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Effective computation of exact and analytic approximate solutions to singular nonlinear equations of Lane–Emden–Fowler type



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ABSTRACT

The particular motivation of this work is to develop a computational method to calculate exact and analytic approximate solutions to singular strongly nonlinear initial or boundary value problems of Lane–Emden–Fowler type which model many phenomena in mathematical physics and astrophysics. A powerful algorithm is proposed based on the series representation of the solution via suitable base functions. The utilization of such functions converts the solution of a given nonlinear differential equation to the solution of algebraic equations. Error analysis and convergence of the method is presented. Comparisons with the other methods reveal validity, applicability and great potential of the method. Several physical problems are treated to illustrative the good performance and high accuracy of the technique.

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1. Introduction

Some problems occurring in extensive applications in mathematical physics, astrophysics and other fields in engineering are generally reduced to so-called singular nonlinear Lane–Emden–Fowler differential equations through a process of mathematical modeling. Therefore, there has been a renewed recent interest in the research on the analytical and numerical solution methods for such nonlinear differential equations of the form

$$y'' + \frac{a}{x}y' + f(x, y) = g(x), \quad 0 < x < b, \quad (1.1)$$

subjected to the initial conditions

$$y(0) = \alpha, \quad y'(0) = \beta, \quad (1.2)$$

or accompanied by the boundary conditions

$$y'(0) = \alpha, \quad y(b) = \beta, \quad (1.3)$$

where $g(x)$ is a sufficiently smooth prescribed function physically representing the source term, a, b, α, β are known constants and $y(x)$ is the sought solution depending on x . It is well known that (1.1) has a unique solution if $f(x, y)$ is a continuous function, $\frac{\partial f}{\partial y} \geq 0$ exists and continuous [1]. It is noted that mixed boundary conditions are also allowed within the subsequent analysis. Moreover, linear differential equations can also be efficiently treated by the proposed method in this work.

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Typically, closed-form solutions to (1.1)–(1.3) either do not exist, except for some special cases, or are practically intractable. It hence becomes important to develop analytical or numerical schemes for finding the solution. Accurate and fast numerical solution of two-point singular boundary value problems for ordinary differential equations of the kind (1.1) is necessary in many important scientific and engineering applications, e.g. reactant concentration in a chemical reactor, boundary layer theory, control and optimization theory, and flow networks in biology, areas of astrophysics such as the theory of stellar interiors, the thermal behavior of a spherical cloud of gas, isothermal gas spheres, and the theory of thermionic currents [2–4].

The solution of the Lane–Emden–Fowler Eqs. (1.1)–(1.3), as well as other various linear and non-linear singular initial and boundary value problems in quantum mechanics and astrophysics, is numerically challenging because of the singular behavior at the origin. The approximate analytical solutions to the Lane–Emden equations were presented in [5] using the Adomian decomposition method. Homotopy perturbation method was used in [6] for the analytical solutions of the Emden–Fowler equations. A special class of Emden–Fowler equations was solved analytically in [7] by means of the homotopy analysis method. Variational iteration method for the approximate solutions of singular initial and boundary value problems of Lane–Emden type was used in [8].

On the other hand, there are some deficiencies of the aforementioned methods. For instance, since inner iterations are required in the variational iteration method, it becomes tedious and even impossible to carry out the integrations beyond some iterations. Although the homotopy analysis method is a decent method which contains the Adomian and homotopy perturbation methods, it requires the knowledge of auxiliary parameters and of optimum convergence control parameter, without which a fast convergence (sometimes convergence itself) is not guaranteed. Considering all these, the main objective of the present study is to develop and implement an analytic approximate solution technique for the nonlinear equations of Lane–Emden–Fowler type. The purpose is such that the devised scheme should give rapid solutions, besides its reliability in terms of its accuracy. As opposed to the techniques employed in the literature, it should be straightforward to apply and formulate. In line with this, the approach adopted here is simply based upon the well chosen base functions (the classical polynomials, for instance). The solution to nonlinear differential equations is later presented as linear combinations of these functions, whose coefficients are determined using the Galerkin-like procedure. A rigorous mathematical proof is laid down for the error analysis and convergence of the method. Unlike the aforementioned methods, the present approach yields a detailed solution in a straightforward fashion without any discretization. Application of the developed method to well-documented nonlinear equations in the literature justifies the success of the introduced method. In addition to this, the method is capable of capturing the exact analytical solutions whenever they are basically classical polynomials.

The remainder of this paper is organized as follows. In Section 2 the new approach is described. The accuracy, error and convergence of the method are discussed in Section 3. Section 4 contains several physical problems and analyzes the method in terms of its accuracy and convergence. Finally, our conclusions follow in Section 5.

2. Description of the method

We assume that differential Eq. (1.1) under the initial and boundary conditions (1.2) and (1.3) has a unique solution. In addition to this, the interval of interest $[0, b]$ is simply taken as $[0, 1]$, which can be achieved by an appropriate linear transformation. Considering the base functions

$$\mathbf{Y} = \{Y_0(x), Y_1(x), \dots, Y_N(x), \dots\},$$

which reside in the solution space of Eq. (1.1), let the solution be expressed by the series expansion in terms of these base functions

$$y(x) = \sum_{k=0}^{\infty} a_k Y_k(x), \quad (2.4)$$

where the coefficients a_k 's are to be determined. By means of the definitions

$$\mathbf{X} = [Y_0(x) \ Y_1(x) \ Y_2(x) \ Y_3(x) \ \dots \ Y_N(x)], \quad \mathbf{A} = [a_0 \ a_1 \ a_2 \ a_3 \ \dots \ a_N]^T,$$

the series solution (2.4) can be approximated at the N th order by the dot product

$$y_N(x) = \sum_{k=0}^N a_k Y_k(x) = \mathbf{X} \cdot \mathbf{A}. \quad (2.5)$$

It is reminded here that the matrix Φ in the below-given notations and theorems is set to \mathbf{X} . The higher derivatives of the solution can also be obtained in the form

$$y_N^{(n)}(x) = \mathbf{X} \cdot \mathbf{B}^n \cdot \mathbf{A}, \quad n \geq 1, \quad (2.6)$$

where B is the operational matrix depending on the choice of \mathbf{Y} .

Consequently, substituting (2.5) and (2.6) into differential Eq. (1.1), the following matrix equation reads

$$x\mathbf{X}\mathbf{B}^2 \cdot \mathbf{A} + a\mathbf{X} \cdot \mathbf{B} \cdot \mathbf{A} + xf(x, \mathbf{X} \cdot \mathbf{A}) = xg(x). \tag{2.7}$$

Having donated the working Hilbert space $H = L^2[0, 1]$ with the inner product

$$\langle f, g \rangle = \int_0^1 f(x)g(x)dx,$$

consider the linearly independent set of functions in H

$$\chi = \{\chi_0(x), \chi_1(x), \dots, \chi_N(x)\}, \tag{2.8}$$

whose entries might be the classical monomial standard polynomials or any other polynomials type, such as, the polynomials of Chebyshev, Legendre, Bernstein and so on [9]. Afterwards, within a Galerkin-like approach, taking the inner product of Eq. (2.7) with the elements of χ results in $(N + 1)$ nonlinear equations for the unknowns of \mathbf{A} , whose matrix equation is given by

$$\mathbf{W} = \mathbf{G}, \tag{2.9}$$

where the i th row of \mathbf{W} and \mathbf{G} , respectively consists of

$$\langle \chi_i, x\mathbf{X} \cdot \mathbf{B}^2 \cdot \mathbf{A} + a\mathbf{X} \cdot \mathbf{B} \cdot \mathbf{A} + xf(x, \mathbf{X} \cdot \mathbf{A}) \rangle, \quad \langle \chi_i, xg(x) \rangle, \quad 0 \leq i \leq N.$$

The initial or boundary conditions (1.2) and (1.3) only modify 2 rows of the matrix \mathbf{W} and the corresponding parts of right hand side matrix \mathbf{G} , due to the restrictions

$$\mathbf{X}(0) \cdot \mathbf{B}^k \cdot \mathbf{A} = \alpha_k, \quad k = 0, 1 \tag{2.10}$$

and

$$\begin{aligned} \mathbf{X}(0) \cdot \mathbf{B} \cdot \mathbf{A} &= \alpha, \\ \mathbf{X}(1) \cdot \mathbf{A} &= \beta. \end{aligned} \tag{2.11}$$

Therefore, taking the inner product of (2.10) or (2.11) with χ_i ($0 \leq i \leq N$) will modify n rows of (2.9). Eq. (2.9) is later solved either numerically or by any routine solver mounted in object-oriented programs so that the elements $a_0, a_1, a_2, \dots, a_N$ of \mathbf{A} are uniquely determined. Eventually, substituting this into (2.5) yields the required analytic approximate solution to the nonlinear differential Eq. (1.1).

3. Error analysis and convergence of the method

Before we approximate a function, we take into account the following preliminaries. Suppose that $H = L^2[0, 1]$, $P_m = \{\phi_0, \phi_1, \dots, \phi_m\} \subset H$ be the set of polynomials of m th-degree and $Y = Span(P_m)$. If f is an arbitrary element in H , due to Y being a finite dimensional vector space, f has the unique best approximation out of Y such as $y_0 \in Y$, that is,

$$\|f - y_0\|_2 \leq \|f - y\|_2, \quad \text{for all } y \in Y,$$

where $\|f\|_2^2 = \langle f, f \rangle$. Since $y_0 \in Y$, there exist unique coefficients $\mathbf{A} = [a_0 \ a_1 \ \dots \ a_m]$ such that

$$f \simeq y_0 = \sum_{k=0}^m a_k \phi_k = \mathbf{A} \cdot \Phi,$$

where $\Phi = [\phi_0 \ \phi_1 \ \dots \ \phi_m]^T$ and \mathbf{A} can be obtained by

$$\mathbf{A} \langle \Phi, \Phi \rangle = \langle f, \Phi \rangle,$$

where

$$\langle f, \Phi \rangle = \int_0^1 f(x)\Phi(x)^T dx = [\langle f, \phi_0 \rangle \quad \langle f, \phi_1 \rangle \quad \dots \quad \langle f, \phi_m \rangle]$$

and $\langle \Phi, \Phi \rangle$ is an $(m + 1) \times (m + 1)$ matrix which is said to be the dual matrix denoted by φ given by

$$\varphi = \langle \Phi, \Phi \rangle = \int_0^1 \Phi(x)\Phi(x)^T dx,$$

then

$$\mathbf{A} = \left(\int_0^1 f(x)\Phi(x)^T dx \right) \cdot \varphi^{-1}.$$

Keeping in mind this information, the subsequent theorems and lemmas can be outlined:

Theorem 1. Suppose that H is an Hilbert space, Y is a closed subspace of H such that $\dim Y$ is finite and $\{y_1, y_2, \dots, y_N\}$ is any basis for Y . Let f be an arbitrary element in H and y_0 be the unique best approximation to f out of Y . Then we have

$$\|f - y_0\|_2^2 = \frac{D(f, y_1, y_2, \dots, y_N)}{D(y_1, y_2, \dots, y_N)}, \quad (3.12)$$

where

$$D(f, y_1, y_2, \dots, y_N) = \begin{vmatrix} \langle f, f \rangle & \langle f, y_1 \rangle & \cdots & \langle f, y_N \rangle \\ \langle y_1, f \rangle & \langle y_1, y_1 \rangle & \cdots & \langle y_1, y_N \rangle \\ \vdots & \vdots & \cdots & \vdots \\ \langle y_N, f \rangle & \langle y_N, y_1 \rangle & \cdots & \langle y_N, y_N \rangle \end{vmatrix}$$

Proof. Refer to [10].

We define the inner product in H by $\langle f, g \rangle = \int_0^1 f(x)g(x)dx$ and the subspace $Y = \text{Span}(P_m)$, so the presented absolute error (3.12) in Theorem 1 can be written

$$\|f - y_0\|_2 = \frac{\det[\int_0^1 \Psi(x)\Psi(x)^T dx]}{\det[\int_0^1 \Phi(x)\Phi(x)^T dx]}$$

for which $\Phi^T = [\phi_0 \ \phi_1 \ \dots \ \phi_m]$ and $\Psi^T = [f \ \phi_0 \ \phi_1 \ \dots \ \phi_m]$. An upper bound for the estimation of approximate error is presented in the following lemma.

Lemma 1. Suppose that function $g: [0, 1] \rightarrow R$ is $m+1$ times continuously differentiable, $g \in C^{m+1}[0, 1]$, and $Y = \text{Span}\{\phi_0, \phi_1, \dots, \phi_m\}$. If $\mathbf{A}\Phi$ is the best approximation to g out of Y , then a bound for the absolute error is presented by

$$\|g - \mathbf{A} \cdot \Phi\|_2 \leq \frac{M}{(m+1)! \sqrt{2m+3}}, \quad (3.13)$$

where $M = \max_{x \in [0, 1]} |g^{(m+1)}(x)|$

Proof. Taking into consideration the Taylor polynomial

$$y_1(x) = g(0) + xg'(0) + \dots + g^{(m)}(0) \frac{x^m}{m!} \quad (3.14)$$

from which it is known

$$|g(x) - y_1(x)| \leq R_m(x) = |g^{(m+1)}(\eta)| \frac{x^{m+1}}{(m+1)!}, \quad (3.15)$$

where $\eta \in [0, 1]$. Since $\mathbf{A}\Phi$ is the best approximation to g out of Y , considering $y_1 \in Y$ and using (3.15) we have

$$\|g - \mathbf{A}\Phi\|_2^2 \leq \|g - y_1\|_2^2 = \int_0^1 |g(x) - y_1(x)|^2 dx \leq \int_0^1 R_m^2(x) dx \leq \frac{M^2}{[(m+1)!]^2 (2m+3)}$$

and taking the square roots we have the bound (3.13).

Now it is intended to prove that this approximation is convergent to f when $m \rightarrow \infty$ only under the continuity constraint of f .

Definition. Let us define the modulus of continuity $\omega(f, \delta)$ of a general function f on $[0, 1]$ by

$$\omega(f, \delta) = \sup_{x, y \in [0, 1] \text{ and } |y-x| \leq \delta} |f(x) - f(y)|.$$

Lemma 2. A function $f(x)$ is continuous on $[0, 1]$ if and only if

$$\lim_{\delta \rightarrow 0} \omega(f, \delta) = 0.$$

Proof. Refer to [11]

Theorem 2. If $f(x)$ is bounded on $[0, 1]$, then

$$\|f - p(f, m)\|_\infty \leq \frac{3}{2} \omega\left(f, \sqrt{\frac{1}{m}}\right),$$

where $p(f, m) = \sum_{k=0}^m f\left(\frac{k}{m}\right) \phi_k$ and $\|f\|_\infty = \sup\{|f(x)| : x \in [0, 1]\}$. If f also satisfies a Lipschitz condition of order α on $[0, 1]$, then

$$\|f - p(f, m)\|_\infty \leq \frac{3}{2} L m^{-\frac{\alpha}{2}},$$

where L is a Lipschitz constant.

Proof. Refer to [11]

Lemma 3. If $f(x)$ is bounded on $[0, 1]$ and $Y = \text{Span}\{\phi_0, \phi_1, \dots, \phi_m\}$, then

$$\|f - \mathbf{A}\Phi\|_2 \leq \frac{3}{2} \omega\left(f, \sqrt{\frac{1}{m}}\right),$$

where $\mathbf{A}\Phi$ is the best approximation to f out of Y .

Proof. Since $\mathbf{A}\Phi$ is the best approximation to f out of Y , $p(f, m) \in Y$ and using $\|f\|_2 \leq \|f\|_\infty$, it is straightforward to write

$$\|f - \mathbf{A}\Phi\|_2 \leq \|f - p(f, m)\|_2 \leq \|f - p(f, m)\|_\infty \leq \frac{3}{2} \omega\left(f, \sqrt{\frac{1}{m}}\right).$$

It is noted that whenever f is defined on $[a, b]$, the interval $[a, b]$ can be transformed onto $[0, 1]$, and if f is also continuous on $[0, 1]$, by virtue of Lemma 2 we get

$$\lim_{m \rightarrow \infty} \omega\left(f, \sqrt{\frac{1}{m}}\right) \rightarrow 0.$$

It is further shown that the approach defined in section 2 generates the exact solution when the solution consists of polynomials of finite degree.

Lemma 4. Let $f(x)$ be a polynomial of finite degree M . Then, the best approximation $\mathbf{A}\Phi$ with degree $m \geq M$ coincides with f .

Proof. Because the Taylor expansion is unique and from Lemma 1, the proof is complete.

4. Application of the method

To illustrate the underlying ideas, validity, effectiveness, accuracy and performance of the proposed technique, we analyze several nonlinear Lane–Emden–Fowler equations. Throughout the calculations, the absolute error is defined by

$$err = |y(x) - y_e(x)|, \tag{4.16}$$

where $y_e(x)$ denotes the exact solution. We incorporate the following simple polynomials as the base functions

$$\mathbf{X} = [1 \ x \ x^2 \ x^3 \ \dots \ x^N]$$

for which the entries of the operational matrix $\mathbf{B}_{(N+1) \times (N+1)}$ are expressed as $b_{i(i+1)} = i$, $1 \leq i \leq N$ and $b_{ij} = 0$ for other $1 \leq i, j \leq N + 1$. All calculations in this paper were made using Mathematica 7.

Example 1. To demonstrate that the presented approach can handle exact solutions when they are polynomials in compliance with the Lemma 4, let us first consider the nonlinear differential equations [12]

$$y'' + \frac{1}{x}y' + y = 4 - 9x + x^2 - x^3, \quad y(0) = y(1) = 0, \quad x \in [0, 1] \tag{4.17}$$

and

$$y'' + \frac{2}{x}y' + y^3 = 12x + (1 + x^3)^3, \quad y(0) - 1 = y'(0) = 0, \quad x \in [0, 1], \tag{4.18}$$

which correspond to linear and nonlinear singular initial boundary value problems. The following approximations are then found using the present algorithm, respectively

$$\begin{aligned}
 y(x) &= 0, \quad n = 0, 1, \\
 y(x) &= -\frac{53}{47}(-1+x)x, \quad n = 2, \\
 y(x) &= -(-1+x)x^2, \quad n \geq 3,
 \end{aligned} \tag{4.19}$$

$$\begin{aligned}
 y(x) &= 1, \quad n = 0, 1, \\
 y(x) &= 1 + \frac{1}{4} \left(-5 - 245 \left(\frac{91}{3(1392921 + 8\sqrt{36024580869})} \right)^{1/3} + \frac{\left(\frac{1}{91} (1392921 + 8\sqrt{36024580869}) \right)^{1/3}}{3^{2/3}} \right) x^2, \quad n = 2, \\
 y(x) &= 1 + x^3, \quad n \geq 3.
 \end{aligned} \tag{4.20}$$

It is fascinating that exact solutions are obtained at all the approximation levels $n \geq 3$ for (4.17) and (4.18). However, [12] used reproducing kernel space approach for (4.17) employing 51 and 250 terms to get a maximum absolute error of only $O(10^{-8})$. Moreover, the variational iteration method used in [13] does also fail to get the exact solution in finite number of iterations (see Example 1 in [13]).

Example 2. Let us consider now the radial stress on a rotationally symmetric shallow membrane cap [14]

$$y'' + \frac{3}{x}y' + \frac{1}{8}y^2 = \frac{1}{2}, \quad y'(0) = y(1) - 1 = 0, \quad x \in [0, 1]. \tag{4.21}$$

No an exact solution exists for this equation, hence it was solved numerically. Table 1 lists the values obtained from the present method at the order of approximation $n = 10$. It is seen from the table that the proposed method can resolve singular nonlinear problem (4.21) accurately for small levels of approximations. In fact, at this level the maximum absolute error is found to be order of magnitude $O(10^{-16})$. However, the variational iteration method used in [13] for the solution of (4.21) is not as accurate as the present one. Even though no exact solution exists, the following 5th order approximation has an accuracy of $O(10^{-9})$ and can be used in place of the exact solution for practical purposes

$$\begin{aligned}
 y &= 0.9521484205 + 0.04833591932x^2 - 6.774976833 \times 10^{-6}x^3 - 0.0004663349433x^4 \\
 &\quad - 0.00001122989264x^5.
 \end{aligned} \tag{4.22}$$

Example 3. Let us consider now the gravitational potential of the degenerate white-dwarf stars which can be modeled by the so-called white-dwarf equation [3]

$$y'' + \frac{2}{x}y' + y^3 = 0, \quad y(0) = 1, \quad y'(0) = 0, \quad x \in [0, R]. \tag{4.23}$$

A closed-form solution to this equation can not be written down, hence a numerical solution is preferred. To approximate the solution to (4.23), we first convert the interval $[0, R]$ to $[0, 1]$ by the linear mapping $x = Rt$ and fix R at 5. Fig. 1 demonstrates numerical (exact) and approximate solutions and the absolute errors at two levels of approximations, $n = 4$ and $n = 12$, respectively. Figs. 1(a-d) clearly show that the accuracy of our method is getting better as the approximation level is increasing. There appears an exponential decrease in the maximum absolute error for increasing n and thus the proposed method converges rapidly to the exact solution. However, even the 20th order approximate solution given in [7] by means of the homotopy analysis method seems not to converge to the exact solution as quickly as the proposed method here. This can

Table 1
Comparison between the present and variational iteration method in [13].

x	Exact solution	Present method	Method in [13]
0.0	0.95214843206703	0.95214843206703	0.95214843208264
0.1	0.95263172996208	0.95263172996208	0.95263172997768
0.2	0.95408104815379	0.95408104815379	0.95408104816938
0.3	0.95649465883179	0.95649465883179	0.95649465884735
0.4	0.95986967789500	0.95986967789500	0.95986967791053
0.5	0.96420205838500	0.96420205838500	0.96420205840048
0.6	0.96948658141026	0.96948658141026	0.96948658142561
0.7	0.97571684467904	0.97571684467904	0.97571684469400
0.8	0.98288524879211	0.98288524879211	0.98288524880584
0.9	0.99098298147963	0.99098298147963	0.99098298148966
1.0	1.00000000000000	1.00000000000000	1.00000000000000

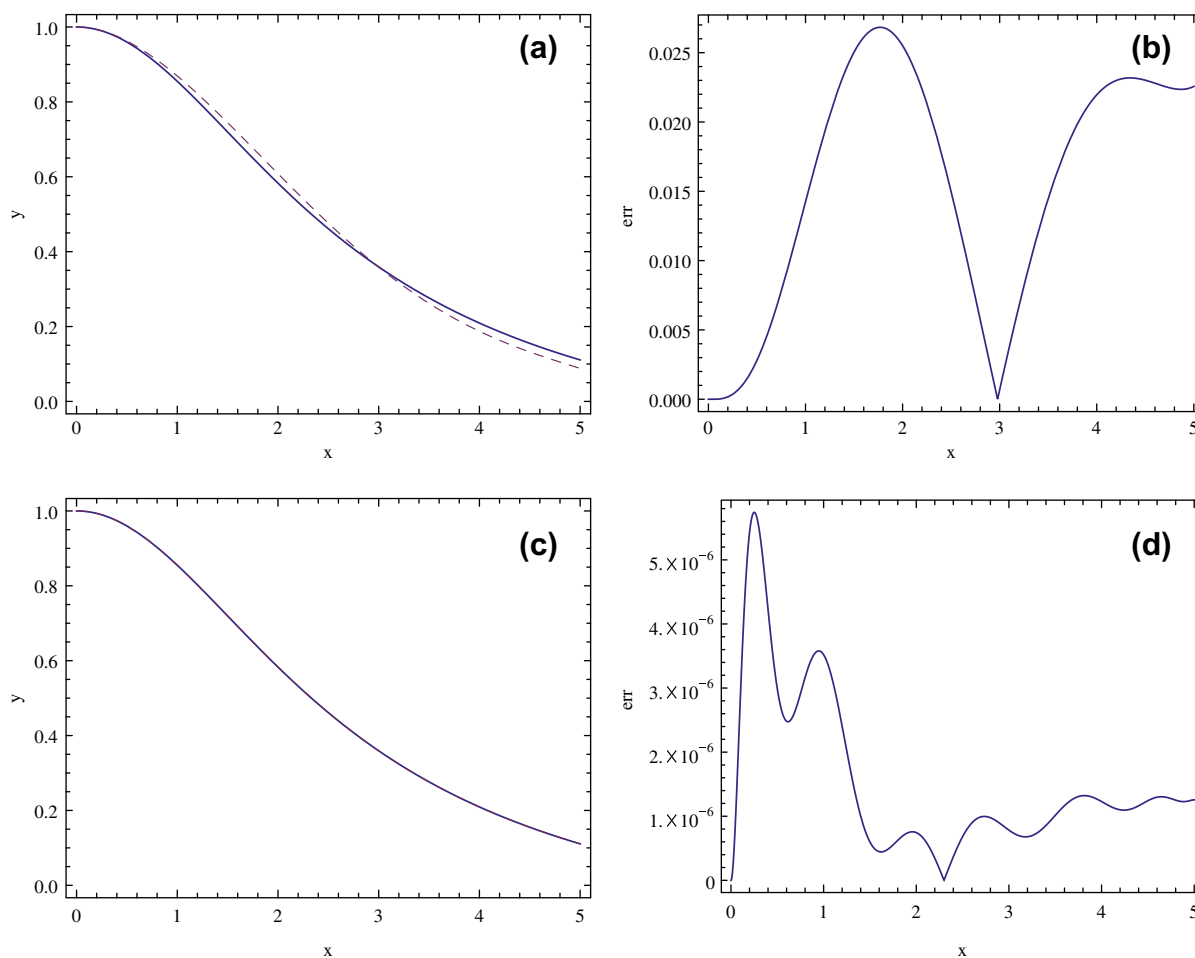


Fig. 1. Exact and approximate solutions and absolute errors computed from the present method for (4.23). (a-b) $n = 4$ and (c-d) $n = 12$. Unbroken curves represent exact and dashed present solutions in (a-c).

Table 2

The theoretical error TE and maximum absolute errors of present method and of [13] are tabulated with various n for the solutions of (4.24).

	$n = 3$	$n = 6$	$n = 8$
TE	1.3888×10^{-2}	1.3893×10^{-3}	3.7606×10^{-4}
Present <i>err</i>	1.1100×10^{-3}	5.5622×10^{-6}	5.2440×10^{-8}
[13] <i>err</i>	3.9200×10^{-4}	–	–

be easily judged from the Figs. 5 and 6 drawn in [7]. A quantification can not be made for this example since no data was presented in [7]. Moreover, the Adomian decomposition method implemented in [5] for this problem was reported to diverge in [7].

Example 4. We consider now a nonlinear singular two-point boundary value problem arising in astronomy, the equilibrium of isothermal gas spheres described by [1]

$$y'' + \frac{2}{x}y' + y^5 = 0, \quad y'(0) = 0, \quad y(1) = \frac{\sqrt{3}}{2}, \quad x \in [0, 1]. \tag{4.24}$$

Eq. (4.24) has an exact solution $y = 1/\sqrt{1 + \frac{x^2}{3}}$. In Table 2 the accuracy and convergence history of the approximate solutions are presented. The maximum absolute errors are observed to diminish fast at an exponential rate as compared with the theoretical estimate TE (3.13) as seen in Table 2. Table also shows the errors of variational iteration method of [13], which are

Table 3

The theoretical error TE and maximum absolute errors of present method are tabulated with various n for the solutions of (4.25).

TE	$n = 5$	$n = 10$	$n = 15$
TE	2.1101×10^{-1}	1.9751×10^{-2}	8.7827×10^{-4}
err	8.0229×10^{-3}	1.2038×10^{-6}	1.4881×10^{-10}

Table 4

Comparison between the present and methods in [13,17].

x	Exact solution	Present method	Method in [13]	Method in [17]
0.0	0.82848329035969	0.82848329035969	0.82848355162932	0.82848327295802
0.1	0.82970609243381	0.82970609243381	0.82970635371727	0.82970607521884
0.2	0.83337473359101	0.83337473359101	0.83337499490687	0.83337471691089
0.3	0.83948991395371	0.83948991395371	0.83949017524076	0.83948989814383
0.4	0.84805278499607	0.84805278499607	0.84805304589079	0.84805277036165
0.5	0.85906492716924	0.85906492716924	0.85906518654929	0.85906491397434
0.6	0.87252831995829	0.87252831995829	0.87252857519543	0.87252830841853
0.7	0.88844530562320	0.88844530562320	0.88844555152002	0.88844529589927
0.8	0.90681854806681	0.90681854806681	0.90681877548439	0.90681854026297
0.9	0.92765098836558	0.92765098836558	0.92765118257926	0.92765098252660
1.0	0.95094579849648	0.95094579849648	0.95094593734191	0.95094579461056

again higher than the presented errors from the current method for increasing approximation levels. It should be noticed that [13] did not give error beyond the iteration number 3 because we believe that the time for integrations beyond the iteration number 3 considerably increases in variational iteration approach.

Example 5. Consider now isothermal gas spheres which are modeled by [2]

$$y'' + \frac{2}{x}y' + e^y = 0, \quad y(0) = y'(0) = 0, \quad x \in [0, R]. \quad (4.25)$$

After transforming with

$$u = e^{-y}, \quad x = Rt,$$

Eq. (4.25) reduces to

$$u \left(u'' + \frac{2}{t}u' - R^2 \right) - u'2 = 0, \quad u(0) = 1, \quad u'(0) = 0, \quad t \in [0, 1]. \quad (4.26)$$

Again no an exact solution can be found for this problem. We now solve Eq. (4.26) using our method and compare the results with the numerical (exact) ones for $R = 5$. For this purpose, in Table 3 the theoretical estimate TE (3.13) for the error and maximum absolute error (4.16) are presented for changing approximation levels. The fast convergence of our method to the exact solution can be observed, quadratic decrease of the maximum absolute error when n is doubled exhibits how accurate the method is for this real hard singular nonlinear physical model. On the other hand, no such a fast convergence can be observed from the 20th order homotopy solutions in [7] (see Fig. 7 in [7]), in which it was also demonstrated that the Adomian decomposition method [5] fails to converge for this problem.

Example 6. Consider now the nonlinear singular boundary value problem arising in oxygen tension in a cell with Michaelis-Menten oxygen uptake kinetics [15,16]

$$y'' + \frac{2}{x}y' - \frac{\alpha y}{y + \beta} = 0, \quad y'(0) = 0, \quad 5y(1) + y'(1) = 5, \quad x \in [0, 1], \quad (4.27)$$

where $\alpha = 0.76129$ and $\beta = 0.03119$. No exact solution is available for this nonlinear problem. For comparison purposes, Table 4 tabulates the values found from the present approach at $n = 12$, the variational iteration approach of [13] and the cubic spline approach of [17]. The superiority of our method over the latter is clear from this table. The 12th order approximate analytic solution to (4.27) results in an accuracy of order $O(10^{-16})$ by the present approach which is adequate for the practical purposes.

Fig. 2 also demonstrates numerical (exact) and approximate solutions and the absolute errors at two levels of approximations, $n = 2$ and $n = 12$, respectively. Figs. 2(a-d) clearly show that the accuracy of our method is getting better as the approximation level is increasing. There appears an exponential decrease in the maximum absolute error for increasing n and thus the proposed method converges rapidly to the exact solution. The convergence of proposed technique for this singular nonlinear problem is summarized in Table 5, which is in line with the results shown in Fig. 2. The 5th order approximate solution having an accuracy of order $O(10^{-8})$ is expressed by

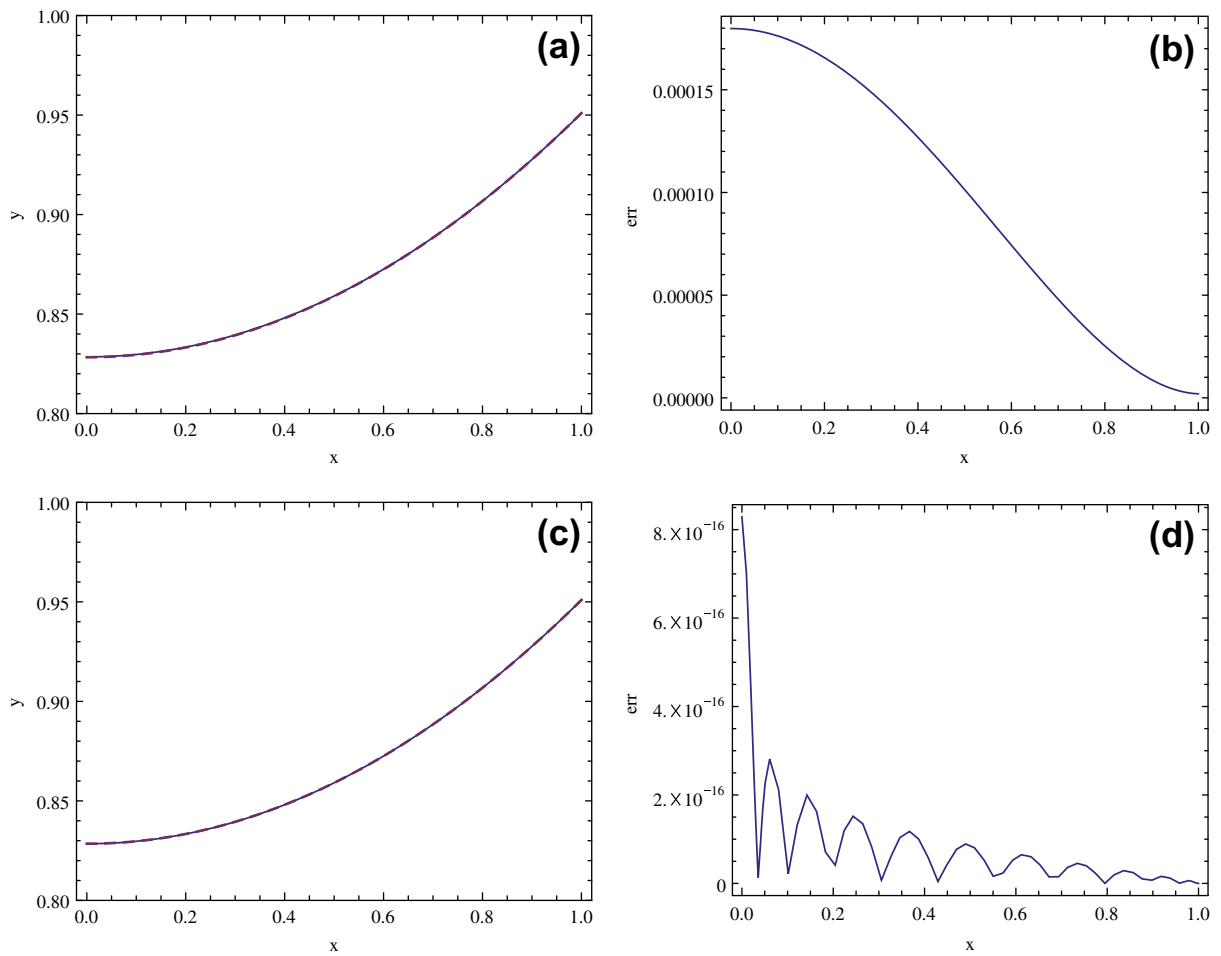


Fig. 2. Exact and approximate solutions and absolute errors computed from the present method for (4.27). (a-b) $n = 2$ and (c-d) $n = 12$. Unbroken curves represent exact and dashed present solutions in (a-c).

Table 5
The theoretical error TE and maximum absolute errors of present method and of [13,17] are tabulated with various n for the solutions of (4.27).

	$n = 4$	$n = 8$	$n = 12$
TE	1.1796×10^{-5}	1.8393×10^{-7}	8.8765×10^{-9}
Present <i>err</i>	8.7984×10^{-7}	2.9079×10^{-11}	8.2903×10^{-16}
[13] <i>err</i>	2.1311×10^{-6}	–	–
[17] <i>err</i>	1.0432×10^{-7}	–	–

$$y = 0.8284832475 + 0.1222817603x^2 - 0.00001764304867x^3 + 0.0002317526606x^4 - 0.00003331890487x^5. \tag{4.28}$$

5. Conclusions

A feasible series expansion technique has been introduced in this study to compute exact and analytic approximate solutions for the singular nonlinear Lane–Emden–Fowler type differential equations arising from the models of mathematical physics and astrophysics. Matrix formulation is used throughout the entire procedure. The presented algorithm expands the desired solution in terms of a set of continuous functions (the simplest are polynomials) over a closed interval and then makes use of the Galerkin-like method to determine the expansion coefficients to construct a solution. Hence, within this new approach, the problem is reduced to the solution of a system of nonlinear algebraic equations. The method is simple and easy to implement, since it is generally based on the well selected base functions. A mathematical analysis regarding

the error and convergence of the technique is provided. The success of the method is verified against comparisons with some of the available methods, such as the homotopy analysis method, the reproducing kernel space method, the cubic spline method and the variational iteration method. Several physical singular nonlinear examples treated by the method exhibit that the proposed method is reliable, efficient and of high accuracy as compared to variants. Therefore, the proposed scheme is an effective and highly promising method for treating various classes of both linear and nonlinear singular boundary value problems.

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