and the set of *closed sets* κ satisfies

(1) $S, \emptyset \in \kappa$,
(2) $K_1, K_2 \in \kappa \Rightarrow K_1 \cup K_2 \in \kappa$ and
(3) $K_i \in \kappa,\ i \in I \Rightarrow \bigcap K_i \in \kappa$.

If (τ, κ) is a ditopology on a complemented texture $(S, \mathcal{S}, \sigma)$ we say (τ, κ) *is complemented* if $\kappa = \sigma[\tau]$.

Let $(S, \mathcal{S})$, $(T, \mathcal{T})$ be textures. In the following definition we consider $\mathcal{P}(S) \otimes \mathcal{T}$. To avoid confusion $\overline{P}_{(s,t)}$, $\overline{Q}_{(s,t)}$ are used to denote the p-sets and q-sets for $(S \times T, \mathcal{P}(S) \otimes \mathcal{T})$. Hence (see [4]) we have $\overline{P}_{(s,t)} = \{s\} \times P_t$ and $\overline{Q}_{(s,t)} = [(S \setminus \{s\}) \times T] \cup [S \times Q_t]$. Now let us recall

Direlations. ([4]) Let $(S, \mathcal{S})$, $(T, \mathcal{T})$ be textures.

(1) $r \in \mathcal{P}(S) \otimes \mathcal{T}$ is called a *relation from* $(S, \mathcal{S})$ *to* $(T, \mathcal{T})$ if it satisfies
R1 $r \not\subseteq \overline{Q}_{(s,t)}$, $P_{s'} \not\subseteq Q_s \Rightarrow r \not\subseteq \overline{Q}_{(s',t)}$.
R2 $r \not\subseteq \overline{Q}_{(s,t)} \Rightarrow \exists s' \in S$ such that $P_s \not\subseteq Q_{s'}$ and $r \not\subseteq \overline{Q}_{(s',t)}$.
(2) $R \in \mathcal{P}(S) \otimes \mathcal{T}$ is called a *co-relation from* $(S, \mathcal{S})$ *to* $(T, \mathcal{T})$ if it satisfies
CR1 $\overline{P}_{(s,t)} \not\subseteq R$, $P_s \not\subseteq Q_{s'} \Rightarrow \overline{P}_{(s',t)} \not\subseteq R$.
CR2 $\overline{P}_{(s,t)} \not\subseteq R \Rightarrow \exists s' \in S$ such that $P_{s'} \not\subseteq Q_s$ and $\overline{P}_{(s',t)} \not\subseteq R$.

A pair (r, R) consisting of a relation r and co-relation R is now called a *direlation*. We will denote by $\mathcal{DR}$ the family of all direlations on a given texture.

Direlations are ordered by $(r_1, R_1) \sqsubseteq (r_2, R_2) \Leftrightarrow r_1 \subseteq r_2$ and $R_2 \subseteq R_1$.

For a general texture $(S, \mathcal{S})$ we define

$$i = i_S = \bigvee \{\overline{P}_{(s,s)} \mid s \in S\} \quad \text{and} \quad I = I_S = \bigcap \{\overline{Q}_{(s,s)} \mid s \in S\}.$$

We refer to (i, I) as the *identity direlation* on $(S, \mathcal{S})$.

A direlation (r, R) on $(S, \mathcal{S})$ (that is, on $(S, \mathcal{S})$ to $(S, \mathcal{S})$) is *reflexive* if r and R are reflexive, that is if $(i, I) \sqsubseteq (r, R)$. We will denote by $\mathcal{RDR}$ the family of reflexive direlations on a given texture.

Inverse of a direlation. ([4]) The *inverse* of (r, R) from $(S, \mathcal{S})$ to $(T, \mathcal{T})$ is the direlation $(r, R)^{\leftarrow} = (R^{\leftarrow}, r^{\leftarrow})$ from $(T, \mathcal{T})$ to $(S, \mathcal{S})$ given by

$$r^{\leftarrow} = \bigcap \{\overline{Q}_{(t,s)} \mid r \not\subseteq \overline{Q}_{(s,t)}\}, \qquad R^{\leftarrow} = \bigvee \{\overline{P}_{(t,s)} \mid \overline{P}_{(s,t)} \not\subseteq R\}.$$

***A*-section of *r*.** ([4]) Let $(S, \mathcal{S})$ and $(T, \mathcal{T})$ be texture spaces and (r, R) a direlation from $(S, \mathcal{S})$ to $(T, \mathcal{T})$. The *A-section of r* is the element $r^{\rightarrow} A$ of $\mathcal{T}$ defined by

$$r^{\rightarrow} A = \bigcap \{Q_t \mid \forall s,\ r \not\subseteq \overline{Q}_{(s,t)} \Rightarrow A \subseteq Q_s\}.$$

***A*-section of *R*.** *A-section of R* is the element $R^{\rightarrow} A$ of $\mathcal{T}$ defined by

$$R^{\rightarrow} A = \bigvee \{P_t \mid \forall s,\ \overline{P}_{(s,t)} \not\subseteq R \Rightarrow P_s \subseteq A\}.$$

Complement of a direlation. ([4]) Let (r, R) be a direlation between the complemented textures $(S, \mathcal{S}, \sigma)$ and $(T, \mathcal{T}, \theta)$.

(1) The *complement r' of the relation r* is the co-relation

$$r' = \bigcap \left\{\overline{Q}_{(s,t)} \,\middle|\, \exists u, v \text{ with } r \not\subseteq \overline{Q}_{(u,v)},\ \sigma(Q_s) \not\subseteq Q_u \text{ and } P_v \not\subseteq \theta(P_t)\right\}.$$

(2) The *complement R' of the co-relation R* is the relation

$$R' = \bigvee \left\{\overline{P}_{(s,t)} \,\middle|\, \exists u, v \text{ with } \overline{P}_{(u,v)} \not\subseteq R,\ P_u \not\subseteq \sigma(P_s) \text{ and } \theta(Q_t) \not\subseteq Q_v\right\}.$$

(3) The *complement $(r, R)'$ of the direlation (r, R)* is the direlation

$$(r, R)' = (R', r').$$

A direlation (r, R) on $(S, \mathcal{S})$ is said to be *complemented* if $(r, R)' = (r, R)$.

Direlational uniformity. ([8]) Let $(S, \mathcal{S})$ be a texture and $\mathcal{U}$ a family of direlations from $(S, \mathcal{S})$ to $(S, \mathcal{S})$. If $\mathcal{U}$ satisfies the conditions

(1) $(i, I) \sqsubseteq (d, D)$ for all $(d, D) \in \mathcal{U}$. That is, $\mathcal{U} \subseteq \mathcal{RDR}$.
(2) $(d, D) \in \mathcal{U}$, $(e, E) \in \mathcal{DR}$ and $(d, D) \sqsubseteq (e, E)$ implies $(e, E) \in \mathcal{U}$.
(3) $(d, D), (e, E) \in \mathcal{U}$ implies $(d, D) \sqcap (e, E) \in \mathcal{U}$.
(4) Given $(d, D) \in \mathcal{U}$ there exists $(e, E) \in \mathcal{U}$ satisfying $(e, E) \circ (e, E) \sqsubseteq (d, D)$.
(5) Given $(d, D) \in \mathcal{U}$ there exists $(c, C) \in \mathcal{U}$ satisfying $(c, C)^{\leftarrow} \sqsubseteq (d, D)$.

Then $\mathcal{U}$ is called a *direlational uniformity* on $(S, \mathcal{S})$, and $(S, \mathcal{S}, \mathcal{U})$ is known as a *direlational uniform texture space*.

For a given direlational uniformity $\mathcal{U}$ on $(S, \mathcal{S}, \sigma)$ the direlational uniformity $\mathcal{U}' = \{(d, D)' \mid (d, D) \in \mathcal{U}\}$ is called the *complement* of $\mathcal{U}$. The di-uniformity $\mathcal{U}$ is said to be *complemented* if $\mathcal{U}' = \mathcal{U}$.

Example 1.2. ([8]) Let $(\mathbb{I}, \mathcal{I})$ be the unit interval texture and for $\epsilon > 0$ define $d_\epsilon = \{(r, s) \mid r, s \in \mathbb{I},\ s < r + \epsilon\}$, $D_\epsilon = \{(r, s) \mid r, s \in \mathbb{I},\ s \leqslant r - \epsilon\}$. Clearly (d_ϵ, D_ϵ) is a reflexive, symmetric direlation on $(\mathbb{I}, \mathcal{I})$ and

$$\mathcal{U}_{\mathbb{I}} = \left\{(d, D) \,\middle|\, (d, D) \in \mathcal{DR} \text{ and } \exists \epsilon > 0 \text{ with } (d_\epsilon, D_\epsilon) \sqsubseteq (d, D)\right\}$$

is a direlational uniformity on $(\mathbb{I}, \mathcal{I})$. We will call $\mathcal{U}_{\mathbb{I}}$ *the usual direlational uniformity on* $(\mathbb{I}, \mathcal{I})$.

Dicovers. A difamily $\mathcal{C} = \{(A_j, B_j) \mid j \in J\}$ of elements of $\mathcal{S} \times \mathcal{S}$ which satisfies $\bigcap_{j \in J_1} B_j \subseteq \bigvee_{j \in J_2} A_j$ for all partitions (J_1, J_2) of J, including the trivial partitions, is called a *dicover* of $(S, \mathcal{S})$. If $\mathcal{D}$ is a dicover we often write $L\mathcal{D}M$ in place of $(L, M) \in \mathcal{D}$. We recall the following notions for dicovers.

$\mathcal{C}$ is a *refinement* of $\mathcal{D}$ if given $j \in J$ we have $L\mathcal{D}M$ so that $A_j \subseteq L$ and $M \subseteq B_j$. In this case we write $\mathcal{C} \prec \mathcal{D}$.

Let (d, D) be a reflexive direlation on $(S, \mathcal{S})$ and for $s \in S$ let $d[s] = d^{\to} P_s$ and $D[s] = D^{\to} Q_s$. Then

$$\gamma(d, D) = \{(d[s], D[s]) \mid s \in S^{\flat}\}$$

is an anchored dicover of $(S, \mathcal{S})$. In this way we may obtain a dicovering uniformity corresponding to a direlational uniformity. The term *di-uniformity* applies to both direlational and dicovering uniformities.

Uniform ditopology. ([10]) Just as a uniformity in the classical sense determines a topology called the uniform topology, so a di-uniformity determines a ditopology called the *uniform ditopology*. Let $(S, \mathcal{S}, \mathcal{U})$ be a direlational uniform texture space with uniform ditopology $(\tau_{\mathcal{U}}, \kappa_{\mathcal{U}})$.

(i) $G \in \tau_{\mathcal{U}} \Leftrightarrow (G \not\subseteq Q_s \Rightarrow \exists (d, D) \in \mathcal{U}$ with $d[s] \subseteq G)$.
(ii) $K \in \kappa_{\mathcal{U}} \Leftrightarrow (P_s \not\subseteq K \Rightarrow \exists (d, D) \in \mathcal{U}$ with $K \subseteq D[s])$.

2. Lebesgue di-uniformities

In this section we introduce the notion of Lebesgue di-uniformity and co-Lebesgue di-uniformity. We will also investigate the effect of a complementation on a direlational uniformity and see the relation between Lebesgue di-uniformities and co-Lebesgue di-uniformities.

We recall [5] that a quasi uniformity $\mathcal{U}$ on a set X is a *Lebesgue quasi uniformity* provided that for each $\tau(\mathcal{U})$-open cover $\mathcal{G}$ of X there is $U \in \mathcal{U}$ such that the cover $\{U(x)\colon x \in X\}$ refines $\mathcal{G}$. The pair $(X, \mathcal{U})$ is then called a Lebesgue quasi uniform space.

Let (τ, κ) be a ditopology on the texture space $(S, \mathcal{S})$. The family $\{G_i \mid i \in I\}$ is said to be an *open cover* [2] of S if $G_i \in \tau$ for all $i \in I$ and $S = \bigvee_{i \in I} G_i$. Dually we may speak of a *closed cocover* of $\emptyset$, namely a family $\{F_i \mid i \in I\}$ with $F_i \in \kappa$ for all $i \in I$ satisfying $\bigcap_{i \in I} F_i = \emptyset$. For the cocovers we need a notion of dual refinement.

Definition 2.1. Let $\mathcal{K}_1, \mathcal{K}_2$ be cocovers. Then $\mathcal{K}_1$ will be called a *dual refinement* of $\mathcal{K}_2$, and write $\mathcal{K}_1 \lhd \mathcal{K}_2$ if for a given $K_2 \in \mathcal{K}_2$ there exists $K_1 \in \mathcal{K}_1$ such that $K_1 \subseteq K_2$.

Definition 2.2. A di-uniformity $\mathcal{U}$ on a texture space $(S, \mathcal{S})$ is called

(1) A *Lebesgue direlational uniformity* provided that for each cover $\mathcal{C}$ of S which is open for the uniform ditopology there is a direlation $(r, R) \in \mathcal{U}$ such that $\{r[s] \mid s \in S^{\flat}\}$ is a refinement of $\mathcal{C}$.
(2) A *co-Lebesgue direlational uniformity* provided that for each cocover $\mathcal{K}$ of $\emptyset$ which is closed for the uniform ditopology there is a direlation $(r, R) \in \mathcal{U}$ such that $\mathcal{K}$ is a dual refinement of $\{R[s] \mid s \in S^{\flat}\}$.

By identifying direlational uniformities on the discrete texture $(X, \mathcal{P}(X), \pi_X)$, $\pi_X(Y) = X \setminus Y$, with diagonal quasi-uniformities on X, as is done on [5], we now show that the above definitions do indeed generalize the classical ones.

Let $d \subseteq X \times X$ be a point relation. Recall [5] that $u(d) = (d, d^{\leftarrow})$ is a direlation on $(X, \mathcal{P}(X))$ and if $\mathcal{Q}$ is a diagonal quasi uniformity on X then the family

$$u(\mathcal{Q}) = \{(e, E) \mid \exists d \in \mathcal{Q} \text{ and } u(d) \sqsubseteq (e, E)\}$$

is a direlational uniformity on the discrete texture $(X, \mathcal{P}(X))$. Indeed, u sets up a bijection between the diagonal quasi-uniformities on S and the direlational uniformities on $(X, \mathcal{P}(X))$ since it is a bijection between the binary point relations on X and the symmetric direlations on $(X, \mathcal{P}(X))$.

If $\mathcal{Q}$ be a quasi-uniformity on X then $\mathcal{Q}^{-1} = \{d^{-1} \mid d \in \mathcal{Q}\}$ is also a quasi-uniformity on X, called the *conjugate* of $\mathcal{Q}$. Note that a quasi-uniformity $\mathcal{Q}$ on X gives rise to a bitopological space $(X, \mathcal{T}_{\mathcal{Q}}, \mathcal{T}_{\mathcal{Q}^{-1}})$, where $\mathcal{T}_{\mathcal{Q}}$ is the topology generated by $\mathcal{Q}$ and $\mathcal{T}_{\mathcal{Q}^{-1}}$ that generated by $\mathcal{Q}^{-1}$. As shown in [10] it follows that $(\mathcal{T}_{\mathcal{Q}}, \mathcal{T}^c_{\mathcal{Q}^{-1}})$, $\mathcal{T}^c_{\mathcal{Q}^{-1}} = \pi_X[\mathcal{T}_{\mathcal{Q}^{-1}}]$, is the uniform ditopology of $u(\mathcal{Q})$.

Now we have the following theorems.

Theorem 2.3. *Let $\mathcal{Q}$ be a Lebesgue quasi uniformity on X. Then the corresponding di-uniformity $u(\mathcal{Q})$ on $(X, \mathcal{P}(X), \pi_X)$ is a Lebesgue direlational uniformity.*

Conversely if $\mathcal{U}$ is a Lebesgue direlational uniformity on $(X, \mathcal{P}(X), \pi_X)$ then $u^{-1}(\mathcal{U})$ is a Lebesgue quasi uniformity on X.

Proof. Since $\mathcal{Q}$ is a quasi uniformity on X, $u(\mathcal{Q})$ is a di-uniformity on $(X, \mathcal{P}(X), \pi_X)$ by [10, Theorem 3.3]. Let $\mathcal{C}$ be an open cover of X. Since $\mathcal{Q}$ is a Lebesgue quasi uniformity there exists $r \in \mathcal{Q}$ such that $\{r(x) \mid x \in X\} \prec \mathcal{C}$. If $r \in \mathcal{P}(X \times X)$ is regarded as a relation then $r^{\rightarrow} P_x = r[x] = r(x)$ by [10, Lemma 3.1] and $u(r) = (r, r^{\leftarrow})$ is a direlation on $(X, \mathcal{P}(X), \pi_X)$ by [10, Definition 3.2] so it follows that $\{r[x] \mid x \in X\}$ refines $\mathcal{C}$.

Conversely, the proof of $u^{-1}(\mathcal{U})$ is a Lebesgue quasi uniformity on X when $\mathcal{U}$ is a Lebesgue direlational uniformity is dual to above and is omitted. □

Proposition 2.4. *([10, Proposition 3.4]) Let $\mathcal{Q}$ be a quasi-uniformity on X and $\mathcal{Q}^{-1}$ its conjugate. Then the direlational uniformity on $(X, \mathcal{P}(X), \pi_X)$ corresponding to $\mathcal{Q}^{-1}$ is the complement of the direlational uniformity corresponding to $\mathcal{Q}$. That is,*

$$u(\mathcal{Q}^{-1}) = u(\mathcal{Q})'.$$

Theorem 2.5. *Let $\mathcal{Q}$ be Lebesgue quasi uniformity on X. Then the complement of the direlational uniformity corresponding to $\mathcal{Q}$, that is $u(\mathcal{Q})'$, is a co-Lebesgue direlational uniformity on $(X, \mathcal{P}(X), \pi_X)$.*

Conversely, if $\mathcal{U}$ is the co-Lebesgue direlational uniformity corresponding to $\mathcal{Q}^{-1}$, then $u^{-1}(\mathcal{U}')$ is a Lebesgue quasi uniformity on X.

Proof. To show $u(\mathcal{Q})'$ is a co-Lebesgue direlational uniformity on $(X, \mathcal{P}(X), \pi_X)$ it will suffice to show that $u(\mathcal{Q}^{-1})$ is a co-Lebesgue direlational uniformity since $u(\mathcal{Q}^{-1}) = u(\mathcal{Q})'$ by [10, Proposition 3.4].

Let $\mathcal{F} = \{F_i \mid i \in I\}$ be a family of closed sets with $\bigcap_{i \in I} \mathcal{F} = \emptyset$. For $F_i \in \kappa_{u(\mathcal{Q}^{-1})}$ we have $X \setminus F_i \in \tau_{u(\mathcal{Q}^{-1})}$ and $\mathcal{G} = \{X \setminus F_i \mid i \in I\}$ is an open cover of $(X, \tau_{\mathcal{Q}^{-1}})$ since

$$\bigvee \mathcal{G} = \bigvee \{X \setminus F_i \mid i \in I\} = X \setminus \bigcap \{F_i \mid i \in I\} = X \setminus \emptyset = X.$$

If $\mathcal{Q}$ is a Lebesgue quasi uniformity there exists $r \in \mathcal{Q}$ such that $r[x] \subseteq X \setminus F_i$ for $x \in X$ and $i \in I$. We also have $r^{-1} \in Q^{-1}$ such that $u(r^{-1}) = (r^{-1}, (r^{-1})^{\leftarrow})$ is a direlation by [10, Definition 3.2]. Hence

$$F_i \subseteq X \setminus r[x] = (r^{-1})^{\leftarrow}[x]$$

which shows $u(\mathcal{Q}^{-1})$ is a co-Lebesgue direlational uniformity on $(X, \mathcal{P}(X), \pi_X)$.

Conversely let $\{G_i \mid i \in I\}$ be an open cover of X such that $G_i \in \tau_{u^{-1}(\mathcal{U}')}$ and $\bigcup G_i = X$. Then $\{X \setminus G_i \mid i \in I\}$ is a family of closed sets satisfying $\bigcap (X \setminus G_i) = \emptyset$ and $X \setminus G_i \in \kappa_{u^{-1}(\mathcal{U}')}$. Since $\mathcal{U} = u(\mathcal{Q}^{-1}) = u(\mathcal{Q})'$ is a co-Lebesgue direlational uniformity on X there exists $(r, R) \in \mathcal{U}$ such that $X \setminus G_i \subseteq R[x]$ for $x \in X$. Since $(r, R) \in u(\mathcal{Q}^{-1})$ we have $d^{-1} \in \mathcal{Q}^{-1}$ satisfying $(d^{-1}, (d^{-1})^{\leftarrow}) \sqsubseteq (r, R)$. Hence we have

$$d[x] = X \setminus (d^{-1})^{\leftarrow}[x] \subseteq X \setminus R[x] \subseteq G_i,$$

which shows $u^{-1}(\mathcal{U}')$ is a Lebesgue quasi uniformity on X. □

Proposition 2.6. *Let $(S, \mathcal{S})$ be a texture.*

(1) *If $\mathcal{U}$ is a direlational uniformity on $(S, \mathcal{S})$ for which $\tau_{\mathcal{U}}$ is compact then $\mathcal{U}$ is a Lebesgue direlational uniformity on $(S, \mathcal{S})$.*
(2) *If $\mathcal{U}$ is a direlational uniformity on $(S, \mathcal{S})$ such that $\kappa_{\mathcal{U}}$ co-compact then $\mathcal{U}$ is a co-Lebesque direlational uniformity on $(S, \mathcal{S})$.*

Proof. (1) Let $\mathcal{C}$ be an open cover of $(S, \mathcal{S}, \tau_{\mathcal{U}})$. For $s \in S^\flat$ there exists $C_s \in \mathcal{C}$ with $C_s \not\subseteq Q_s$, and since $C_s \in \tau_{\mathcal{U}}$ there exists $(d_s, D_s) \in \mathcal{U}$ with $d_s[s] \subseteq C_s$. We may choose $(e_s, E_s) \in \mathcal{U}$ with $(e_s, E_s)^2 \sqsubseteq (d_s, D_s)$. By the proof of [8, Proposition 2.7] we have $P_s \subseteq]e_s[s][$, where for $A \in \mathcal{S}$, $]A[$ denotes the interior of A, so $\{]e_s[s][\mid s \in S^\flat\}$ is an open cover of $(S, \mathcal{S}, \tau_{\mathcal{U}})$. By compactness we have $s_1, s_2, \ldots, s_n \in S^\flat$ for which $S = \bigcup_{k=1}^{n}]e_{s_k}[s_k][= \bigcup_{k=1}^{n} e_{s_k}[s_k]$.

Define $(e, E) = \prod_{k=1}^{n} (e_{s_k}, E_{s_k}) \in \mathcal{U}$. For $s \in S^\flat$ we have k, $1 \leqslant k \leqslant n$ with $e_{s_k}[s_k] \not\subseteq Q_s$. We will complete the proof by showing $e[s] \subseteq d_{s_k}[s_k] \subseteq C_{s_k} \in \mathcal{C}$, whence $\{e[s] \mid s \in S^\flat\} \prec \mathcal{C}$. Hence, suppose that $e[s] \not\subseteq d_{s_k}[s_k]$ and take $u \in S$ with $e[s] \not\subseteq Q_u$ and $P_u \not\subseteq d_{s_k}[s_k]$. Since $e = \prod_{i=1}^{n} e_{s_i}$ we have $e \subseteq e_{s_k}$, whence $e[s] \subseteq e_{s_k}[s]$ and we have $e_{s_k}[s] \not\subseteq Q_u$. Hence $\overline{P}_{(s_k,u)} \subseteq e_{s_k}^2 \subseteq d_{s_k}$.

From $e_{s_k}[s_k] = e_{s_k}^{\rightarrow} P_{s_k} \not\subseteq Q_s$ we deduce $e_{S_k} \not\subseteq \overline{Q}_{(s_k,s)}$, and from $e_{s_k}[s] \not\subseteq Q_u$ we deduce $e_{s_k} \not\subseteq \overline{Q}_{(s,u)}$. On the other hand $P_u \not\subseteq d_{s_k}[s_k] = d_{s_k}^{\rightarrow} P_{s_k}$ gives $P_u \not\subseteq Q_{u'}$ and

$$d_{s_k} \not\subseteq \overline{Q}_{(v,u')} \quad \Rightarrow \quad P_{s_k} \subseteq Q_v. \tag{1}$$

From $\overline{P}_{(s_k,u)} \subseteq d_{s_k}$ and $P_u \not\subseteq Q_{u'}$ we have $d_{s_k} \not\subseteq \overline{Q}'_{(s_k,u)}$, and since d_{s_k} is a relation we have $s'_k \in S$ with $P_{s_k} \not\subseteq Q_{s'_k}$ with $d_{s_k} \not\subseteq \overline{Q}'_{(s'_k,u)}$ by *R2*. Applying the implication (1) with $v = s'_k$ we deduce $P_{s_k} \subseteq Q_{s'_k}$, which is a contradiction.

(2) The proof is dual to (1) and is omitted. □

Example 2.7. Consider the texture $(\mathbb{I}, \mathcal{I})$ of Example 1.1 with the natural ditopology

$$\tau_{\mathbb{I}} = \{[0,r) \mid r \in \mathbb{I}\} \cup \{\mathbb{I}\}, \qquad \kappa_{\mathbb{I}} = \{[0,r] \mid r \in \mathbb{I}\} \cup \{\emptyset\}.$$

The dicovering uniformity $\upsilon_{\mathbb{I}}$ corresponding to the direlational uniformity $\mathcal{U}_{\mathbb{I}}$ of (see [10, Example 3.3]) has a base consisting of the dicovers $\mathcal{D}_\epsilon$, $\epsilon > 0$, where

$$\mathcal{D}_\epsilon = \{([0,r+\epsilon),[0,r-\epsilon]) \mid r \in I\},$$

and $[0,r+\epsilon)$ is understood to be $[0,1]$ when $r+\epsilon > 1$ and $[0,r-\epsilon]$ is $\emptyset$ if $r-\epsilon < 0$. Since $\tau_{\mathcal{U}_{\mathbb{I}}}$ is compact $(\mathbb{I}, \mathcal{I}, \mathcal{U}_{\mathbb{I}})$ is a Lebesgue direlational uniform texture space. Similarly since $\kappa_{\mathcal{U}_{\mathbb{I}}}$ is co-compact it is co-Lebesgue direlational uniform texture space.

We recall that [8] we may associate a dicovering uniformity with a given direlational uniformity. Let us recall the equivalence of these two concepts.

Theorem 2.8. *([8]) Let $(S, \mathcal{S})$ be a texture.*

(1) *To each direlational uniformity $\mathcal{U}$ on $(S,\mathcal{S})$ we may associate a dicovering uniformity $\upsilon = \Gamma(\mathcal{U}) = \{\mathcal{C} \in \mathcal{DC} \mid \exists (c,C) \in \mathcal{U}$ with $\gamma(c,C) \prec \mathcal{C}\}$.*
(2) *To each dicovering uniformity υ on $(S,\mathcal{S})$ we may associate a direlational uniformity $\mathcal{U} = \Delta(\upsilon) = \{(d,D) \in \mathcal{RDR} \mid \exists \mathcal{C} \in \upsilon$ with $\delta(\mathcal{C}) \sqsubseteq (d,D)\}$.*
(3) *$\Delta(\Gamma(\mathcal{U})) = \mathcal{U}$ for every direlational uniformity $\mathcal{U}$ on $(S,\mathcal{S})$.*
(4) *$\Gamma(\Delta(\upsilon)) = \upsilon$ for every dicovering uniformity υ on $(S,\mathcal{S})$.*

Proposition 2.9.

(1) *Let $\mathcal{U}$ be a Lebesgue direlational uniformity on $(S,\mathcal{S})$ and $\upsilon = \Gamma(\mathcal{U})$ the dicovering uniformity corresponding to $\mathcal{U}$. Then υ has the property that for a given open cover $\mathcal{C}$ there exists $\mathcal{D} \in \upsilon$ such that* $\operatorname{dom}\mathcal{D} \prec \mathcal{C}$.
(2) *Let υ be a dicovering uniformity on $(S,\mathcal{S})$ satisfying for a given open cover $\mathcal{C}$ there exists $\mathcal{D} \in \upsilon$ such that* $\operatorname{dom}\mathcal{D} \prec \mathcal{C}$. *Then the corresponding di-uniformity $\Delta(\upsilon) = \mathcal{U}$ is a Lebesgue direlational uniformity.*

Proof. (1) Let $\mathcal{C}$ be an open cover of S. Since $\mathcal{U}$ is a Lebesgue direlational uniformity there exists a direlation $(r,R) \in \mathcal{U}$ such that $\{r[s] \mid s \in S^\flat\}$ is a refinement of $\mathcal{C}$. Since $\gamma(r,R) \in \upsilon$ there exists $(c,C) \in \mathcal{U}$ with $\gamma(c,C) \prec \gamma(r,R)$. Now let $\gamma(c,C) = \mathcal{D}$ so we have $\operatorname{dom}\mathcal{D} \prec \mathcal{C}$.

(2) Let $\mathcal{C}$ be an open cover of S satisfying for $\mathcal{C}$ there exists a dicover $\{(L_j, M_j) \mid j \in J\} = \mathcal{D} \in \upsilon$ such that $\operatorname{dom}\mathcal{D} \prec \mathcal{C}$. Since υ is a dicovering uniformity there exists $(c,C) \in \mathcal{U}$ with $\gamma(c,C) \prec \mathcal{D}$. For $L\mathcal{D}M$ we have $c[s] \subseteq L = \operatorname{dom}\mathcal{D} \prec \mathcal{C}$ which means $\mathcal{U}$ is a Lebesgue direlational uniformity. □

The above proposition justifies the following definition.

Definition 2.10. A dicovering uniformity υ on a texture space $(S,\mathcal{S})$ is called a *Lebesgue dicovering uniformity* provided that for a given open cover $\mathcal{C}$ there exists $\mathcal{D} \in \upsilon$ such that $\operatorname{dom}\mathcal{D} \prec \mathcal{C}$.

Proposition 2.11.

(1) *Let $\mathcal{U}$ be a co-Lebesgue direlational uniformity on $(S,\mathcal{S})$ and $\upsilon = \Gamma(\mathcal{U})$ the dicovering uniformity corresponding to $\mathcal{U}$. Then υ has the property that for a given closed cocover $\mathcal{C}$ there exists $\mathcal{D} \in \upsilon$ such that $\mathcal{C} \lhd$* $\operatorname{ran}\mathcal{D}$.
(2) *Let υ be a dicovering uniformity on $(S,\mathcal{S})$ satisfying for a given closed cocover $\mathcal{C}$ there exists $\mathcal{D} \in \upsilon$ such that $\mathcal{C} \lhd$* $\operatorname{ran}\mathcal{D}$. *Then the corresponding di-uniformity $\Delta(\upsilon) = \mathcal{U}$ is a co-Lebesgue direlational uniformity.*

Proof. (1) Let $\mathcal{C}$ be a closed cocover of S. Since $\mathcal{U}$ is a co-Lebesgue direlational uniformity there exists a direlation $(r,R) \in \mathcal{U}$ such that $\mathcal{C}$ is a dual refinement of $\{R[s] \mid s \in S^\flat\}$. Since $\gamma(r,R) \in \upsilon$ there exists $(c,C) \in \mathcal{U}$ with $\gamma(c,C) \prec \gamma(r,R)$. Now let $\gamma(c,C) = \mathcal{D}$ so we have $\mathcal{C} \lhd \operatorname{ran}\mathcal{D}$.

(2) The proof is similar to (1) and is omitted. □

Definition 2.12. A dicovering uniformity υ on a texture space $(S,\mathcal{S})$ is called a *co-Lebesgue dicovering uniformity* provided that a given closed cocover $\mathcal{C}$ there exists $\mathcal{D} \in \upsilon$ such that $\mathcal{C} \lhd \operatorname{ran}\mathcal{D}$.

The term Lebesgue di-uniformity (co-Lebesgue di-uniformity) will be used to denote both Lebesgue direlational uniformity and Lebesgue dicovering uniformity (co-Lebesgue direlational uniformity and co-Lebesgue dicovering uniformity).

To conclude this section we consider a complemented di-uniformity on a complemented texture space $(S, \mathcal{S}, \sigma)$. We recall [10] that if υ is a dicovering uniformity on $(S, \mathcal{S}, \sigma)$ with uniform ditopology (τ, κ) then the uniform ditopology of the conjugate dicovering uniformity $(\upsilon)'$ is $(\sigma(\kappa), \sigma(\tau))$.

Theorem 2.13. *Let υ be Lebesgue dicovering uniformity on $(S, \mathcal{S}, \sigma)$ with uniform ditopology (τ, κ). Then $(\upsilon)'$ is a co-Lebesgue dicovering uniformity on $(S, \mathcal{S}, \sigma)$.*

Proof. Let $\mathcal{F} = \{\sigma(G_i) \mid G_i \in \sigma(\tau)\}$ be a $\sigma(\tau)$ closed cocover of S then $\mathcal{G} = \{G_i \mid i \in I\}$ is a τ-open cover of S. Since υ is Lebesgue dicovering uniformity there exists $\mathcal{D} \in \upsilon$ such that $\operatorname{dom}\mathcal{D} \prec \mathcal{G}$. This implies $\mathcal{F} \lhd \operatorname{ran}(\mathcal{D})'$, and we see that $(\upsilon)'$ is a co-Lebesgue dicovering uniformity on $(S, \mathcal{S}, \sigma)$. □

Corollary 2.14. *Let υ be a complemented dicovering uniformity on $(S, \mathcal{S}, \sigma)$. Then υ is a Lebesgue dicovering uniformity if and only if υ is a co-Lebesgue dicovering uniformity on $(S, \mathcal{S}, \sigma)$.*

Proof. Clear. □

The previous theorem shows that the notions of Lebesgue di-uniformity and co-Lebesgue di-uniformity coincide for a complemented di-uniformity. We recall [10] that on the discrete texture $(X, \mathcal{P}(X), \pi_X)$ a complemented di-uniformity is just a uniformity on X hence these concepts coincide also for uniformities.

3. Lebesgue quasi di-uniform spaces

The notion of quasi di-uniformity was introduced in [11] by removing the symmetry condition in the definition of direlational uniformity. Equivalently using the dual dicovers the notion of dual dicovering quasi uniformity was also introduced.

We may now give

Definition 3.1. A quasi di-uniformity $\mathcal{U}^q$ on a texture space $(S, \mathcal{S})$ is a Lebesque quasi di-uniformity provided that for each $(\tau_{\mathcal{U}^q}, \kappa_{\mathcal{U}^q})$ open co-closed dicover $\mathcal{C}$ of $(S, \mathcal{S})$ there is a direlation $(r, R) \in \mathcal{U}^q$ such that the dicover $\gamma(r, R) = \{(r[s], R[s]) \mid s \in S\}$ refines $\mathcal{C}$ and $(S, \mathcal{S}, \mathcal{U}^q)$ is called a Lebesgue quasi di-uniform texture space.

In order to define dual dicovering Lebesgue quasi uniformity it will be necessary to recall the definition of a dual dicover.

Definition 3.2. ([11]) A dual difamily

$$\mathcal{C}_d = \{((C_j^{1,1}, C_j^{1,2}),\ (C_j^{2,1}, C_j^{2,2})) \mid j \in J\}$$

of elements of $(\mathcal{S} \times \mathcal{S}) \times (\mathcal{S} \times \mathcal{S})$ is called a *dual dicover of* $(S, \mathcal{S})$ if

$$\{(C_j^{1,1} \cap C_j^{2,1},\ C_j^{1,2} \cup C_j^{2,2}) \mid j \in J\}$$

is a dicover of $(S, \mathcal{S})$. Clearly a dual dicover $\mathcal{C}_d$ satisfying $(C_j^{1,1}, C_j^{1,2}) \in (\tau_{\mathcal{U}^q}, \kappa_{\mathcal{U}^q})$ and $(C_j^{2,1}, C_j^{2,2}) \in (\tau_{(\mathcal{U}^q)^{\leftarrow}}, \kappa_{(\mathcal{U}^q)^{\leftarrow}})$ is called open co-closed.

Proposition 3.3. *([11]) Let (r, R) be a reflexive direlation on $(S, \mathcal{S})$ with $r[s] = r^{\rightarrow} P_s$; $R[s] = R^{\rightarrow} Q_s$ and $r^{\leftarrow}[s] = (r^{\leftarrow})^{\rightarrow} Q_s$; $R^{\leftarrow}[s] = (R^{\leftarrow})^{\rightarrow} P_s$. The family*

$$\gamma^q(r, R) = \{(\gamma(r, R), \gamma(r, R)^{\leftarrow}) \mid s \in S\},$$

where $\gamma(r, R) = \{(r[s], R[s]) \mid s \in S\}$ and $\gamma(r, R)^{\leftarrow} = \{(R^{\leftarrow}[s], r^{\leftarrow}[s]) \mid s \in S\}$ is an anchored dual dicover.

Definition 3.4. ([11]) Let $\mathcal{C}_d = \{((C_j^{1,1}, C_j^{1,2}), (C_j^{2,1}, C_j^{2,2})) \mid j \in J\}$ and $\mathcal{D}_d$ be dual dicovers. Then $\mathcal{C}_d$ is a refinement of $\mathcal{D}_d$, written $\mathcal{C}_d \prec \mathcal{D}_d$, if given $j \in J$ we have $((D^{1,1}, D^{1,2}), (D^{2,1}, D^{2,2})) \in \mathcal{D}_d$ so that

$$\begin{aligned} &(C_j^{1,1}, C_j^{1,2}) \sqsubseteq (D^{1,1}, D^{1,2}) \quad \text{and} \quad (C_j^{2,1}, C_j^{2,2}) \sqsubseteq (D^{2,1}, D^{2,2}) \\ &\quad \Leftrightarrow \quad C_j^{1,1} \subseteq D^{1,1}; \quad D^{1,2} \subseteq C_j^{1,2} \quad \text{and} \quad C_j^{2,1} \subseteq D^{2,1}; \quad D^{2,2} \subseteq C_j^{2,2}. \end{aligned}$$

Now we introduce the notion of a dual dicovering Lebesgue quasi di-uniformity.

Definition 3.5. Let $(S, \mathcal{S}, \mathcal{U}^q)$ be a quasi di-uniform space. $\mathcal{U}^q$ is a dual dicovering Lebesgue quasi uniformity if for each open co-closed dual dicover

$$\mathcal{C}_d = \left\{\left(\left(C_j^{1,1}, C_j^{1,2}\right),\ \left(C_j^{2,1}, C_j^{2,2}\right)\right) \,\middle|\, j \in J\right\}$$

of $(S, \mathcal{S}, \mathcal{U}^q)$ there is a direlation $(r, R) \in \mathcal{U}^q$ such that $\gamma^q(r, R)$ refines $\mathcal{C}_d$.

Proposition 3.6. *Let $(S, \mathcal{S}, \mathcal{U}^q)$ be a dual dicovering Lebesgue quasi uniform space. Then $\mathcal{U}^q$ and $(\mathcal{U}^q)^{\leftarrow}$ are Lebesgue quasi di-uniformities.*

Proof. Let $\mathcal{C} = \{(C_j^{1,1}, C_j^{1,2}) \mid j \in J\}$ be a $(\tau_{\mathcal{U}^q}, \kappa_{\mathcal{U}^q})$ open co-closed dicover. For each $j \in J$ let $(C_j^{2,1}, C_j^{2,2}) = (\mathcal{S}, \emptyset)$ then $\mathcal{C}_d = \{((C_j^{1,1}, C_j^{1,2}), (C_j^{2,1}, C_j^{2,2})) \mid j \in J\}$ is an open co-closed dual dicover. Since $\mathcal{U}^q$ is a dual dicovering Lebesgue quasi uniformity there exists $(r, R) \in \mathcal{U}^q$ such that $\gamma^q(r, R)$ refines $\mathcal{C}_d$. Since $(r[s], R[s]) \sqsubseteq (C_j^{1,1}, C_j^{1,2})$ we conclude that $\gamma(r, R)$ refines $(C_j^{1,1}, C_j^{1,2})$ which gives $\mathcal{U}^q$ is a Lebesgue quasi di-uniformity.

The proof of $(\mathcal{U}^q)^{\leftarrow}$ is a Lebesgue quasi di-uniformity can be done similarly. □

We recall [11] that the direlational uniformity with subbase $\mathcal{U}^q \cup (\mathcal{U}^q)^{\leftarrow}$ is called the *direlational uniformity associated with* $\mathcal{U}^q$ and is denoted by $\mathcal{U}^q \vee (\mathcal{U}^q)^{\leftarrow}$ then

$$\mathcal{U}^q \vee \left(\mathcal{U}^q\right)^{\leftarrow} = \left\{(d, D) \,\middle|\, \exists (r, R) \in \mathcal{U}^q \text{ such that } \left((r, R) \sqcap (r, R)^{\leftarrow}\right) \sqsubseteq (d, D)\right\}$$

is a direlational uniformity on $(S, \mathcal{S})$.

Theorem 3.7. *Let $\mathcal{U}^q$ be a dual dicovering Lebesgue quasi uniformity on $(S, \mathcal{S})$. Then $(S, \mathcal{S}, \mathcal{U}^q \vee (\mathcal{U}^q)^{\leftarrow})$ is a Lebesgue di-uniform texture space.*

Proof. Let $\{U_i \mid i \in I\}$ be a $\tau_{\mathcal{U}^q \vee (\mathcal{U}^q)^{\leftarrow}}$-open cover of $(S, \mathcal{S})$. Let $U_i = G_i \cap H_i$ with $G_i \in \tau_{\mathcal{U}^q}$ and $H_i \in \tau_{(\mathcal{U}^q)^{\leftarrow}}$. Then for $G_i \not\subseteq Q_s$ there exists $(d, D) \in \mathcal{U}^q$ satisfying $d[s] \subseteq G_i$. Similarly for $H_i \not\subseteq Q_s$ there exists $(d, D)^{\leftarrow} \in (\mathcal{U}^q)^{\leftarrow}$ with $D^{\leftarrow}[s] \subseteq H_i$. It is easy to verify that $\mathcal{C}_d = \{((G_i, \emptyset), (H_i, \emptyset)) \mid i \in I\}$ is an open co-closed dual dicover. Now since $\mathcal{U}^q$ is a dual dicovering Lebesgue quasi uniformity there exists $(f, F) \in \mathcal{U}^q$ such that $\gamma^q(f, F) \prec \{((G_i, \emptyset), (H_i, \emptyset)) \mid i \in I\}$. Hence for $i_0 \in I$ we have $f[s] \subseteq G_{i_0}$ and $F^{\leftarrow}[s] \subseteq H_{i_0}$ by [11, Proposition 3.10]. Thus we obtain $f[s] \cap F^{\leftarrow}[s] \subseteq G_i \cap H_i$ which gives $(S, \mathcal{S}, \mathcal{U}^q \vee (\mathcal{U}^q)^{\leftarrow})$ is a Lebesgue di-uniform texture space. □

Theorem 3.8. *Let $\mathcal{U}^q$ be a dual dicovering Lebesgue quasi uniformity on $(S, \mathcal{S})$. Then $(S, \mathcal{S}, \mathcal{U}^q \vee (\mathcal{U}^q)^{\leftarrow})$ is a co-Lebesgue di-uniform texture space.*

Proof. Let $\{F_i \mid i \in I\}$ be a $\kappa_{\mathcal{U}^q \vee (\mathcal{U}^q)^{\leftarrow}}$ closed-cocover of $(S, \mathcal{S})$. Let $F_i = M_i \cup K_i$ with $M_i \in \kappa_{\mathcal{U}^q}$ and $K_i \in \kappa_{(\mathcal{U}^q)^{\leftarrow}}$. Then $M_i \subseteq d[s]$ and $K_i \subseteq d^{\leftarrow}[s]$. Now $\mathcal{C}_d = \{((S, M_i), (S, K_i)) \mid i \in I\}$ is an open co-closed dual dicover. Since $\mathcal{U}^q$ is a dual dicovering Lebesgue quasi uniformity there exists $(v, V) \in \mathcal{U}^q$ such that $\gamma^q(f, F) \prec \mathcal{C}_d$ the we obtain $M_i \cup K_i \subseteq V[s] \cup v^{\leftarrow}[s]$ which establish that $(S, \mathcal{S}, \mathcal{U}^q \vee (\mathcal{U}^q)^{\leftarrow})$ is a co-Lebesgue di-uniform texture space. □

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