Global attractors for von Karman equations with nonlinear interior dissipation

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Abstract

In this paper we study the asymptotic behavior of weak solutions for von Karman equations with nonlinear interior dissipation. We prove the existence of a global attractor in the space $\dot{W}^2_2(\Omega) \times L^2(\Omega)$.

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1. Introduction

Let $\Omega$ be a bounded smooth domain in $\mathbb{R}^2$ with boundary $\partial \Omega$. We consider the following von Karman system with the homogeneous boundary conditions:

\begin{align*}
& w_{tt} + \Delta^2 w + g(w_t) = [\mathcal{F}(w), w] + h \quad \text{in} \ (0, +\infty) \times \Omega, \\
& \Delta^2 \mathcal{F}(w) = -[w, w] \quad \text{in} \ (0, +\infty) \times \Omega, \\
& \frac{\partial w}{\partial \nu} = \mathcal{F} = \frac{\partial \mathcal{F}}{\partial \nu} = 0 \quad \text{on} \ (0, +\infty) \times \partial \Omega,
\end{align*}

where $\mathcal{F}(w)$ is the strain energy function.


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\[ w(0, \cdot) = w_0, \quad w_t(0, \cdot) = w_1 \quad \text{in } \Omega, \]  
(1.4)

where \( h \in L^2(\Omega) \), the vector \( \nu \) denotes an outward normal and von Karman bracket is given by

\[ [u, v] = \frac{\partial^2 u}{\partial x_1^2} \frac{\partial^2 v}{\partial x_2^2} + \frac{\partial^2 u}{\partial x_2^2} \frac{\partial^2 v}{\partial x_1^2} - 2 \frac{\partial^2 u}{\partial x_1 \partial x_2} \frac{\partial^2 v}{\partial x_1 \partial x_2}. \]

The damping function \( g \in C^1(R) \) satisfies the condition

\[ g(0) = 0, \quad g \text{ strictly increasing}, \quad \liminf_{|s| \to \infty} g'(s) > 0. \]  
(1.5)

The long-time behavior of solutions for von Karman equations with interior dissipations were studied in [1–7] and references therein. The wellposedness of weak solutions of problem (1.1)–(1.4) has been established in [3] (see also [6]). The problem of existence of weak attractors for (1.1)–(1.4) in the case when \( g(\cdot) \) is linear, was studied in [2]. In the case of nonlinear dissipation, the most general treatment for the problem (1.1)–(1.4) to our knowledge is given in [7]. In that article the authors have proved the existence of a global attractor in \( W^2_2(\Omega) \times L^2(\Omega) \) for large values of the damping parameter.

Our main goal in this paper is to prove the existence of a global attractor for the problem (1.1)–(1.4) without assuming large values for the damping parameter. The sharp regularity of Airy’s stress function obtained in [8] plays a key role in our result.

2. Preliminaries

Denote the spaces \( \tilde{W}^r_2(\Omega) \), \( W^r_2(\Omega) \) and \( L^2(\Omega) \), by \( H^r_0 \), \( H^r \), and \( H \), respectively. The norm and scalar product in \( H \) are denoted by \( \| \cdot \| \) and \( (,.) \), respectively. It is known that under condition (1.5) the solution operator \( S(t)(w_0, w_1) = (w(t), w_t(t)) \), \( t \geq 0 \), of problem (1.1)–(1.4) generates a \( C^0 \)-semigroup on the energy space \( H^2_0 \times H \) (see [3,6]) in which

\[ E(w(t)) + \frac{1}{4} \| \Delta F(w(t)) \|^2 + \int_s^t \int_{\Omega} g(w_t(\tau, x)) w_t(\tau, x) \, dx \, d\tau - \langle h, w(t) \rangle \]

\[ \leq E(w(s)) + \frac{1}{4} \| \Delta F(w(s)) \|^2 - \langle h, w(s) \rangle \]  
(2.1)

and

\[ E(w(t) - u(t)) + \int_s^t \int_{\Omega} (g(w_t(\tau, x)) - g(u_t(\tau, x)))(w_t(\tau, x) - u_t(\tau, x)) \, dx \, d\tau \]

\[ \leq E(w(s) - u(s)) + \int_s^t \left( \langle [F(w(\tau)), w(\tau)] - [F(u(\tau)), u(\tau)], w_t(\tau) - u_t(\tau) \rangle \right) \, d\tau, \]  
(2.2)

hold for \( (w(t), w_t(t)) = S(t)(w_0, w_1) \) and \( (u(t), u_t(t)) = S(t)(u_0, u_1) \), where \( E(v(t)) = \frac{1}{2}(\| \Delta v(t) \|^2 + \| v_t(t) \|^2) \) and \( t \geq s \geq 0 \).
Denote by \( G(u, v) \) a solution to a biharmonic problem:

\[
z \equiv G(u, v) \iff \Delta^2 z = [u, v] \text{ in } \Omega \quad \text{and} \quad z = \frac{\partial}{\partial \nu} z = 0 \text{ on } \partial \Omega.
\]

We will use the following theorem on sharp regularity of Airy’s stress function from [8], and prove some lemmas in order to show asymptotic compactness of \( S(t) \).

**Theorem 1.** [8] The map \((u, v) \to G(u, v)\) is bounded from \( H^2 \times H^2 \to H^3 \cap W^{2, \infty}_\infty(\Omega) \).

**Lemma 1.** Let \( g(\cdot) \) satisfy condition (1.5). Then for any \( \delta > 0 \) there exists \( c(\delta) > 0 \), such that

\[
|u - v|^2 \leq \delta + c(\delta)(g(u) - g(v))(u - v) \quad \text{for } u, v \in \mathbb{R}. \tag{2.3}
\]

**Proof.** Assume (2.3) does not hold. Then there exist \( \delta_0 > 0, c_n \to +\infty \), and \( u_n \in \mathbb{R}, v_n \in \mathbb{R} \) such that

\[
|u_n - v_n|^2 > \delta_0 + c_n(g(u_n) - g(v_n))(u_n - v_n)
\]

from which we obtain

\[
|u_n - v_n|^2 > \delta_0 \quad \text{and} \quad \frac{1}{u_n - v_n} \int_{v_n}^{u_n} g'(s) \, ds \to 0,
\]

which contradicts (1.5). \( \Box \)

**Lemma 2.** Assume that \( w \in L_\infty(0, T; H^2_0) \) and \( w_t \in L_\infty(0, T; H) \). Then \( \mathcal{F}(w) \in C(0, T; H^2_0) \) and

\[
\frac{1}{4} \| \Delta \mathcal{F}(w(t)) \|^2 = -\int_0^t \left[ \left[ \mathcal{F}(w(\tau)), w(\tau), (w_t(\tau)) \right] d\tau + \frac{1}{4} \| \Delta \mathcal{F}(w(s)) \|^2, \tag{2.4}
\]

for every \( t, s \in [0, T] \).

**Proof.** Since \( w \in L_\infty(0, T; H^2_0) \) and \( w_t \in L_\infty(0, T; H) \), we have \( w \in C(0, T; H^1_0) \) and consequently \( w \in C_c(0, T; H^2_0) \) (see [9, Lemma 8.1, p. 275]). It means that if \( t_n \to t_0 \), then \( w(t_n) \to w(t_0) \) weakly in \( H^2_0 \). So by Theorem 1 and the compact embedding theorems we obtain

\[
\mathcal{F}(w(t_n)) \to \mathcal{F}(w(t_0)) \quad \text{strongly in } H^2_0.
\]

Hence \( \mathcal{F}(w) \in C(0, T; H^2_0) \).

Let the sequence \( w^n \in C^\infty_c((0, T) \times \Omega) \) be such that

\[
w^n \to w \quad \text{strongly in } L_4(0, T; H^2_0)
\]

and

\[
w^n_t \to w_t \quad \text{strongly in } L_4(0, T; H)
\]
as \( n \) tends to infinity. Then by Theorem 1 we have

\[
\mathcal{F}(w^n) \rightarrow \mathcal{F}(w) \quad \text{strongly in } L_2(0, T; H^3 \cap W^{2,\infty}_\infty(\Omega))
\]  

(2.5)

and

\[
\langle \left[ \mathcal{F}(w^n), w^n \right], w^n_t \rangle \rightarrow \langle \left[ \mathcal{F}(w), w \right], w_t \rangle \quad \text{strongly in } L_1(0, T).
\]  

(2.6)

Taking into account \( \frac{\partial}{\partial t} \| \Delta \mathcal{F}(w^n(t)) \|^2 = -4\langle [\mathcal{F}(w^n(t)), w^n(t)], w^n_t(t) \rangle \) from (2.5)–(2.6) we find that

\[
\frac{\partial}{\partial t} \| \Delta \mathcal{F}(w(t)) \|^2 = -4\langle [\mathcal{F}(w(t)), w(t)], w(t) \rangle \in L_\infty(0, T),
\]

which implies (2.4).

Lemma 3. Assume \( \{ w^n(t) \} \) and \( \{ w^n_t(t) \} \) are weakly star convergent in \( L_\infty(0, T; H^2_0) \) and \( L_\infty(0, T; H) \), respectively. Then

\[
\lim_{n \rightarrow \infty} \lim_{m \rightarrow \infty} \int_0^T \left[ \langle \mathcal{F}(w^n(t)), w^n(t) \rangle - \langle \mathcal{F}(w^m(t)), w^m(t) \rangle, w^n_t(t) - w^m_t(t) \rangle \right] dt = 0.
\]  

(2.7)

Proof. Let

\[
\begin{align*}
\{ w^n \} & \rightarrow w \quad \text{weakly star in } L_\infty(0, T; H^2_0), \\
\{ w^n_t \} & \rightarrow w_t \quad \text{weakly star in } L_\infty(0, T; H).
\end{align*}
\]  

(2.8)

By the compact embedding theorem (see [10, Theorem 5.1, p. 58]) from (2.8) we have

\[
w^n \rightarrow w \quad \text{strongly in } L_p(0, T; H^{2-\varepsilon}_0)
\]  

(2.9)

for \( 1 \leq p < \infty \) and \( \varepsilon > 0 \).

Using (2.8)1, (2.9) and the property of the von Karman bracket, we obtain

\[
[w^n, w^n] \rightarrow [w, w] \quad \text{weakly star in } L_\infty(0, T; H^{-2})
\]

and consequently

\[
\mathcal{F}(w^n) \rightarrow \mathcal{F}(w) \quad \text{weakly star in } L_\infty(0, T; H^2_0).
\]  

(2.10)

From (2.8)1, (2.9) and (2.10) we have

\[
\langle \left[ \mathcal{F}(w^n), w^n \right], w \rangle \rightarrow \langle \left[ \mathcal{F}(w), w \right], w \rangle \quad \text{weakly star in } L_\infty(0, T; H^{-2}).
\]  

(2.11)

On the other hand, by (2.8)1 and Theorem 1 we find that \( \{ \left[ \mathcal{F}(w^n), w^n \right] \} \) is bounded in \( L_\infty(0, T; H) \), which together with (2.11) gives

\[
\langle \left[ \mathcal{F}(w^n), w^n \right], \mathcal{F}(w) \rangle \rightarrow \langle \left[ \mathcal{F}(w), w \right], \mathcal{F}(w) \rangle \quad \text{weakly star in } L_\infty(0, T; H).
\]  

(2.12)

From (2.8), also follows that

\[
w^n \rightarrow w \quad \text{weakly in } C(0, T; H^1_0)
\]  

(2.13)
which according to [9, Lemma 8.1, p. 275], together with (2.8)\(_1\) yields \(w^n \in C(0, T; H^2_0)\). So \(\langle w^n(\cdot), \varphi \rangle \in C[0, T]\) and

\[
\left\| \langle w^n(t), \varphi \rangle \right\|_{C[0, T]} \leq \| w^n \|_{L^\infty(0, T; H^2_0)} \| \varphi \|_{H^{-2}}, \tag{2.14}
\]

for every \(t \in [0, T]\) and \(\varphi \in H^{-2}\).

From (2.13) and (2.14) we obtain

\[w^n(t) \to w(t)\]

weakly in \(H^2_0\) for every \(t \in [0, T]\). Thus by Theorem 1 we find that

\[
F(w^n(t)) \to F(w(t))\]

weakly in \(H^3\) \(\tag{2.15}\)

for every \(t \in [0, T]\).

By Lemma 2, we have

\[
\begin{aligned}
\int_0^T \left[ \langle F(w^n(t)), w^n(t) \rangle - \langle F(w^m(t)), w^m(t) \rangle, w^n(t) - w^m(t) \rangle dt \\
= \frac{1}{4} \left[ \left\| \Delta F(w^n(0)) \right\|^2 + \left\| \Delta F(w^m(0)) \right\|^2 - \left\| \Delta F(w^n(T)) \right\|^2 - \left\| \Delta F(w^m(T)) \right\|^2 \right] \\
- \int_0^T \left[ \langle F(w^n(t)), w^n(t) \rangle, w^n(t) \rangle dt - \int_0^T \left[ \langle F(w^m(t)), w^m(t) \rangle, w^m(t) \rangle dt. \right.
\end{aligned}
\]

Taking into account (2.8)\(_2\), (2.12), (2.15) and passing to limit in the last equality, we obtain

\[
\lim_{n \to \infty} \lim_{m \to \infty} \int_0^T \left[ \langle F(w^n(t)), w^n(t) \rangle - \langle F(w^m(t)), w^m(t) \rangle, w^n(t) - w^m(t) \rangle dt \\
= \frac{1}{2} \left[ \left\| \Delta F(w(0)) \right\|^2 - \left\| \Delta F(w(T)) \right\|^2 \right] - 2 \int_0^T \left[ \langle F(w(t)), w(t) \rangle, w(t) \rangle dt,
\]

which together with Lemma 2 imply (2.7). •

**Lemma 4.** Assume the condition (1.5) is satisfied, and \(B\) is a bounded subset of \(H^2_0 \times H\). Then for any \(\varepsilon > 0\) there exists \(T = T(\varepsilon, B)\) such that

\[
\limsup_{n \to \infty} \sup_{p \in \mathbb{N}} \left\| S(T)\vartheta_{n+p} - S(T)\vartheta_n \right\|_{H^2_0 \times H} \leq \varepsilon, \tag{2.16}
\]

where \(\{\vartheta_n\}\) is a sequence in \(B\) and \(\{S(t)\vartheta_n\}\) weakly star converges in \(L^\infty(0, \infty; H^2_0 \times H)\).

**Proof.** We will use techniques used in [6, Proof of Lemma 2.5] for similar estimates for von Karman equations (see also [7]). Let \((w^n(t), w^n(t)) = S(t)\vartheta_n\). From (2.2) we have
\[
\int_0^T \int_\Omega \left( g\left( w^n(t,x) \right) - g\left( w^m(t,x) \right) \right) \left( w^n(t,x) - w^m(t,x) \right) \, dx \, dt \\
\leq \tilde{c} \left( \| B \|_{H^2_0 \times H^2_0} \right) + \int_0^T \left[ \mathcal{F}(w^n(t)) - \mathcal{F}(w^m(t)) \right] \left( w^n(t) - w^m(t) \right) dt,
\]
for \( T \geq 0 \), where \( \| B \|_{H^2_0 \times H^2_0} = \sup_{v \in B} \| v \|_{H^2_0 \times H^2_0} \). Taking into account (2.3) in the last, inequality we obtain
\[
\int_0^T \| w^n(t) - w^m(t) \|^2 dt \\
\leq \delta T \text{mes} \, \Omega + c(\delta) \tilde{c} \left( \| B \|_{H^2_0 \times H^2_0} \right) \\
+ c(\delta) \int_0^T \left[ \mathcal{F}(w^n(t)) - \mathcal{F}(w^m(t)) \right] \left( w^n(t) - w^m(t) \right) dt,
\]
for every \( \delta > 0 \). On the other hand, multiplying both sides of
\[
(w^n - w^m)_{tt} + \Delta^2 (w^n - w^m) + g(w^n_t) - g(w^m_t) = \left[ \mathcal{F}(w^n), w^n \right] - \left[ \mathcal{F}(w^m), w^m \right]
\]
by \( (w^n - w^m) \), integrating over \([0, T] \times \Omega\) and taking into account (2.1), we find that
\[
\int_0^T \| \Delta (w^n(t) - w^m(t)) \|^2 dt \\
\leq \tilde{c} \left( \| B \|_{H^2_0 \times H^2_0} \right) + \int_0^T \| w^n_t(t) - w^m_t(t) \|^2 dt \\
+ \int_0^T \left[ \mathcal{F}(w^n(t)) - \mathcal{F}(w^m(t)) \right] \left( w^n(t) - w^m(t) \right) dt \\
+ \int_0^T \int_\Omega \left( g(w^n_t(t,x)) - g(w^m_t(t,x)) \right) \left( w^n(t,x) - w^m(t,x) \right) \, dx \, dt,
\]
for \( T \geq 0 \). (2.18)
Thus by (2.17) and (2.18) we have
\[
\int_0^T E(w^n(t) - w^m(t)) \, dt
\]
\[ \leq \delta T \mes \Omega + \tilde{c}(\| B \|_{H_0^2 \times H}) \left( c(\delta) + \frac{1}{2} \right) \]
\[ + c(\delta) \int_0^T \left( [\mathcal{F}(w^n(t)), w^n(t)] - [\mathcal{F}(w^m(t)), w^m(t)] \right) dt \]
\[ + \frac{1}{2} \int_0^T \left( [\mathcal{F}(w^n(t)), w^n(t)] - [\mathcal{F}(w^m(t)), w^m(t)] \right) dt \]
\[ + \frac{1}{2} \int_0^T \int_\Omega \left( g(w^n(t,x)) - g(w^m(t,x)) \right) (w^n(t,x) - w^m(t,x)) dx dt, \]
for \( T \geq 0, \)

which together with (2.2) implies

\[ \begin{aligned}
E(w^n(T) - w^m(T)) & \leq \delta \mes \Omega + \frac{1}{T} \tilde{c}(\| B \|_{H_0^2 \times H}) \left( c(\delta) + \frac{1}{2} \right) \\
& \quad + \frac{1}{T} c(\delta) \int_0^T \left( [\mathcal{F}(w^n(t)), w^n(t)] - [\mathcal{F}(w^m(t)), w^m(t)] \right) dt \\
& \quad + \frac{1}{T} \int_0^T \int_0^T \left( [\mathcal{F}(w^n(s)), w^n(s)] - [\mathcal{F}(w^m(s)), w^m(s)] \right) ds dt \\
& \quad + \frac{1}{2T} \int_0^T \int_\Omega \left( g(w^n(t,x)) - g(w^m(t,x)) \right) (w^n(t,x) - w^m(t,x)) dx dt \\
& \quad + \frac{1}{2T} \int_0^T \left( [\mathcal{F}(w^n(\tau)), w^n(\tau)] - [\mathcal{F}(w^m(\tau)), w^m(\tau)] \right) d\tau \\
& \equiv \delta \mes \Omega + \frac{1}{T} \tilde{c}(\| B \|_{H_0^2 \times H}) \left( c(\delta) + \frac{1}{2} \right) + K_1 + K_2 + K_3 + K_4. \quad (2.19)
\end{aligned} \]

By Lemma 3 we have

\[ \lim_{n \to \infty} \lim_{m \to \infty} K_1 = 0 \quad \text{and} \quad \lim_{n \to \infty} \lim_{m \to \infty} K_2 = 0. \quad (2.20) \]

Since \( \{w^n, w^m\}_{n=1}^\infty \) is bounded in \( C(0, T; H_0^2 \times H) \) and the embedding \( H_0^2 \subset C(\bar{\Omega}) \) is compact, by Arzela theorem \( \{w^n\}_{n=1}^\infty \) is compact in \( C(0, T; C(\bar{\Omega})) \). On the other hand, \( \{w^n\}_{n=1}^\infty \) converges weakly star in \( L_\infty(0, T; H_0^2) \). Thus \( \{w^n\}_{n=1}^\infty \) strongly converges in \( C(0, T; C(\bar{\Omega})). \)

Since by (1.5) and (2.1)
we have

\[ |K_3| \leq \frac{1}{T} \| w^n - w^m \|_{C(0,T;\mathcal{C}(\overline{\Omega}))} \left( T \text{mes } \Omega \left( g(1) + |g(-1)| \right) + \tilde{c}(\| B \|_{H_0^2 \times H}) \right). \]

(2.21)

On the other hand, for \( K_4 \) we find that

\[ |K_4| \leq \tilde{c}(\| B \|_{H_0^2 \times H}) \| w^n - w^m \|_{C(0,T;\mathcal{C}(\overline{\Omega}))}. \]

(2.22)

From (2.21) and (2.22) we obtain

\[ \lim_{n \to \infty} \lim_{m \to \infty} K_3 = 0 \quad \text{and} \quad \lim_{n \to \infty} \lim_{m \to \infty} K_4 = 0. \]  

(2.23)

Thus by (2.19), (2.20) and (2.23) we get

\[ \lim_{n \to \infty} \sup_{m \to \infty} E\left( w^n(T) - w^m(T) \right) \leq \delta \text{mes } \Omega + \frac{1}{T} \tilde{c}(\| B \|_{H_0^2 \times H}) \left( c(\delta) + \frac{1}{2} \right), \]

consequently

\[ \lim_{n \to \infty} \sup_{p \in \mathbb{N}} E\left( w^{n+p}(T) - w^n(T) \right) \]

\[ \leq 2 \lim_{n \to \infty} \sup_{p \in \mathbb{N}} \lim_{m \to \infty} E\left( w^{n+p}(T) - w^m(T) \right) \]

\[ + 2 \lim_{n \to \infty} \sup_{m \to \infty} E\left( w^m(T) - w^n(T) \right) \]

\[ \leq 4 \left( \delta \text{mes } \Omega + \frac{1}{T} \tilde{c}(\| B \|_{H_0^2 \times H}) \left( c(\delta) + \frac{1}{2} \right) \right), \]

which yields (2.16). \( \Box \)

3. Global attractors

In this section, we shall show the existence of the global attractor. To this end, we first prove the asymptotic compactness of \( S(t) \) in \( H_0^2 \times H \), which is given in the following theorem:
Theorem 2. Assume the condition (1.5) holds. Then for any bounded subset $B$ of $H^2_0 \times H$, the set $\{S(t_n)\theta_n\}_{n=1}^{\infty}$ is relatively compact in $H^2_0 \times H$, where $t_n \to \infty$ and $\{\theta_n\}_{n=1}^{\infty} \subset B$.

Proof. Since $B$ is bounded, by (2.1) we have $\sup_{t \geq 0} \sup_{\theta \in B} \|S(t)\theta\|_{H^2_0 \times H} < \infty$. Therefore there exists a bounded subset $B_0$ of $H^2_0 \times H$ such that $S(t)\theta \in B_0$, for every $t \geq 0$ and $\theta \in B$. Let $\varepsilon_m > 0$ and $\varepsilon_m \to 0$. By Lemma 4, for every $\varepsilon_m$ there exists $T_m = T_m(B_0) > 0$ such that

$$\limsup_{k \to \infty} \sup_{p \in \mathbb{N}} \|S(T_m)\varphi_{k+p} - S(T_m)\varphi_k\|_{H^2_0 \times H} \leq \varepsilon_m,$$

where $\{\varphi_k\}_{n=1}^{\infty}$ is a sequence in $B_0$ and $\{S(t)\varphi_k\}_{n=1}^{\infty}$ weakly star converges in $L_\infty(0, \infty; H^2_0 \times H)$.

Now for $\varepsilon_1$, choose a subsequence $\{n_k^{(1)}\} \subset \{n\}$ such that $t_{n_k^{(1)}} \geq T_1$ and $\{S(t)S(t_{n_k^{(1)}} - T_1)\theta_{n_k^{(1)}}\}_{k=1}^{\infty}$ weakly star converges in $L_\infty(0, \infty; H^2_0 \times H)$. For $\varepsilon_2$, choose a subsequence $\{n_k^{(2)}\} \subset \{n_k^{(1)}\}$ such that $t_{n_k^{(2)}} \geq T_2$ and $\{S(t)S(t_{n_k^{(2)}} - T_2)\theta_{n_k^{(2)}}\}_{k=1}^{\infty}$ weakly star converges in $L_\infty(0, \infty; H^2_0 \times H)$. Continuing this procedure we have $\{n_k^{(1)}\} \supset \{n_k^{(2)}\} \supset \cdots \supset \{n_k^{(m)}\} \supset \cdots$, such that $t_{n_k^{(m)}} \geq T_m$ and $\{S(t)S(t_{n_k^{(m)}} - T_m)\theta_{n_k^{(m)}}\}_{k=1}^{\infty}$ weakly star converges in $L_\infty(0, \infty; H^2_0 \times H)$. Taking $\varphi_k = S(t_{n_k^{(m)}} - T_m)\theta_{n_k^{(m)}}$ in (3.1), we obtain

$$\limsup_{k \to \infty} \sup_{p \in \mathbb{N}} \|S(t_{n_k^{(m)}p})\theta_{n_k^{(m)}p} - S(t_{n_k^{(m)})}\theta_{n_k^{(m)}}\|_{H^2_0 \times H} \leq \varepsilon_m,$$

for every $m \in \mathbb{N}$.

Now we construct the diagonal subsequence $\{S(t_{n_k^{(k)}p})\theta_{n_k^{(k)}p}\}$. Since for every $m \in \mathbb{N}$, the sequence $\{S(t_{n_k^{(k)}p})\theta_{n_k^{(k)}p}\}_{k=m}$ is a subsequence of $\{S(t_{n_k^{(m)}p})\theta_{n_k^{(m)}p}\}_{k=1}^{\infty}$, by (3.2) we have

$$\limsup_{k \to \infty} \sup_{p \in \mathbb{N}} \|S(t_{n_k^{(k)}p})\theta_{n_k^{(k)}p} - S(t_{n_k^{(k)}p})\theta_{n_k^{(k)}p}\|_{H^2_0 \times H} \leq \varepsilon_m.$$

Since $\varepsilon_m \to 0$, the last inequality means that the sequence $\{S(t_{n_k^{(k)}p})\theta_{n_k^{(k)}p}\}_{k=1}^{\infty}$ is a Cauchy sequence in $H^2_0 \times H$ and consequently this sequence strongly converges in $H^2_0 \times H$. In other words, the sequence $\{S(t_{n_k^{(k)}p})\theta_{n_k^{(k)}p}\}_{n=1}^{\infty}$ has a subsequence which is strongly convergent in $H^2_0 \times H$. It can be seen in a similar way that every subsequence of $\{S(t_n)\theta_n\}_{n=1}^{\infty}$ has a subsequence strongly convergent in $H^2_0 \times H$. Thus the set $\{S(t_n)\theta_n\}_{n=1}^{\infty}$ is relatively compact in $H^2_0 \times H$. \qed

Since by (2.1) the problem (1.1)–(1.4) admits a “good” Lyapunov function (see [11, p. 41]) $L(w(t)) = E(w(t)) + \frac{1}{2} \|\Delta F(w(t))\|^2 - \langle h, w(t) \rangle$ and since the set of stationary solutions is bounded in $H^2_0$, using the results of [11, pp. 49–50], we can formulate our main result.

Theorem 3. Assume that (1.5) holds. Then problem (1.1)–(1.4) has a global attractor in $H^2_0 \times H$, which is invariant and compact.
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References