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# Di-uniformities and Hutton uniformities☆

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**Abstract**

The authors characterize di-uniformities on a texture $(S, \mathscr{S})$ in the sense of Özçağ and Brown (Di-uniform texture spaces, Appl. Gen. Top. 4(1) (2003), 157–192) in terms of functions on the texturing $\mathscr{S}$. This characterization enables quasi-uniformities in the sense of Hutton (Uniformities on Fuzzy Topological Spaces, J. Math. Anal. Appl. 58 (1977) 559–571) to be regarded as di-uniformities on the corresponding Hutton Texture, thereby revealing di-uniformities as a generalization of Hutton quasi-uniformities. The effect of imposing a complementation on $(S, \mathscr{S})$ is also considered and several important results established.


*Keywords:* Topology; Texture; Di-uniformity; Quasi di-uniformity; Hutton uniformity; Hutton quasi-uniformity; Category theory

## 1. Introduction

Di-uniform texture spaces were introduced in [14], and their study continued in [15], where the relation with classical uniformities and quasi-uniformities was considered. More recently, the concept of quasi-di-uniformity has been introduced in [17]. The most useful representations to date have been the direlational and dicovering approaches, although the use of dimetrics has also been considered. This paper is based on the direlational representation, which is recalled below:

**Definition 1.1** (*Özçağ and Brown [14]*). Let $(S, \mathcal{S})$ be a texture and $\mathcal{U}$ a family of direlations on $(S, \mathcal{S})$. If $\mathcal{U}$ satisfies the conditions:

(1) $(i, I) \sqsubseteq (d, D)$ for all $(d, D) \in \mathcal{U}$. That is, $\mathcal{U} \subseteq \mathcal{RDR}$.
(2) $(d, D) \in \mathcal{U}$, $(e, E) \in \mathcal{DR}$ and $(d, D) \sqsubseteq (e, E)$ implies $(e, E) \in \mathcal{U}$.
(3) $(d, D), (e, E) \in \mathcal{U}$ implies $(d, D) \sqcap (e, E) \in \mathcal{U}$.
(4) Given $(d, D) \in \mathcal{U}$ there exists $(e, E) \in \mathcal{U}$ satisfying $(e, E) \circ (e, E) \sqsubseteq (d, D)$.
(5) Given $(d, D) \in \mathcal{U}$ there exists $(c, C) \in \mathcal{U}$ satisfying $(c, C)^{\leftarrow} \sqsubseteq (d, D)$.

then $\mathcal{U}$ is called a *direlational uniformity* on $(S, \mathcal{S})$, and $(S, \mathcal{S}, \mathcal{U})$ is known as a *direlational uniform texture space*.

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Clearly, this definition is formally the same as the usual definition of diagonal uniformity, although the symmetry condition (5) is of a different nature from the symmetry condition for entourages, and it is shown in [15] that a direlational uniformity on the discrete texture $(X, \mathcal{P}(X))$ corresponds not to a uniformity but to a quasi-uniformity, the distinction between the two being a matter of complementation. When the symmetry condition (5) is removed, one obtains a direlational quasi-uniformity [17].

In [11], Hutton gave the definition of uniformities and quasi-uniformities on a Hutton algebra $\mathbb{L}^X$ that involves functions on $\mathbb{L}^X$, and it is natural to ask if a similar representation is possible for di-uniformities and quasi di-uniformities. It is the aim of this paper to give a positive answer to this question, showing in particular that di-uniformities stand in the same relation to Hutton uniformities and quasi-uniformities as they do to uniformities and quasi-uniformities in the classical sense.

The layout of the paper is as follows. In Section 2 we give a point-free representation of direlational uniformities, called difunctional uniformities. Our motivation is from [11]. Specifically, Hutton considers functions $g$ on a Hutton algebra $\mathbb{L}$ (or $\mathbb{L}^X$, as in [11]) satisfying

$(a_1)$ $\alpha \leq g(\alpha) \quad \forall \alpha \in \mathbb{L}$,
$(a_2)$ $g(\bigvee_j \alpha_j) = \bigvee g(\alpha_j) \quad \text{for } \alpha_j \in \mathbb{L},\ j \in J$,

uniformities and quasi-uniformities being appropriate subsets of the set $\mathcal{Q}$ of functions on $\mathbb{L}$. We begin by showing that functions on the texturing $\mathcal{S}$ of a general texture $(S, \mathcal{S})$ satisfying conditions corresponding to $(a_1)$ and $(a_2)$ are in one-to-one correspondence with the reflexive (textural) relations on $(S, \mathcal{S})$, while functions satisfying conditions dual to $(a_1)$, $(a_2)$ likewise characterize the reflexive corelations. This leads to the required representation, and by expressing uniform bicontinuity in similar terms it is shown that we obtain a category that is concretely isomorphic with the category of direlational uniformities and uniformly bicontinuous difunctions.

Section 3 presents various properties of difunctional uniformities and quasi-uniformities. In particular the uniform ditopology is characterized in difunctional terms, while a generalization of a result in [11] shows that an arbitrary ditopological texture space is difunctionally quasi-uniformizable. This strengthens a result in [17], where only plain ditopological texture spaces were shown to be direlationally quasi-uniformizable. In this section the existence of an open, coclosed base for a difunctional (quasi-) uniformity, separation and complementation are also considered.

Finally, Section 4 looks at difunctional (quasi-) uniformities as (quasi-) di-uniformities on a complete, completely distributive lattice $\mathbb{L}$, thereby defining the notion of (quasi-) di-uniform Hutton space. The relations between the category of quasi-di-uniform Hutton spaces and uniformly bicontinuous morphisms, and the category of difunctional quasi-uniform texture spaces and uniformly bicontinuous difunctions are studied, and an alternative representation of (quasi-) di-uniformities on $\mathbb{L}$ is obtained. This permits the quasi-uniformities of Hutton on $\mathbb{L}$ to be regarded as di-uniformities on $\mathbb{L}$, while the Hutton uniformities correspond to the complemented di-uniformities.

This section concludes with some basic definitions from the theory of ditopological texture spaces, and the reader is referred to [3–7] for more background material on textures and [14–17,22] for di-uniformities. A modern introduction to quasi-uniformities is [12]. Our standard reference for category theory is [1], while terms from lattice theory not defined here may be found in [10].

*Texture*: A *texturing* on a set $S$ is a point-separating, complete, completely distributive lattice $\mathcal{S}$ of subsets of $S$ with respect to inclusion, which contains $S$ and $\emptyset$, and for which arbitrary meet coincides with intersection and finite joins coincide with unions. The pair $(S, \mathcal{S})$ is called a *texture*. For $s \in S$ the sets

$$P_s = \bigcap\{A \in \mathcal{S} \mid s \in A\} \quad \text{and} \quad Q_s = \bigvee\{A \in \mathcal{S} \mid s \notin A\}$$

are called, respectively, the *p*-sets and *q*-sets of $(S, \mathcal{S})$. These sets are used in the definition of many textural concepts. A texture need not be closed under set complementation. A mapping $\sigma : \mathcal{S} \to \mathcal{S}$ satisfying $\sigma(\sigma(A)) = A, \forall A \in \mathcal{S}$ and $A \subseteq B \implies \sigma(B) \subseteq \sigma(A),\ \forall A, B \in \mathcal{S}$ is called a *complementation* on $(S, \mathcal{S})$ and $(S, \mathcal{S}, \sigma)$ is then said to be a *complemented texture*.

*Examples*:

(1) For any set $X$, $(X, \mathcal{P}(X), \pi_X)$, $\pi_X(Y) = X \setminus Y$ for $Y \subseteq X$, is the complemented *discrete texture* representing the usual set structure of $X$. Clearly, $P_x = \{x\}$ and $Q_x = X \setminus \{x\}$ for all $x \in X$.

(2) For $\mathbb{I} = [0,1]$ define $\mathcal{I} = \{[0,t] \mid t \in [0,1]\} \cup \{[0,t) \mid t \in [0,1]\}$, $\iota([0,t]) = [0,1-t)$ and $\iota([0,t)) = [0,1-t]$, $t \in [0,1]$. Then $(\mathbb{I}, \mathcal{I}, \iota)$ is a complemented texture, which we will refer to as the *unit interval texture*. Here $P_t = [0,t]$ and $Q_t = [0,t)$ for all $t \in I$.

Textures were introduced as a point-set setting for the study of fuzzy topology, and provide a unified setting for the study of topology, bitopology and fuzzy topology. Some of the links with Hutton spaces, $\mathbb{L}$-fuzzy sets and topologies are expressed in a categorical setting in [4,5]. Let us recall that a Hutton algebra $\mathbb{L}$ is a complete, completely distributive lattice with an order reversing involution $'$. We denote by $M_{\mathbb{L}}$ the set of molecules (non-zero join irreducible elements) in $\mathbb{L}$, set $\hat{a} = \{m \in M_{\mathbb{L}} \mid m \le a\}$ for $a \in \mathbb{L}$ and $\mathcal{M}_{\mathbb{L}} = \{\hat{a} \mid a \in \mathbb{L}\}$. Then $(M_{\mathbb{L}}, \mathcal{M}_{\mathbb{L}})$ is a simple texture. Moreover $\mu_{\mathbb{L}} : \mathcal{M}_{\mathbb{L}} \to \mathcal{M}_{\mathbb{L}}$ defined by $\mu_{\mathbb{L}}(\hat{a}) = \widehat{a'}$ is a complementation on $(M_{\mathbb{L}}, \mathcal{M}_{\mathbb{L}})$. We will refer to $(M_{\mathbb{L}}, \mathcal{M}_{\mathbb{L}}, \mu_{\mathbb{L}})$ as the *Hutton texture* of $(\mathbb{L}, ')$. Conversely every complemented simple texture $(S, \mathcal{S}, \sigma)$ is the Hutton texture of $(\mathcal{S}, \sigma)$, regarded as a Hutton algebra.

*Direlations*: Let $(S, \mathcal{S})$, $(T, \mathcal{T})$ be textures. We denote by $\mathcal{P}(S) \otimes \mathcal{T}$ the textural product of $\mathcal{P}(S)$ and $\mathcal{T}$, and by $\overline{P}_{(s,t)}$, $\overline{Q}_{(s,t)}$ the p-sets and q-sets, respectively, in the product texture $(S \times T, \mathcal{P}(S) \otimes \mathcal{T})$. Then:

(1) $r \in \mathcal{P}(S) \otimes \mathcal{T}$ is called a *relation from* $(S, \mathcal{S})$ *to* $(T, \mathcal{T})$ if it satisfies
*R1* $r \not\subseteq \overline{Q}_{(s,t)}, P_{s'} \not\subseteq Q_s \implies r \not\subseteq \overline{Q}_{(s',t)}$.
*R2* $r \not\subseteq \overline{Q}_{(s,t)} \implies \exists s' \in S$ such that $P_s \not\subseteq Q_{s'}$ and $r \not\subseteq \overline{Q}_{(s',t)}$.
(2) $R \in \mathcal{P}(S) \otimes \mathcal{T}$ is called a *corelation from* $(S, \mathcal{S})$ *to* $(T, \mathcal{T})$ if it satisfies
*CR1* $\overline{P}_{(s,t)} \not\subseteq R, P_s \not\subseteq Q_{s'} \implies \overline{P}_{(s',t)} \not\subseteq R$.
*CR2* $\overline{P}_{(s,t)} \not\subseteq R \implies \exists\, s' \in S$ such that $P_{s'} \not\subseteq Q_s$ and $\overline{P}_{(s',t)} \not\subseteq R$.

A pair (*r*,*R*) consisting of a relation *r* and corelation *R* is now called a *direlation*.
(*i*,*I*) is called the *identity direlation* on $(S, \mathcal{S})$ where

$$i = i_S = \bigvee \{\overline{P}_{(s,s)} \mid s \in S\} \quad \text{and} \quad I = I_S = \bigcap \{\overline{Q}_{(s,s)} \mid s \in S\}.$$

A direlation (*r*,*R*) on $(S, \mathcal{S})$ (that is, on $(S, \mathcal{S})$ to $(S, \mathcal{S})$) is *reflexive* if *r* and *R* are reflexive, that is if $(i, I) \sqsubseteq (r, R)$. Let us denote by $\mathcal{RR}$ the set of reflexive relations and by $\mathcal{RCR}$ the set of reflexive corelations.

Now let (*r*,*R*) be a direlation from $(S, \mathcal{S})$ to $(T, \mathcal{T})$. The inverses of *r* and *R* are given by

$$r^{\leftarrow} = \bigcap \{\overline{Q}_{(t,s)} \mid r \not\subseteq \overline{Q}_{(s,t)}\}, \quad R^{\leftarrow} = \bigvee \{\overline{P}_{(t,s)} \mid \overline{P}_{(s,t)} \not\subseteq R\},$$

where $R^{\leftarrow}$ is a relation and $r^{\leftarrow}$ a corelation. The direlation $(r, R)^{\leftarrow} = (R^{\leftarrow}, r^{\leftarrow})$ from $(T, \mathcal{T})$ to $(S, \mathcal{S})$ is called the *inverse* of (*r*,*R*).

For $A \subseteq S$ the *A-section of a relation r* and *A-section of a corelation R* are defined by

$$r^{\rightarrow} A = \bigcap \{Q_t \mid \forall s, r \not\subseteq \overline{Q}_{(s,t)} \implies A \subseteq Q_s\},$$

$$R^{\rightarrow} A = \bigvee \{P_t \mid \forall s, \overline{P}_{(s,t)} \not\subseteq R \implies P_s \subseteq A\}.$$

For $B \subseteq T$ the *B-presections of r and R* are defined by

$$r^{\leftarrow} B = \bigvee \{P_s \mid \forall t, r \not\subseteq \overline{Q}_{(s,t)} \implies P_t \subseteq B\} \in \mathcal{S},$$

$$R^{\leftarrow} B = \bigcap \{Q_s \mid \forall t, \overline{P}_{(s,t)} \not\subseteq R \implies B \subseteq Q_t\} \in \mathcal{S}.$$

Let $(S, \mathcal{S})$, $(T, \mathcal{T})$, $(U, \mathcal{U})$ be textures. If *p* is a relation on $(S, \mathcal{S})$ to $(T, \mathcal{T})$ and *q* a relation on $(T, \mathcal{T})$ to $(U, \mathcal{U})$ then their *composition* is the relation $q \circ p$ on $(S, \mathcal{S})$ to $(U, \mathcal{U})$ defined by

$$q \circ p = \bigvee \{\overline{P}_{(s,u)} \mid \exists t \in T \text{ with } p \not\subseteq \overline{Q}_{(s,t)} \text{ and } q \not\subseteq \overline{Q}_{(t,u)}\}.$$

If *P* is a corelation on $(S, \mathcal{S})$ to $(T, \mathcal{T})$ and *Q* a corelation on $(T, \mathcal{T})$ to $(U, \mathcal{U})$ then their *composition* is the corelation $Q \circ P$ on $(S, \mathcal{S})$ to $(U, \mathcal{U})$ defined by

$$Q \circ P = \bigcap \{\overline{Q}_{(s,u)} \mid \exists t \in T \text{ with } \overline{P}_{(s,t)} \not\subseteq P \text{ and } \overline{P}_{(t,u)} \not\subseteq Q\}.$$

The *composition* of the direlations $(p,P)$, $(q,Q)$ is the direlation

$$(q, Q) \circ (p, P) = (q \circ p, Q \circ P).$$

A direlation $(f,F)$ on $(S, \mathcal{S})$ to $(T, \mathcal{T})$ is called a *difunction* if it satisfies the following two conditions.

$DF1$ For $s, s' \in S$, $P_s \not\subseteq Q_{s'} \Longrightarrow \exists t \in T$ with $f \not\subseteq \overline{Q}_{(s,t)}$ and $\overline{P}_{(s',t)} \not\subseteq F$.
$DF2$ For $t, t' \in T$ and $s \in S$, $f \not\subseteq \overline{Q}_{(s,t)}$ and $\overline{P}_{(s,t')} \not\subseteq F \Longrightarrow P_{t'} \not\subseteq Q_t$.

## 2. A point-free characterization of di-uniformities

**Definition 2.1.** Let $(S, \mathcal{S})$ be a texture.

(1) We denote by $\mathcal{F}_{\mathcal{R}}$ the family of functions $\varphi : \mathcal{S} \to \mathcal{S}$ satisfying

(i) $\varphi\left(\bigvee_{j \in J} A_j\right) = \bigvee_{j \in J} \varphi(A_j), \forall A_j \in \mathcal{S},\ j \in J$

and by $\mathcal{F}_{\mathcal{R}\mathcal{R}}$ those functions that satisfy (i) and

(ii) $A \subseteq \varphi(A),\ \forall A \in \mathcal{S}$.

(2) We denote by $\mathcal{F}_{\mathcal{C}\mathcal{R}}$ the family of functions $\psi : \mathcal{S} \to \mathcal{S}$ satisfying

(i) $\psi\left(\bigcap_{j \in J} A_j\right) = \bigcap_{j \in J} \psi(A_j), \forall A_j \in \mathcal{S},\ j \in J$

and by $\mathcal{F}_{\mathcal{R}\mathcal{C}\mathcal{R}}$ those functions that satisfy (i) and

(ii) $\psi(A) \subseteq A,\ \forall A \in \mathcal{S}$.

The following proposition shows that the elements of $\mathcal{F}_{\mathcal{R}}$ $(\mathcal{F}_{\mathcal{R}\mathcal{R}})$ correspond to the (reflexive) relations on $(S, \mathcal{S})$, and that dually those of $\mathcal{F}_{\mathcal{C}\mathcal{R}}$ $(\mathcal{F}_{\mathcal{R}\mathcal{C}\mathcal{R}})$ correspond to the (reflexive) corelations on $(S, \mathcal{S})$.

**Proposition 2.2.** *Let $(S, \mathcal{S})$ be a texture.*

(1) *Let $r$ be a relation on $(S, \mathcal{S})$, and define the function $\varphi_r : \mathcal{S} \to \mathcal{S}$ by $\varphi_r(A) = r^{\to}A$ for all $A \in \mathcal{S}$. Then $\varphi_r \in \mathcal{F}_{\mathcal{R}}$, while if $r$ is reflexive then $\varphi_r \in \mathcal{F}_{\mathcal{R}\mathcal{R}}$.*
*Conversely, if $\varphi \in \mathcal{F}_{\mathcal{R}}$ then $r_\varphi \in \mathcal{P}(S) \otimes \mathcal{S}$ defined by*

$$r_\varphi = \bigvee \{\overline{P}_{(s,t)} \mid \exists u, v \in S\ with\ P_s \not\subseteq Q_u\ and\ P_v \not\subseteq Q_t\ so\ that\ \varphi(B) \subseteq Q_v \Longrightarrow B \subseteq Q_u \forall B \in \mathcal{S}\}$$

*is a relation on $(S, \mathcal{S})$, while if $\varphi \in \mathcal{F}_{\mathcal{R}\mathcal{R}}$ then it is a reflexive relation.*
*Moreover, $\varphi_{r_\varphi} = \varphi, \forall \varphi \in \mathcal{F}_{\mathcal{R}}$, and $r_{\varphi_r} = r, \forall r \in \mathcal{R}$.*

(2) *Let $R$ be a corelation on $(S, \mathcal{S})$, and define the function $\psi_R : \mathcal{S} \to \mathcal{S}$ by $\psi_R(A) = R^{\to}A$ for all $A \in \mathcal{S}$. Then $\psi_R \in \mathcal{F}_{\mathcal{C}\mathcal{R}}$, while if $R$ is a reflexive corelation then $\psi_R \in \mathcal{F}_{\mathcal{R}\mathcal{C}\mathcal{R}}$.*
*Conversely, if $\psi \in \mathcal{F}_{\mathcal{C}\mathcal{R}}$ then $R_\psi \in \mathcal{P}(S) \otimes \mathcal{S}$ defined by*

$$R_\psi = \bigcap \{\overline{Q}_{(s,t)} \mid \exists u, v \in S\ with\ P_u \not\subseteq Q_s\ and\ P_t \not\subseteq Q_v\ so\ that\ P_v \subseteq \psi(B) \Longrightarrow P_u \subseteq B \forall B \in \mathcal{S}\}$$

*is a corelation on $(S, \mathcal{S})$, while if $\psi \in \mathcal{F}_{\mathcal{R}\mathcal{C}\mathcal{R}}$ then it is a reflexive corelation.*
*Moreover, $\psi_{R_\psi} = \psi,\ \forall \psi \in \mathcal{F}_{\mathcal{C}\mathcal{R}}$, and $R_{\psi_R} = R, \forall R \in \mathcal{C}\mathcal{R}$.*

**Proof.** We prove (1), leaving the proof of the dual statements (2) to the interested reader.

If $r \in \mathcal{R}$ then $\varphi_r$ satisfies Definition 2.1(1(i)) by [5, Corollary 2.12(2)], while if $r \in \mathcal{R}\mathcal{R}$ and $A \in \mathcal{S}$, then $A = \iota_S^{\to} A \subseteq r^{\to} A = \varphi_r(A)$ since $\iota_S \subseteq r$ by the reflexivity of $r$, so $\varphi_r$ also satisfies Definition 2.1(1(ii)) and hence $\varphi_r \in \mathcal{F}_{\mathcal{R}\mathcal{R}}$.

It is trivial to check conditions *R1* and *R2* of [5, Definition 2.1(1)] for $\varphi_r$, even if $\varphi$ is arbitrary, and we omit the details. Suppose that $\varphi$ satisfies Definition 2.1(1(ii)). To prove the relation $r_\varphi$ is reflexive suppose that $\iota_S \not\subseteq r_\varphi$. Then for some $s, t \in S$ we have $\iota_S \not\subseteq \overline{Q}_{(s,t)}$ and $\overline{P}_{(s,t)} \not\subseteq r_\varphi$. The first result gives $P_s \not\subseteq Q_t$, whence we may choose $u, v \in S$

with $P_s \not\subseteq Q_u$, $P_u \not\subseteq Q_v$ and $P_v \not\subseteq Q_t$. But now for $B \in \mathcal{S}$ we have $\varphi(B) \subseteq Q_v \implies P_u \not\subseteq \varphi(B) \implies B \subseteq \varphi(B) \subseteq Q_u$ by Definition 2.1(1(ii)), so $\overline{P}_{(s,t)} \subseteq r_\varphi$, which is a contradiction. Hence $r_\varphi$ is a reflexive relation, as required.

Now for $\varphi \in \mathcal{F}_{\mathcal{R}}$ let us prove $\varphi_{r_\varphi} = \varphi$. To this end, suppose first that $\varphi_{r_\varphi}(A) \not\subseteq \varphi(A)$ for some $A \in \mathcal{S}$. Then there exists $t \in S$ with $\varphi_{r_\varphi}(A) \not\subseteq Q_t$ and $P_t \not\subseteq \varphi(A)$. Now $\varphi_{r_\varphi}(A) = r_\varphi^{\to} A \not\subseteq Q_t$, whence by [5, Definition 2.5(1)] there exists $z \in S$ with $r_\varphi \not\subseteq \overline{Q}_{(z,t)}$ and $A \not\subseteq Q_z$. By the definition of $r_\varphi$ we now have $t' \in S$ with $\overline{P}_{(z,t')} \not\subseteq \overline{Q}_{(z,t)}$, and $u, v \in S$ satisfying $P_z \not\subseteq Q_u$, $P_v \not\subseteq Q_{t'}$ for which

$$\varphi(B) \subseteq Q_v \implies B \subseteq Q_u, \quad \forall B \in \mathcal{S}. \tag{1}$$

Clearly $P_t \subseteq P_{t'} \subseteq P_v$, so $P_t \not\subseteq \varphi(A)$ gives $P_v \not\subseteq \varphi(A)$, whence $\varphi(A) \subseteq Q_v$. Applying the implication (1) with $B = A$ now gives $A \subseteq Q_u$. Since $Q_u \subseteq Q_z$, this contradicts $A \not\subseteq Q_z$.

Now suppose that $\varphi(A) \not\subseteq \varphi_{r_\varphi}(A)$ for some $A \in \mathcal{S}$. Now we have $t \in S$ with $\varphi(A) \not\subseteq Q_t$ and $P_t \not\subseteq r_\varphi^{\to} A$, and hence $t' \in S$ with $P_t \not\subseteq Q_{t'}$ for which

$$r_\varphi \not\subseteq \overline{Q}_{(z,t')} \implies A \subseteq Q_z, \quad \forall z \in S. \tag{2}$$

Now choose $v, t'' \in S$ satisfying $P_t \not\subseteq Q_v$, $P_v \not\subseteq Q_{t''}$ and $P_{t''} \not\subseteq Q_{t'}$, and set

$$B_0 = \bigvee \{B \in \mathcal{S} \mid \varphi(B) \subseteq Q_v\}.$$

By Definition 2.1(1(i)) we have $\varphi(B_0) = \bigvee\{\varphi(B) \mid B \in \mathcal{S}, \varphi(B) \subseteq Q_v\} \subseteq Q_v \subseteq Q_t$, whence $\varphi(A) \not\subseteq \varphi(B_0)$. Again using Definition 2.1(1(i)) we see that $\varphi$ preserves inclusion, so $A \not\subseteq B_0$ and we may choose $s, u \in S$ satisfying $A \not\subseteq Q_s$, $P_s \not\subseteq Q_u$ and $P_u \not\subseteq B_0$. We now obtain $\overline{P}_{(s,t'')} \not\subseteq \overline{Q}_{(s,t')}$, $P_s \not\subseteq Q_u$, $P_v \not\subseteq Q_{t''}$ and for $B \in \mathcal{S}$, $\varphi(B) \subseteq Q_v \implies B \subseteq B_0 \subseteq Q_u$, so $r_\varphi \not\subseteq \overline{Q}_{(s,t')}$. Finally, applying implication (2) with $z = s$ gives $A \subseteq Q_s$, which is a contradiction. This completes the proof that $\varphi_{r_\varphi} = \varphi$. $\square$

To obtain the second equality we note the following:

**Lemma 2.3.** *If $p, q \in \mathcal{R}$ and $\varphi_p$, $\varphi_q$ are the corresponding functions on $\mathcal{S}$ then $p \subseteq q \iff \varphi_p \le \varphi_q$, where $\varphi_p \le \varphi_q \iff \varphi_p(A) \subseteq \varphi_q(A)\ \forall A \in \mathcal{S}$.*

*Likewise, for $P, Q \in \mathcal{CR}$ we have $P \subseteq Q \iff \psi_P \le \psi_Q$.*

**Proof.** By [5, Lemma 2.7(1)] we have $p \subseteq q \iff (p^{\to} A \subseteq q^{\to} A\ \forall A \in \mathcal{S})$, and the right hand condition is just $\varphi_p \le \varphi_q$. In just the same way, the second result follows from [5, Lemma 2.7(2)]. $\square$

Replacing $\varphi$ by $\varphi_r$ for $r \in \mathcal{R}$ in the equality $\varphi_{r_\varphi} = \varphi$ gives $\varphi_{r_{\varphi_r}} = \varphi_r$, which by Lemma 2.3 is equivalent to $r_{\varphi_r} = r$. $\square$

In this paper we will be concerned mainly with reflexive relations and reflexive corelations, but we pause to note that Proposition 2.2 establishes the important fact that a relation from $(S, \mathcal{S})$ to $(T, \mathcal{T})$ may be regarded as a join-preserving mapping $\varphi : \mathcal{T} \to \mathcal{S}$, and a corelation as a meet-preserving mapping $\psi : \mathcal{T} \to \mathcal{S}$. It follows that the category of textures and relations is equivalent to the category of completely distributive lattices and join-preserving functions, while the lattice of textures and corelations is equivalent to the category of completely distributive lattices and meet-preserving functions.

These results should be compared with [6, Proposition 4.1] which shows that a difunction $(f, F) : (S, \mathcal{S}) \to (T, \mathcal{T})$ may be regarded as a complete lattice morphism $\theta : \mathcal{T} \to \mathcal{S}$, where $f$ is a co-adjoint of $\theta$ and $F$ an adjoint of $\theta$. This gives a dual equivalence between **dfTex**, the category of textures and difunctions and the category of completely distributive lattices and complete lattice morphisms, hence an equivalence between **dfTex** and the category of completely distributive lattices and generalized order homomorphisms of Wang [20].

It is known [5] that the inverse of a relation is a corelation, so the above theorem implies the existence of a bijection between $\mathcal{F}_{\mathcal{R}}$ and $\mathcal{F}_{\mathcal{CR}}$. If $r$ is reflexive then $i_S \subseteq r$, whence $r^{\leftarrow} \subseteq i_S^{\leftarrow}$ by [5, Lemma 2.4(2)], and since $i_S^{\leftarrow} = I_S$ we have $r^{\leftarrow} \subseteq I_S$, so $r^{\leftarrow}$ is also reflexive. In just the same way, the inverse of a reflexive corelation is a reflexive relation. Hence the above bijection will restrict to a bijection between $\mathcal{F}_{\mathcal{RR}}$ and $\mathcal{F}_{\mathcal{RCR}}$. The following proposition makes this explicit.

**Proposition 2.4.** *For $\varphi = \varphi_r \in \mathcal{F}_{\mathcal{R}}$ define $\overleftarrow{\varphi} : \mathcal{S} \to \mathcal{S}$ by $\overleftarrow{\varphi} = \psi_{r^{\leftarrow}} \in \mathcal{F}_{\mathcal{CR}}$. Then*

$$\overleftarrow{\varphi}(A) = \bigvee\{B \in \mathcal{S} \mid \varphi(B) \subseteq A\}, \quad A \in \mathcal{S},$$

*and if $\varphi \in \mathcal{F}_{\mathcal{RR}}$ we have $\overleftarrow{\varphi} \in \mathcal{F}_{\mathcal{RCR}}$.*

*Dually, for $\psi = \psi_R \in \mathcal{F}_{\mathcal{CR}}$ define $\overleftarrow{\psi} : \mathcal{S} \to \mathcal{S}$ by $\overleftarrow{\psi} = \varphi_{R^{\leftarrow}} \in \mathcal{F}_{\mathcal{R}}$. Then*

$$\overleftarrow{\psi}(B) = \bigcap\{A \in \mathcal{S} \mid B \subseteq \psi(A)\}, \quad B \in \mathcal{S},$$

*and if $\psi \in \mathcal{F}_{\mathcal{RCR}}$ we have $\overleftarrow{\psi} \in \mathcal{F}_{\mathcal{RR}}$.*

**Proof.** Take $\varphi = \varphi_r \in \mathcal{F}_{\mathcal{R}}$. By Proposition 2.2 we need only verify the formula $\overleftarrow{\varphi}(A) = \bigvee\{B \in \mathcal{S} \mid \varphi(B) \subseteq A\}$. Clearly it is sufficient to show that $B \subseteq \overleftarrow{\varphi}(A) \iff \varphi(B) \subseteq A$. Regarding the set $\mathcal{S}$ as a category in the usual way, $\overleftarrow{\varphi} : \mathcal{S} \to \mathcal{S}$ is an adjoint. Indeed, for $B \in \mathcal{S}$, $([B, \overleftarrow{\varphi}(\varphi(B))], \varphi(B))$ is a $\overleftarrow{\varphi}$-universal arrow with domain $B$ by [5, Lemma 2.10]. It follows that if $B \subseteq \overleftarrow{\varphi}(A)$ then $\varphi(B) \subseteq A$ by the universal property.

$$\begin{tikzcd}
B \arrow[r, "{[B,\ \overleftarrow{\varphi}(\varphi(B))]}"] \arrow[dr, "{[B,\ \overleftarrow{\varphi}(A)]}"'] & \overleftarrow{\varphi}(\varphi(B)) \arrow[d, dashed, "{\overleftarrow{\varphi}([\varphi(B),\ A])}"] \\
 & \overleftarrow{\varphi}(A)
\end{tikzcd}$$

Conversely, since $B \subseteq \overleftarrow{\varphi}(\varphi(B))$ then $\varphi(B) \subseteq A$ immediately gives $B \subseteq \overleftarrow{\varphi}(A)$.

This proves the first equality, and the proof of the second is dual and is omitted. $\square$

Since $(r^{\leftarrow})^{\leftarrow} = r$ and $(R^{\leftarrow})^{\leftarrow} = R$ it is clear from Proposition 2.4 that $\overleftarrow{\overleftarrow{\varphi}} = \varphi$ and $\overleftarrow{\overleftarrow{\psi}} = \psi$. Since adjoints are unique in the category $\mathcal{S}$ it follows that $\varphi \mapsto \overleftarrow{\varphi}$ is a bijection from $\mathcal{F}_{\mathcal{RR}}$ to $\mathcal{F}_{\mathcal{RCR}}$ with inverse $\psi \mapsto \overleftarrow{\psi}$.

Finally, to give our point-free characterization of di-uniformities we require the following.

**Lemma 2.5.**

*Let $(S, \mathcal{S})$ be a texture, $p, q \in \mathcal{RR}$ and $P, Q \in \mathcal{RCR}$.*

(1) *$\varphi_{p \sqcap q} = \varphi_p \wedge \varphi_q$, $\psi_{P \sqcup Q} = \psi_P \vee \psi_Q$, where $\wedge$ denotes the greatest lower bound in $(\mathcal{F}_{\mathcal{RR}}, \leq)$ and $\vee$ the least upper bound in $(\mathcal{F}_{\mathcal{RCR}}, \leq)$.*

(2) *$\varphi_{p \circ q} = \varphi_p \circ \varphi_q$, $\psi_{P \circ Q} = \psi_P \circ \psi_Q$, where $\circ$ denotes functional composition in both $\mathcal{F}_{\mathcal{RR}}$ and $\mathcal{F}_{\mathcal{RCR}}$.*

**Proof.** (1) It is clear that since $p$ and $q$ are reflexive then so is $p \sqcap q$. Hence by [15, Proposition 1.9(1)] it is the greatest lower bound of $p$, $q$ in $(\mathcal{RR}, \subseteq)$. In view of Lemma 2.3 we deduce that $\varphi_{p \sqcap q}$ is the greatest lower bound of $\varphi_p$, $\varphi_q$ in $(\mathcal{F}_{\mathcal{RR}}, \leq)$. The proof for $\psi_{P \sqcup Q}$ follows in the same way from [15, Proposition 1.9(2)].

(2) It is easy to verify that $p \circ q$ is reflexive since $p$, $q$ are. For $A \in \mathcal{S}$ we have

$$\varphi_{p \circ q}(A) = (p \circ q)^{\to} A = p^{\to}(q^{\to} A)) = \varphi_p(\varphi_q(A)) = (\varphi_p \circ \varphi_q)(A)$$

by [5, Lemma 2.16(1)], whence $\varphi_{p \circ q} = \varphi_p \circ \varphi_q$ as required. The proof for $\psi_{P \circ Q}$ is dual, and is omitted. $\square$

Since a direlational uniformity on $(S, \mathcal{S})$ is a family of direlations we shall consider pairs $(\varphi, \psi)$, where $\varphi \in \mathcal{F}_{\mathcal{RR}}$, $\psi \in \mathcal{F}_{\mathcal{RCR}}$. Here we will write $(\varphi_1, \psi_1) \leq (\varphi_2, \psi_2)$ if and only if $\varphi_1 \leq \varphi_2$ and $\psi_2 \leq \psi_1$. Hence, by Lemma 2.3 we have

$$(r_1, R_1) \sqsubseteq (r_2, R_2) \iff (\varphi_{r_1}, \psi_{R_1}) \leq (\varphi_{r_2}, \psi_{R_2}),$$

and so by Lemma 2.5(1), $(r_1, R_1) \sqcap (r_2, R_2)$ in $\mathcal{RDR}$ corresponds to $(\varphi_{r_1}, \psi_{R_1}) \wedge (\varphi_{r_2}, \psi_{R_2})$ in $\mathcal{F}_{\mathcal{RDR}} = \mathcal{F}_{\mathcal{RR}} \times \mathcal{F}_{\mathcal{RCR}}$. Next we will set $(\varphi_1, \psi_1) \circ (\varphi_2, \psi_2) = (\varphi_1 \circ \varphi_2, \psi_1 \circ \psi_2)$ in $\mathcal{F}_{\mathcal{RDR}}$, whence by Lemma 2.5(2), $(r_1, R_1) \circ (r_2, R_2)$ in

$\mathcal{RDR}$ corresponds to $(\varphi_{r_1}, \psi_{R_1}) \circ (\varphi_{r_2}, \psi_{R_2})$ in $\mathcal{F}_{\mathcal{RDR}}$. For simplicity we set $(\varphi, \psi) \circ (\varphi, \psi) = (\varphi, \psi)^2$. Finally we will use $(\varphi, \psi)^{\leftarrow}$ to denote $(\overleftarrow{\psi}, \overleftarrow{\varphi})$, whence $(\varphi, \psi)^{\leftarrow} \in \mathcal{F}_{\mathcal{RDR}}$ by Proposition 2.4. Clearly $(r, R)^{\leftarrow}$ corresponds to $(\varphi_r, \psi_R)^{\leftarrow}$, so we will call $(\varphi, \psi)$ *symmetric* if $(\varphi, \psi)^{\leftarrow} = (\varphi, \psi)$, that is if $\overleftarrow{\varphi} = \psi$, or equivalently, $\overleftarrow{\psi} = \varphi$.

The proof of the following lemma is now straightforward, and is omitted.

**Lemma 2.6.** *Let $(S, \mathcal{S}, \mathcal{U})$ be a direlational uniform texture space and define $\mathcal{U}_{\mathcal{F}} \subseteq \mathcal{F}_{\mathcal{RDR}}$ by*

$$\mathcal{U}_{\mathcal{F}} = \{(\varphi_r, \psi_R) \mid (r, R) \in \mathcal{U}\}.$$

*Then $\mathcal{U}_{\mathcal{F}}$ is non-empty and has the following properties*:

(1) $(\varphi, \psi) \in \mathcal{U}_{\mathcal{F}}, (\varphi, \psi) \preceq (\varphi_1, \psi_1) \in \mathcal{F}_{\mathcal{RDR}} \Longrightarrow (\varphi_1, \psi_1) \in \mathcal{U}_{\mathcal{F}}$.
(2) $(\varphi_1, \psi_1), (\varphi_2, \psi_2) \in \mathcal{U}_{\mathcal{F}} \Longrightarrow (\varphi_1, \psi_1) \wedge (\varphi_2, \psi_2) \in \mathcal{U}_{\mathcal{F}}$.
(3) *For $(\varphi, \psi) \in \mathcal{U}_{\mathcal{F}}$ there exists $(\varphi_1, \psi_1) \in \mathcal{U}_{\mathcal{F}}$ with $(\varphi_1, \psi_1)^2 \preceq (\varphi, \psi)$.*
(4) *For $(\varphi, \psi) \in \mathcal{U}_{\mathcal{F}}$ there exists $(\varphi_1, \psi_1) \in \mathcal{U}_{\mathcal{F}}$ with $(\varphi_1, \psi_1)^{\leftarrow} \preceq (\varphi, \psi)$.* $\square$

**Definition 2.7.** Let $(S, \mathcal{S})$ be a texture. A subset $\mathcal{U}_{\mathcal{F}}$ of $\mathcal{F}_{\mathcal{RDR}}$ that satisfies conditions (1)–(4) of Lemma 2.6 is called a *difunctional uniformity* on $(S, \mathcal{S})$, and the triple $(S, \mathcal{S}, \mathcal{U}_{\mathcal{F}})$ a *difunctional uniform texture space*.

The mapping $\mathcal{U} \mapsto \mathcal{U}_{\mathcal{F}}$ defined above is clearly a bijection between the direlational uniformities $\mathcal{U}$ on $(S, \mathcal{S})$ and the difunctional uniformities $\mathcal{U}_{\mathcal{F}}$ on $(S, \mathcal{S})$. To view this as a functor we need to characterize the uniformly continuous difunctions in terms of difunctional uniformities. Given a difunction $(f, F) : (S, \mathcal{S}) \to (T, \mathcal{T})$ and $(\varphi, \psi) \in \mathcal{F}^T_{\mathcal{RDR}}$ let us consider the following mappings on $\mathcal{S}$:

$$((f, F)^{-1}(\varphi))(A) = F^{\leftarrow}(\varphi(f^{\rightarrow}A)), \quad A \in \mathcal{S},$$

$$((f, F)^{-1}(\psi))(A) = f^{\leftarrow}(\psi(F^{\rightarrow}A)), \quad A \in \mathcal{S}.$$

**Lemma 2.8.** *With the notation as above, $(f, F)^{-1}(\varphi) \in \mathcal{F}^S_{\mathcal{RR}}$ and $(f, F)^{-1}(\psi) \in \mathcal{F}^S_{\mathcal{RCR}}$.*

**Proof.** For $A \in \mathcal{S}$ we have $A \subseteq F^{\leftarrow}(f^{\rightarrow}A) \subseteq F^{\leftarrow}(\varphi(f^{\rightarrow}A)) = ((f, F)^{-1}(\varphi))(A)$ by [5, Theorem 2.24(2a)] and Definition 2.1(1a). Also, for $A_j \in \mathcal{S}$, $j \in J$, we have

$$\begin{aligned} ((f, F)^{-1}(\varphi))\left(\bigvee_{j\in J} A_j\right) &= F^{\leftarrow}\left(\varphi\left(f^{\rightarrow}\bigvee_{j\in J} A_j\right)\right) = F^{\leftarrow}\left(\varphi\left(\bigvee_{j\in J} f^{\rightarrow}A_j\right)\right) \\ &= F^{\leftarrow}\left(\bigvee_{j\in J} \varphi(f^{\rightarrow}A_j)\right) = \bigvee_{j\in J} F^{\leftarrow}(\varphi(f^{\rightarrow}A_j)) \\ &= \bigvee_{j\in J} ((f, F)^{-1}(\varphi))(A_j) \end{aligned}$$

by [5, Corollary 2.12(2)], Definition 2.1(1b) and [5, Corollary 2.12(2)] again. This shows that $(f, F)^{-1}(\varphi) \in \mathcal{F}^S_{\mathcal{RR}}$, and the proof of the second result is dual to this and therefore omitted. $\square$

Setting $(f, F)^{-1}(\varphi, \psi) = ((f, F)^{-1}(\varphi), (f, F)^{-1}(\psi))$ now gives an element of $\mathcal{F}^S_{\mathcal{RDR}}$. Hence we have a mapping from $\mathcal{F}^T_{\mathcal{RDR}}$ to $\mathcal{F}^S_{\mathcal{RDR}}$, and we may make the following definition:

**Definition 2.9.** Let $(S, \mathcal{S}, \mathcal{U}_{\mathcal{F}}), (T, \mathcal{T}, \mathcal{V}_{\mathcal{F}})$ be difunctional uniform texture spaces and $(f, F) : (S, \mathcal{S}, \mathcal{U}_{\mathcal{F}}) \to (T, \mathcal{T}, \mathcal{V}_{\mathcal{F}})$ a difunction. Then $(f,F)$ is called $\mathcal{U}_{\mathcal{F}}$–$\mathcal{V}_{\mathcal{F}}$ *uniformly bicontinuous* if $(\varphi, \psi) \in \mathcal{V}_{\mathcal{F}} \Longrightarrow (f, F)^{-1}(\varphi, \psi) \in \mathcal{U}_{\mathcal{F}}$.

In order to compare uniform bicontinuity for a difunction between direlational uniformities in the sense of [15, Definition 5.9] with that for the same difunction between the corresponding difunctional uniformities it will suffice to compare the mapping $(f,F)^{-1}$ defined above with the mapping of the same name defined in [15] that maps the reflexive direlations on $(S,\mathcal{S})$ to those on $(T,\mathcal{T})$.

**Proposition 2.10.** *Let* $(f,F):(S,\mathcal{S})\to(T,\mathcal{T})$ *be a difunction and (r,R) a reflexive direlation on* $(T,\mathcal{T})$. *Then*

$$(f,F)^{-1}(\varphi_r,\psi_R)=(\varphi_{(f,F)^{-1}(r)},\psi_{(f,F)^{-1}(R)}).$$

**Proof.** We show that $(f,F)^{-1}(\varphi_r)=\varphi_{(f,F)^{-1}(r)}$, leaving the dual proof of the second equality to the interested reader.

First suppose that for some $A\in\mathcal{S}$ we have $(f,F)^{-1}(\varphi_r)(A)\not\subseteq\varphi_{(f,F)^{-1}(r)}(A)$, that is $F^{\leftarrow}(\varphi_r(f^{\rightarrow}A))\not\subseteq(f,F)^{-1}(r)^{\rightarrow}A$, where $(f,F)^{-1}(r)$ is given by [15, Definition 5.1]. Now we have $s\in S$ with $F^{\leftarrow}(\varphi_r(f^{\leftarrow}A))\not\subseteq Q_s$, $P_s\not\subseteq(f,F)^{-1}(r)^{\rightarrow}A$, and it follows from [5, Definition 2.8(2)] that for some $u\in S$ we have $\overline{P}_{(s,u)}\not\subseteq F$, $\varphi_r(f^{\rightarrow}A)\not\subseteq Q_u$, while from [5, Definition 2.5(1)] we have $P_s\not\subseteq Q_{s'}$, $s'\in S$ with

$$(f,F)^{-1}(r)\not\subseteq\overline{Q}_{(z,s')}\Longrightarrow A\subseteq Q_z,\quad\forall z\in S. \tag{3}$$

Now from $r^{\rightarrow}(f^{\rightarrow}A)=\varphi_r(f^{\rightarrow}A)\not\subseteq Q_u$ we have $v\in S$ with $r\not\subseteq\overline{Q}_{(v,u)}$, $f^{\rightarrow}A\not\subseteq Q_v$; hence $w\in S$ with $f\not\subseteq\overline{Q}_{(w,v)}$, $A\not\subseteq Q_w$. By condition R2 for the relation $f$ we may choose $w'\in S$ with $P_w\not\subseteq Q_{w'}$ and $f\not\subseteq\overline{Q}_{(w',v)}$. Applying the implication (3) with $z=w$ we obtain $(f,F)^{-1}(r)\subseteq\overline{Q}_{(w,s')}$, and $P_s\not\subseteq Q_{s'}$ gives $\overline{P}_{(w,s)}\not\subseteq(f,F)^{-1}(r)$. Now using [15, Definition 5.1] we have $t_1,t_2\in S$ with $\overline{P}_{(w',t_1)}\not\subseteq F$, $f\not\subseteq\overline{Q}_{(s,t_2)}$ and $\overline{P}_{(t_1,t_2)}\not\subseteq r$.

By condition $DF2$ for the difunction (*f,F*), from $\overline{P}_{(s,u)}\not\subseteq F$, $f\not\subseteq\overline{Q}_{(s,t_2)}$ we obtain $P_u\not\subseteq Q_{t_2}$, and from $f\not\subseteq\overline{Q}_{(w',v)}$, $\overline{P}_{(w',t_1)}\not\subseteq F$ we obtain $P_{t_1}\not\subseteq Q_v$. Since *r* is a relation and $r\not\subseteq\overline{Q}_{(v,u)}$ we easily obtain $r\not\subseteq\overline{Q}_{(t_1,t_2)}$, which gives the contradiction $\overline{P}_{(t_1,t_2)}\subseteq r$.

Now suppose that we have $A\in\mathcal{S}$ and $s\in S$ with $(f,F)^{-1}(r)^{\rightarrow}A\not\subseteq Q_s$ and $P_s\not\subseteq F^{\leftarrow}(r^{\rightarrow}(f^{\rightarrow}A))$. The first statement gives us $u\in S$ with $(f,F)^{-1}(r)\not\subseteq\overline{Q}_{(u,s)}$, $A\not\subseteq Q_u$, and hence $s'\in S$, $u'\in S$ with $\overline{P}_{(u,s')}\not\subseteq\overline{Q}_{(u,s)}$, $P_u\not\subseteq Q_{u'}$, for which

$$\overline{P}_{(u',t_1)}\not\subseteq F,\ f\not\subseteq\overline{Q}_{(s',t_2)}\Longrightarrow r\not\subseteq\overline{Q}_{(t_1,t_2)},\quad\forall t_1,t_2\in T, \tag{4}$$

by the comment following [15, Definition 5.1]. Applying condition $DF1$ for (*f,F*) to $P_u\not\subseteq Q_{u'}$ gives $w_1\in T$ with $f\not\subseteq\overline{Q}_{(u,w_1)}$, $\overline{P}_{(u',w_1)}\not\subseteq F$, and we may choose $w_1'\in T$ with $\overline{P}_{(u',w_1)}\not\subseteq\overline{Q}_{(u',w_1')}$ and $\overline{P}_{(u',w_1')}\not\subseteq F$.

On the other hand the second statement gives $v\in S$, $P_s\not\subseteq Q_v$, for which

$$\overline{P}_{(v,t)}\not\subseteq F\Longrightarrow r^{\rightarrow}(f^{\rightarrow}A)\subseteq Q_t,\quad\forall t\in T. \tag{5}$$

Applying condition $DF1$ for (*f,F*) to $P_s\not\subseteq Q_v$ gives $w_2\in T$ with $f\not\subseteq\overline{Q}_{(s,w_2)}$, $\overline{P}_{(v,w_2)}\not\subseteq F$, and we may choose $w_2'\in T$ with $f\not\subseteq\overline{Q}_{(s,w_2')}$ and $\overline{P}_{(s,w_2')}\not\subseteq\overline{Q}_{(s,w_2)}$.

Setting $t=w_2$ in (5) now gives $r^{\rightarrow}(f^{\rightarrow}A)\subseteq Q_{w_2}$, and $P_{w_2'}\not\subseteq Q_{w_2}$ so $P_{w_2'}\not\subseteq r^{\rightarrow}(f^{\rightarrow}A)$. Thus we have $w_2''\in T$ with $P_{w_2'}\not\subseteq Q_{w_2''}$ for which

$$r\not\subseteq\overline{Q}_{(z,w_2'')}\Longrightarrow f^{\rightarrow}A\subseteq Q_z,\quad\forall z\in T. \tag{6}$$

Now $\overline{P}_{(u',w_1')}\not\subseteq F$ and $f\not\subseteq\overline{Q}_{(s,w_2')}$, whence $P_{s'}\not\subseteq Q_s$ gives $f\not\subseteq\overline{Q}_{(s',w_2')}$ by $R1$, while $P_{w_2'}\not\subseteq Q_{w_2''}$ implies $Q_{w_2''}\subseteq Q_{w_2'}$ and hence $f\not\subseteq\overline{Q}_{(s',w_2'')}$. Now we may apply the implication (4) with $t_1=w_1'$, $t_2=w_2''$ to give $r\not\subseteq\overline{Q}_{(w_1',w_2'')}$, and then (6) with $z=w_1'$ to give $f^{\rightarrow}A\subseteq Q_{w_1'}$. Since $P_{w_1}\not\subseteq Q_{w_1'}$ we now have $P_{w_1}\not\subseteq f^{\rightarrow}A$, and so there exists $w\in T$ with $P_{w_1}\not\subseteq Q_w$ so that

$$f\not\subseteq\overline{Q}_{(z,w)}\Longrightarrow A\subseteq Q_z,\quad\forall z\in S. \tag{7}$$

However, $f\not\subseteq\overline{Q}_{(u,w_1)}$, and $P_{w_1}\not\subseteq Q_w$ gives $f\not\subseteq\overline{Q}_{(u,w)}$, so we may apply (7) with *z*=*u* to give the contradiction $A\subseteq Q_u$. □

**Corollary 2.11.** *Let* $(f,F):(S,\mathcal{S})\to(T,\mathcal{T})$ *be a difunction,* $\mathcal{U}$ *a direlational uniformity on* $(S,\mathcal{S})$ *and* $\mathcal{V}$ *a direlational uniformity on* $(T,\mathcal{T})$. *Then (f,F) is* $\mathcal{U}$–$\mathcal{V}$ *uniformly bicontinuous if and only if it is* $\mathcal{U}_{\mathcal{F}}$–$\mathcal{V}_{\mathcal{F}}$ *uniformly bicontiuous.*

**Proof.** Straightforward by Proposition 2.10. □

Let us denote by $\mathcal{A}$ the category of direlational uniformities and uniformly bicontinuous difunctions in the sense of [15, Definition 5.9], by $\mathcal{B}$ the category of difunctional uniformities and uniformly bicontinuous difunctions in the sense of Definition 2.9, and define the functor $\mathfrak{F} : \mathcal{A} \to \mathcal{B}$ by

$$\mathfrak{F}((S, \mathcal{S}, \mathcal{U}) \overset{(f,F)}{\longrightarrow} (T, \mathcal{T}, \mathcal{V})) = (S, \mathcal{S}, \mathcal{U}_{\mathcal{F}}) \overset{(f,F)}{\longrightarrow} (T, \mathcal{T}, \mathcal{V}_{\mathcal{F}}).$$

It is immediate from Proposition 2.10 that $\mathfrak{F}$ is an isomorphism. Moreover, both $\mathcal{A}$ and $\mathcal{B}$ may be regarded as concrete categories over **dfTex** via the forgetful functors $\mathfrak{U} : \mathcal{A} \to$ **dfTex**, $\mathfrak{V} : \mathcal{B} \to$ **dfTex**, and it is clear that $\mathfrak{U} = \mathfrak{V} \circ \mathfrak{F}$ whence $\mathfrak{F}$ is a concrete isomorphism [1, Remark 5.10]. Hence we have proved:

**Corollary 2.12.** *The categories $\mathcal{A}$ and $\mathcal{B}$ are concretely isomorphic.*

In view of Corollary 2.12 we may extend the term di-uniformity to include also difunctional uniformities.

We recall from [17] that a direlational quasi-uniformity on $(S, \mathcal{S})$ is defined by removing the symmetry condition from the definition of direlational uniformity. In that paper characterizations of direlational quasi-uniformities were given in terms of quasi-pseudodimetrics and in terms of a notion of dual dicover, the term quasi di-uniformity being used to denote any of these equivalent concepts. It is clear that removing the symmetry condition (4) from the definition of difunctional uniformity gives yet another characterization of direlational quasi-uniformities, so we extend the term quasi di-uniformity to include difunctional quasi-uniformities also.

## 3. Properties of difunctional (quasi-) uniformities

In this section we characterize various concepts and results relating to (quasi) di-uniformities in terms of difunctional uniformities.

The definition of a direlational (quasi-) uniformity $\mathcal{U}$ on $(S, \mathcal{S})$ has been introduced in Definition 1.1. It will be noted that this definition is formally the same as the usual definition of a diagonal uniformity, and the notions of base and subbase may be defined in the obvious way.

We begin by recalling that by [15, Lemma 4.3], the *uniform ditopology* $(\tau_{\mathcal{U}}, \kappa_{\mathcal{U}})$ of a direlational (quasi-) uniformity $\mathcal{U}$ on $(S, \mathcal{S})$ may be defined by the following conditions:

(i) $G \in \tau_{\mathcal{U}} \iff (G \not\subseteq Q_s \Longrightarrow \exists (r, R) \in \mathcal{U}$ with $r[s] \subseteq G)$.
(ii) $K \in \kappa_{\mathcal{U}} \iff (P_s \not\subseteq K \Longrightarrow \exists (r, R) \in \mathcal{U}$ with $K \subseteq R[s])$.

Here $r[s] = r^{\to} P_s = \varphi_r(P_s)$, $R[s] = R^{\to} Q_s = \psi_R(Q_s)$, so the following gives the corresponding definition of the uniform ditopology of a difunctional uniformity.

**Definition 3.1.** Let $(S, \mathcal{S}, \mathcal{U}_{\mathcal{F}})$ be a difunctional (quasi-) uniform texture space. Then the *uniform ditopology* $(\tau_{\mathcal{U}_{\mathcal{F}}}, \kappa_{\mathcal{U}_{\mathcal{F}}})$ of $\mathcal{U}_{\mathcal{F}}$ on $(S, \mathcal{S})$ is characterized by:

(i) $G \in \tau_{\mathcal{U}_{\mathcal{F}}} \iff (G \not\subseteq Q_s \Longrightarrow \exists (\varphi, \psi) \in \mathcal{U}_{\mathcal{F}}$ with $\varphi(P_s) \subseteq G)$.
(ii) $K \in \kappa_{\mathcal{U}_{\mathcal{F}}} \iff (P_s \not\subseteq K \Longrightarrow \exists (\varphi, \psi) \in \mathcal{U}_{\mathcal{F}}$ with $K \subseteq \psi(Q_s))$.

When we speak of the ditopology of $(S, \mathcal{S}, \mathcal{U}_{\mathcal{F}})$ we will always mean the uniform ditopology.

In [17] it was shown that an arbitrary ditopology on a plain texture has a compatible direlational quasi-uniformity. By adapting the construction used in the proof of [11, Theorem 7] we now show that we can omit the restriction to plain textures, thus answering in the affirmative a question posed in [17].

**Theorem 3.2.** *Every ditopological texture space $(S, \mathcal{S}, \tau, \kappa)$ is quasi-di-uniformizable.*

**Proof.** For $G \in \tau$, $K \in \kappa$ define $\varphi_G, \psi_K : \mathcal{S} \to \mathcal{S}$ by

$$\varphi_G(A) = \begin{cases} S & \text{if } A \not\subseteq G, \\ G & \text{if } A \subseteq G, \end{cases} \quad \text{and} \quad \psi_K(A) = \begin{cases} \emptyset & \text{if } K \not\subseteq A, \\ K & \text{if } K \subseteq A. \end{cases}$$

It is trivial to verify that $\varphi_G \in \mathcal{F}_{\mathcal{RR}}$, $\psi_K \in \mathcal{F}_{\mathcal{RCR}}$, whence $(\varphi_G, \psi_K) \in \mathcal{F}_{\mathcal{RDR}}$. Moreover, $(\varphi_G, \psi_K)^2 = (\varphi_G, \psi_K)$, so

$$\{(\varphi_G, \psi_K) \mid G \in \tau, K \in \kappa\}$$

is a subbase for a difunctional quasi-uniformity $\mathcal{U}_{\mathcal{F}}$ on $(S, \mathcal{S})$. Clearly $\tau \subseteq \tau_{\mathcal{U}_{\mathcal{F}}}$, for if $G \in \tau$ and $G \not\subseteq Q_s$ then $P_s \subseteq G$ so $\varphi_G(P_s) = G$.

On the other hand, take $H \in \tau_{\mathcal{U}_{\mathcal{F}}}$ and $H \not\subseteq Q_s$. Then we have $(\varphi, \psi) \in \mathcal{U}_{\mathcal{F}}$ with $\varphi(P_s) \subseteq H$ and $G_1, G_2, \ldots, G_n \in \tau$ with

$$\varphi_{G_1} \wedge \varphi_{G_2} \wedge \cdots \wedge \varphi_{G_n} \leq \varphi. \tag{8}$$

Now it is straightforward to verify that

$$(\varphi_{G_1} \wedge \varphi_{G_2})(A) = \begin{cases} G_1 \cap G_2 & \text{if } A \subseteq G_1 \cap G_2, \\ G_1 & \text{if } A \subseteq G_1 \setminus G_2, \\ G_2 & \text{if } A \subseteq G_2 \setminus G_1, \\ S & \text{otherwise}, \end{cases}$$

with an obvious generalization for more terms, so $(\varphi_{G_1} \wedge \varphi_{G_2} \wedge \cdots \wedge \varphi_{G_n})(A) \in \tau$ for all $A \in \mathcal{S}$. But,

$$P_s \subseteq (\varphi_{G_1} \wedge \varphi_{G_2} \wedge \cdots \wedge \varphi_{G_n})(P_s) \subseteq \varphi(P_s) \subseteq H$$

by (8) and we see that $H \in \tau$ by [6, Theorem 3.2(1(iii))]. Thus $\tau = \tau_{\mathcal{U}_{\mathcal{F}}}$, and the proof of $\kappa = \kappa_{\mathcal{U}_{\mathcal{F}}}$ is dual and is omitted. □

In the case where $(S, \mathcal{S})$ is plain it is interesting to compare the quasi di-uniformity constructed above to that given in [17], which is a direct generalization of the Pervin quasi-uniformity [18]. Using the formula for $(r_{\varphi_G}, R_{\psi_K})$ given in Proposition 2.2 a fairly straightforward calculation shows that

$$r_{\varphi_G} = (S \times G) \cup ((S \setminus G) \times S) \quad \text{and} \quad R_{\psi_K} = (S \setminus K) \times K,$$

whence they coincide [17, Example 2.3]. Hence the construction in the proof of Theorem 3.2 generalizes the Pervin quasi-uniformity to general textures, and we will continue to call it the Pervin quasi di-uniformity on $(S, \mathcal{S}, \tau, \kappa)$. As in the classical case we have:

**Proposition 3.3.** *Bicontinuous difunctions between ditopological texture spaces are uniformly bicontinuous with respect to the corresponding Pervin quasi di-uniformities.*

**Proof.** Let $(f, F) : (S_1, \mathcal{S}_1, \tau_1, \kappa_1) \to (S_2, \mathcal{S}_2, \tau_2, \kappa_2)$ be bicontinuous and denote the corresponding Pervin difunctional quasi-uniformities on $(S_1, \mathcal{S}_1)$, $(S_2, \mathcal{S}_2)$ by $\mathcal{U}_{\mathcal{F}_1}$, $\mathcal{U}_{\mathcal{F}_2}$, respectively. We are to prove that $(f, F) : (S_1, \mathcal{S}_1, \mathcal{U}_{\mathcal{F}_1}) \to (S_2, \mathcal{S}_2, \mathcal{U}_{\mathcal{F}_2})$ is uniformly bicontinuous. Take $G \in \tau_2$, $K \in \kappa_2$. It will clearly suffice to show that $(f, F)^{-1}(\varphi_G, \psi_K) \in \mathcal{U}_{\mathcal{F}_1}$. By bicontinuity $F^{\leftarrow}G \in \tau_1$, $f^{\leftarrow}K \in \kappa_1$, so $(\varphi_{F^{\leftarrow}G}, \psi_{f^{\leftarrow}K}) \in \mathcal{U}_{\mathcal{F}_1}$ and it will enough to verify that

$$(\varphi_{F^{\leftarrow}G}, \psi_{f^{\leftarrow}K}) \sqsubseteq (F^{\leftarrow} \circ \varphi_G \circ f^{\rightarrow},\ f^{\leftarrow} \circ \psi_K \circ F^{\leftarrow}).$$

We prove $\varphi_{F^{\leftarrow}G}(A) \subseteq F^{\leftarrow}(\varphi_G(f^{\rightarrow}A))$ for all $A \in \mathcal{S}_1$, leaving the dual proof of the second inclusion to the interested reader. Now

$$\varphi_{F^{\leftarrow}G}(A) = \begin{cases} F^{\leftarrow}G & \text{if } A \subseteq F^{\leftarrow}G, \\ S_1 & \text{otherwise}, \end{cases}$$

$$F^{\leftarrow}(\varphi_G(f^{\rightarrow}A)) = \begin{cases} F^{\leftarrow}G & \text{if } f^{\rightarrow}A \subseteq G, \\ F^{\leftarrow}S_2 & \text{otherwise}, \end{cases}$$

by [5, Proposition 2.28(c)] we have $F^{\leftarrow}S_2 = S_1$, while by [5, Theorem 2.24(2a)],

$$f^{\rightarrow}A \subseteq G \Longrightarrow A \subseteq F^{\leftarrow}(f^{\rightarrow}A) \subseteq F^{\leftarrow}G.$$

Hence the required inclusion holds. □

As in [17] we denote by **dfQDiU** the category of quasi di-uniform texture spaces and uniformly bicontinuous difunctions, and as usual **dfDitop** is the category of ditopological texture spaces and bicontinuous difunctions. Since uniformly bicontinuous difunctions are bicontinuous for the uniform bitopologies we have a functor $\mathfrak{F}$ : **dfQDiU** → **dfDitop** which associates to each quasi di-uniformity its uniform ditopology, and maps a uniformly bicontinuous difunction to itself [17]. In view of Theorem 3.2 and Proposition 3.3 we may set up a functor $\mathfrak{P}$ : **dfDitop** → **dfQDiU** which maps each ditopological texture space to the corresponding Pervin quasi di-uniform space, and each bicontinuous difunction to itself. Clearly $\mathfrak{F} \circ \mathfrak{P}$ is the identity functor on **dfDitop**.

A well known classical theorem says that (quasi-) uniformities have an open base and a closed base. Moreover, these results make sense for both the diagonal and (dual) covering representations. In the case of (quasi-) di-uniformities we currently have counterparts of these results for the dicovering representation only. Specifically, a dicovering uniformity has a base of open, coclosed dicovers and a base of closed, co-open dicovers [14, Proposition 4.8]. Now we shall give a meaning to $(\varphi, \psi)$ being "open, coclosed", and then prove that difunctional uniformities also have a base of open, coclosed elements. As yet the extension of the result for "closed, co-open" remains open.

**Definition 3.4.** Let $(S, \mathcal{S}, \tau, \kappa)$ be a ditopological texture space, $\varphi \in \mathcal{F}_{\mathcal{R}\mathcal{R}}$ and $\psi \in \mathcal{F}_{\mathcal{R}\mathcal{C}\mathcal{R}}$. Then

(1) $\varphi$ is called *open* if $\varphi(A) \in \tau\ \forall A \in \mathcal{S}$.
(2) $\psi$ is called *closed* if $\psi(A) \in \kappa\ \forall A \in \mathcal{S}$.
(3) $(\varphi, \psi)$ is called *open, coclosed* if $\varphi$ is open and $\psi$ is closed.

**Proposition 3.5.** *A difunctional uniformity on* $(S, \mathcal{S})$ *has a base that is open, coclosed for the uniform ditopology*.

**Proof.** There is no loss of generality in taking the difunctional uniformity in the form $\mathcal{U}_{\mathcal{F}}$, where $\mathcal{U}$ is a direlational uniformity on $(S, \mathcal{S})$. Take $(d, D) \in \mathcal{U}$ and $(e, E) \in \mathcal{U}$ with $(e, E)^2 \sqsubseteq (d, D)$. We recall the following two facts:

(i) For $A \in \mathcal{S}$ we have $A = \bigvee\{P_s | A \not\subseteq Q_s\}$ by [5, Theorem 1.2(7)], whence for any relation $r$,

$$r^{\to} A = r^{\to}\left(\bigvee\{P_s | A \not\subseteq Q_s\}\right) = \bigvee\{r^{\to} P_s | A \not\subseteq Q_s\}$$

by [5, Corollary 2.12(2)].

(ii) For any $A \in \mathcal{S}$,

$$G(L) = \bigvee\{P_u | \exists (r, R) \in \mathcal{U} \quad \text{with } r^{\to} P_u \subseteq L\} \in \tau_{\mathcal{U}},$$

while for a reflexive relation $r$, $A \subseteq G(r^{\to} A) \subseteq r^{\to} A$ (see the proof of [15, Proposition 2.7]), and so $A \subseteq ]r^{\to} A[$.

For $A \in \mathcal{S}$ define $\varphi(A) = \bigvee\{]d^{\to} P_s[ | A \not\subseteq Q_s\}$. Then $\varphi(A) \in \tau_{\mathcal{U}} = \tau_{\mathcal{U}_{\mathcal{F}}}$, that is $\varphi$ is open. Next, $A \subseteq \varphi(A)$ since $P_s \subseteq ]d^{\to} P_s[$ by (ii) with $A = P_s, r = d$. Also, for $A_j \in \mathcal{S}$, $j \in J$, we clearly have $\varphi(\bigvee_{j \in J} A_j) = \bigvee_{j \in J} \varphi(A_j)$. Hence $\varphi \in \mathcal{F}_{\mathcal{R}\mathcal{R}}$. Finally,

$$e^{\to} P_s \subseteq ]e^{\to} e^{\to} P_s[ = ]e^2 P_s[ \subseteq ]d^{\to} P_s[ \subseteq d^{\to} P_s$$

by (ii) with $A = e^{\to} P_s, r = e$ and [5, Lemma 2.16(1)]. Hence, by (i) with $r = e, d$ we have, for $A \in \mathcal{S}$,

$$e^{\to} A = \bigvee\{e^{\to} P_s | A \not\subseteq Q_s\} \subseteq \bigvee\{]d^{\to} P_s[ | A \not\subseteq Q_s\} \subseteq \bigvee\{d^{\to} P_s | A \not\subseteq Q_s\} = d^{\to} A.$$

This gives $\varphi_e \leq \varphi \leq \varphi_d$, and a dual proof shows the existence of a closed $\psi \in \mathcal{F}_{\mathcal{R}\mathcal{C}\mathcal{R}}$ satisfying $\psi_D \leq \psi \leq \psi_E$, whence $\mathcal{U}_{\mathcal{F}}$ has a base of open, coclosed elements, as required. □

**Corollary 3.6.** *Let* $\mathcal{U}_{\mathcal{F}}$ *be a difunctional uniformity with uniform ditopology* $(\tau, \kappa)$. *Then for* $A \in \mathcal{S}$ *the interior* $]A[$ *and closure* $[A]$ *are given by*

$$]A[ = \bigvee\{B \in \mathcal{S} | \exists (\varphi, \psi) \in \mathcal{U}_{\mathcal{F}} \quad with\ \varphi(B) \subseteq A\},\ and$$
$$[A] = \bigcap\{B \in \mathcal{S} | \exists (\varphi, \psi) \in \mathcal{U}_{\mathcal{F}} \quad with\ A \subseteq \psi(B)\},$$

*respectively*.

**Proof.** We prove the first equality, leaving the dual proof of the second equality to the interested reader. Denote the join on the right by $A^*$. Take $B \in \mathcal{S}$ with $\varphi(B) \subseteq A$ for some $(\varphi, \psi) \in \mathcal{U}_{\mathcal{F}}$. Since $B \subseteq \varphi(B) \subseteq A$ it is clear that $A^* \subseteq A$. By Definition 2.7(3) we have $(\varphi_1, \psi_1) \in \mathcal{U}_{\mathcal{F}}$ with $(\varphi_1, \psi_1)^2 \sqsubseteq (\varphi, \psi)$, and without loss of generality we may take $(\varphi_1, \psi_1)$ to be open, coclosed by Proposition 3.5. Since $B \subseteq \varphi_1(B)$ the join of the open sets $\varphi_1(B)$ is a superset of $A^*$, but since $\varphi_1(B)$ satisfies $\varphi_1(\varphi_1(B)) \subseteq A$ it is also a subset, so $A^* \in \tau$.

Finally, take $G \in \tau$ with $G \subseteq A$ and suppose that $G \not\subseteq A^*$. Then we have $s \in S$ with $G \not\subseteq Q_s$, $P_s \not\subseteq A^*$, and by Definition 3.1(i) there exists $(\varphi, \psi) \in \mathcal{U}_{\mathcal{F}}$ with $\varphi(P_s) \subseteq G$. This gives $\varphi(P_s) \subseteq A$, whence $P_s \subseteq A^*$ and we have a contradiction. This establishes that $A^*$ is the interior of $A$. $\square$

We now recall that a direlational uniformity $\mathcal{U}$ on $(S, \mathcal{S})$ is called *separated* [14] if $\sqcap_{(r,R)\in\mathcal{U}}(r, R) = (i_S, I_S)$. Calling a difunctional uniformity separated if the corresponding direlational uniformity is, we see at once that

**Proposition 3.7.** *The difunctional uniformity $\mathcal{U}_{\mathcal{F}}$ is separated if and only if*

$$\bigwedge_{(\varphi,\psi)\in\mathcal{U}_{\mathcal{F}}} (\varphi, \psi) = (\imath_{\mathcal{S}}, \imath_{\mathcal{S}}),$$

*where $\imath_{\mathcal{S}} : \mathcal{S} \to \mathcal{S}$ is the identity mapping.*

We recall from [14, Theorem 4.16] that, exactly as in the classical case, a di-uniformity is separated if and only if its uniform ditopology is $T_0$.

Finally, let $\sigma : \mathcal{S} \to \mathcal{S}$ be a complementation. For $\varphi \in \mathcal{F}_{\mathcal{RR}}$ let us define $\varphi' : \mathcal{S} \to \mathcal{S}$ by $\varphi' = \sigma \circ \varphi \circ \sigma$. It is trivial to verify that $\varphi' \in \mathcal{F}_{\mathcal{RCR}}$. Likewise, for $\psi \in \mathcal{F}_{\mathcal{RCR}}$ we have $\psi' = \sigma \circ \psi \circ \sigma \in \mathcal{F}_{\mathcal{RR}}$, and clearly $(\varphi')' = \varphi$, $(\psi')' = \psi$. Hence, setting

$$(\varphi, \psi)' = (\psi', \varphi')$$

gives us an involution on $\mathcal{F}_{\mathcal{RDR}}$. We now link this with the involution $(r, R) \mapsto (r, R)' = (R', r')$ on the family of reflexive direlations.

**Lemma 3.8.**

(1) *For $r \in \mathcal{RR}$, $R \in \mathcal{RCR}$ we have $(\varphi_r)' = \psi_{r'}$ and $(\psi_R)' = \varphi_{R'}$. Hence, $(\varphi_r, \psi_R)' = (\varphi_{R'}, \psi_{r'})$.*
(2) *For $\varphi \in \mathcal{F}_{\mathcal{RR}}$, $\psi \in \mathcal{F}_{\mathcal{RCR}}$ we have $(r_\varphi)' = R_{\varphi'}$ and $(R_\psi)' = r_{\psi'}$. Hence, $(r_\varphi, R_\psi)' = (r_{\psi'}, R_{\varphi'})$.*

**Proof.** Take $r \in \mathcal{RR}$. Then $(\varphi_r)'(A) = \sigma(\varphi_r(\sigma(A))) = \sigma(r^{\to}\sigma(A))$, and by [5, Lemma 2.20(1)], $\sigma(r^{\to}\sigma(A)) = (r')^{\to}A$, so $(\varphi_r)'(A) = \psi_{r'}$ since $r' \in \mathcal{F}_{\mathcal{RCR}}$. This proves $(\varphi_r)' = \psi_{r'}$, and the second equality in (1) is proved likewise.

For (2), take $\varphi \in \mathcal{F}_{\mathcal{RR}}$ and let $r = r_\varphi$, that is $\varphi = \varphi_r$. By (1) $\varphi' = (\varphi_r)' = \psi_{r'}$, which gives $R_{\varphi'} = r' = (r_\varphi)'$. The second equality may be proved likewise. $\square$

Recalling from [15, Theorem 2.3] that for a direlational uniformity $\mathcal{U}$ on $(S, \mathcal{S})$, $\mathcal{U}' = \{(r, R)' | (r, R) \in \mathcal{U}\}$ is also a direlational uniformity, called the complement of $\mathcal{U}$, we likewise call the corresponding difunctional uniformity $(\mathcal{U}')_{\mathcal{F}}$ the *complement* of $\mathcal{U}_{\mathcal{F}}$, and denote it by $\mathcal{U}'_{\mathcal{F}}$. By Lemma 3.6 we clearly have

$$\mathcal{U}'_{\mathcal{F}} = \{(\varphi, \psi)' | (\varphi, \psi) \in \mathcal{U}_{\mathcal{F}}\}.$$

Again we will call a difunctional uniformity $\mathcal{U}_{\mathcal{F}}$ *complemented* if it is equal to its complement. It is shown in [15] that on the discrete complemented texture $(X, \mathcal{P}(X), \pi_X)$ the complemented di-uniformities correspond precisely to the uniformities on $X$.

## 4. Applications to Hutton uniformities and Hutton quasi-uniformities

Hutton's original setting for (quasi-) uniformities and topology was a Hutton algebra, that is a complete, completely distributive lattice $\mathbb{L}$ on which is defined an order reversing involution $'$. A topology $\tau$ on $\mathbb{L}$ is then regarded as a family of open sets, the corresponding family of closed sets being obtained by applying the operation $'$ to give $\kappa = \tau' = \{\lambda' | \lambda \in \mathbb{L}\}$. If we regard $(\tau, \kappa)$ as a topological structure in which the open and closed sets have equal status, then it is natural to call this a ditopology on $\mathbb{L}$.

More generally, as in [21], we may dispense with $'$ and consider arbitrary pairs $(\tau, \kappa)$ consisting of a topology and cotopology on a complete, completely distributive lattice $\mathbb{L}$, thereby producing the notion of *Hutton dispace*. Given two Hutton dispaces $(\mathbb{L}_i, \tau_i, \kappa_i)$, $i$=1,2, a mapping $\varphi : \mathbb{L}_2 \to \mathbb{L}_1$ preserving arbitrary meets and joins is said to be *continuous* if $\varphi[\tau_2] \subseteq \tau_1$, *cocontinuous* if $\varphi[\kappa_2] \subseteq \kappa_1$, and *bicontinuous* if it is both continuous and cocontinuous. We regard such a bicontinuous mapping as a morphism from $(\mathbb{L}_1, \tau_1, \kappa_1)$ to $(\mathbb{L}_2, \tau_2, \kappa_2)$, so defining the category **diH** of Hutton dispaces and bicontinuous mappings.

If $(\mathbb{L}, \tau, \kappa)$ is a Hutton dispace and $(M_{\mathbb{L}}, \mathcal{M}_{\mathbb{L}})$ the Hutton texture of $\mathbb{L}$, we may set $\tau_{\mathbb{L}} = \{\widehat{\lambda} | \lambda \in \tau\}$, $\kappa_{\mathbb{L}} = \{\widehat{\lambda} | \lambda \in \kappa\}$ to give the Hutton ditopological texture space $(M_{\mathbb{L}}, \mathcal{M}_{\mathbb{L}}, \tau_{\mathbb{L}}, \kappa_{\mathbb{L}})$ of $(\mathbb{L}, \tau, \kappa)$. A mapping $\theta : \mathbb{L}_2 \to \mathbb{L}_1$ preserving arbitrary meets and joins corresponds to the mapping $\widehat{\theta} : \mathcal{M}_{\mathbb{L}_2} \to \mathcal{M}_{\mathbb{L}_1}$, $\widehat{\theta}(\widehat{\lambda}) = \widehat{\theta(\lambda)}$, which also preserves arbitrary meets and joins, and hence by [6, Proposition 4.1] to the difunction $(f^{\widehat{\theta}}, F^{\widehat{\theta}}) : (M_{\mathbb{L}_1}, \mathcal{M}_{\mathbb{L}_1}) \to (M_{\mathbb{L}_2}, \mathcal{M}_{\mathbb{L}_2})$ characterized by $(f^{\widehat{\theta}})^{\leftarrow}\widehat{\lambda} = \widehat{\theta(\lambda)} = (F^{\widehat{\theta}})^{\leftarrow}\widehat{\lambda}$. Moreover, if $\theta$ is bicontinuous then so is $(f^{\widehat{\theta}}, F^{\widehat{\theta}})$, so we have a functor $\mathfrak{H}$ : **diH** $\to$ **dfDitop**.

In the opposite direction, if $(S, \mathcal{S}, \tau, \kappa)$ is a ditopological texture space then $(\mathcal{S}, \tau, \kappa)$ is a Hutton dispace. Moreover, given a bicontinuous difunction $(f, F) : (S_1, \mathcal{S}_1, \tau_1, \kappa_1) \to (S_2, \mathcal{S}_2, \tau_2, \kappa_2)$, the mapping $\theta_{(f,F)} : \mathcal{S}_2 \to \mathcal{S}_1$ defined in [6] by $\theta_{(f,F)}(B) = f^{\leftarrow}B = F^{\leftarrow}B$, $B \in \mathcal{S}_2$ is clearly a **diH**-morphism, and we have a functor $\mathfrak{E}$ : **dfDitop** $\to$ **diH**.

In [21] it is shown that $\mathfrak{E}$ is a co-adjoint with $\mathfrak{H}$ the corresponding adjoint, and that moreover $\mathfrak{E}$ and $\mathfrak{H}$ define an equivalence between the categories **diH** and **dfDitop**.

In view of this equivalence, all the pointfree aspects of the theory of ditopological texture spaces will have an equivalent expression in the context of Hutton dispaces, and conversely. For example, Proposition 2.2 shows that a direlation (*r*,*R*) on $(S, \mathcal{S})$ corresponds to a pair of mappings $(\varphi_r, \psi_R)$ with $\varphi_r \in \mathcal{F}_{\mathcal{R}}$, $\psi_R \in \mathcal{F}_{\mathcal{CR}}$, while the corresponding representation for a difunction used above has been known for some time. In [21] a pointfree generalization of the notion of real dicompactness is obtained and carried over to Hutton dispaces. The aim of this final section is to express the notion of functional (quasi-) di-uniformity in this context, and tie this in with Hutton quasi-uniformities and uniformities as given in [11].

**Definition 4.1.** Let $\mathbb{L}$ be a complete, completely distributive lattice. Denote by $\mathcal{Q} = \mathcal{Q}_{\mathbb{L}}$ the set of mappings $g$ on $\mathbb{L}$ satisfying:

(i) $g(\bigvee_{j \in J} \alpha_j) = \bigvee_{j \in J} g(\alpha_j) \quad \forall \alpha_j \in \mathbb{L},\ j \in J$
(ii) $\alpha \leq g(\alpha) \quad \forall \alpha \in \mathbb{L}$,

and by $\mathcal{P} = \mathcal{P}_{\mathbb{L}}$ the set of mappings $h$ on $\mathbb{L}$ satisfying

(i) $h(\bigwedge_{j \in J} \alpha_j) = \bigwedge_{j \in J} h(\alpha_j) \quad \forall \alpha_j \in \mathbb{L},\ j \in J$
(ii) $h(\alpha) \leq \alpha \quad \forall \alpha \in \mathbb{L}$.

Then a non-empty set $\mathcal{U} \subseteq \mathcal{Q}_{\mathbb{L}} \times \mathcal{P}_{\mathbb{L}}$ is called a *di-uniformity* on $\mathbb{L}$ if it satisfies

(1) $(g, h) \in \mathcal{U}, (g, h) \leq (g_1, h_1) \in \mathcal{Q} \times \mathcal{P} \Longrightarrow (g_1, h_1) \in \mathcal{U}$,
(2) $(g_1, h_1), (g_2, h_2) \in \mathcal{U} \Longrightarrow (g_1, h_1) \wedge (g_2, h_2) \in \mathcal{U}$,
(3) For $(g, h) \in \mathcal{U} \exists (g_1, h_1) \in \mathcal{U}$ with $(g_1, h_1)^2 \leq (g, h)$,
(4) For $(g, h) \in \mathcal{U} \exists (g_1, h_1) \in \mathcal{U}$ with $(g_1, h_1)^{\leftarrow} \leq (g, h)$.

Here the operations on the elements of $\mathcal{Q} \times \mathcal{P}$ are as described following Lemma 2.5. In case the symmetry condition (4) is omitted, $\mathcal{U}$ is called a *quasi-di-uniformity* on $\mathbb{L}$. Also, $(\mathbb{L}, \mathcal{U})$ is called a (*quasi*-) *di-uniform Hutton space*.

Let $(\mathbb{L}, \mathcal{U})$ be a (quasi-) di-uniform Hutton space, and set $\mathcal{U}_{\mathbb{L}} = \{(\hat{g}, \hat{h}) | (g, h) \in \mathcal{U}\}$. Then clearly $(M_{\mathbb{L}}, \mathcal{M}_{\mathbb{L}}, \mathcal{U}_{\mathbb{L}})$ is a difunctional (quasi-) uniform texture space. Now let $\theta : (\mathbb{L}_2, \mathcal{U}_2) \to (\mathbb{L}_1, \mathcal{U}_1)$ preserve arbitrary meets and joins, and consider the corresponding difunction $(f^{\widehat{\theta}}, F^{\widehat{\theta}}) : (M_{\mathbb{L}_1}, \mathcal{M}_{\mathbb{L}_1}, \mathcal{U}_{\mathbb{L}_1}) \to (M_{\mathbb{L}_2}, \mathcal{M}_{\mathbb{L}_2}, \mathcal{U}_{\mathbb{L}_2})$. In order for this difunction to be uniformly bicontinuous we require that

$$((f^{\widehat{\theta}}, F^{\widehat{\theta}})^{-1}(\hat{g}), (f^{\widehat{\theta}}, F^{\widehat{\theta}})^{-1}(\hat{h})) \in \mathcal{U}_{\mathbb{L}_1}$$

for all $(g, h) \in \mathcal{U}_2$. Now for $(g, h) \in \mathcal{U}_2$ we have $(\hat{g}, \hat{h}) \in \mathcal{U}_{\mathbb{L}_2}$ and

$$(f^{\widehat{\theta}}, F^{\widehat{\theta}})^{-1}(\hat{g})(\widehat{\alpha}) = (F^{\widehat{\theta}})^{\leftarrow}(\hat{g})((f^{\widehat{\theta}})^{\rightarrow}\widehat{\alpha}), \quad (f^{\widehat{\theta}}, F^{\widehat{\theta}})^{-1}(\hat{h})(\widehat{\alpha}) = (f^{\widehat{\theta}})^{\leftarrow}(\hat{g})((F^{\widehat{\theta}})^{\rightarrow}\widehat{\alpha})$$

The following lemma will help us to represent $(f^{\widehat{\theta}})^{\rightarrow}\widehat{\alpha}$ and $(F^{\widehat{\theta}})^{\rightarrow}\widehat{\alpha}$.

**Lemma 4.2.** *Let $(f, F) : (S, \mathcal{S}) \to (T, \mathcal{T})$ be a difunction and take $A \in \mathcal{S}$. Then*:

(1) $f^{\rightarrow}A = \bigcap\{B \in \mathcal{T} | A \subseteq F^{\leftarrow}B\}$,
(2) $F^{\rightarrow}A = \bigvee\{B \in \mathcal{T} | f^{\leftarrow}B \subseteq A\}$.

**Proof.** By [5, Theorem 2.24(2a)] we have $A \subseteq F^{\leftarrow}(f^{\rightarrow}A)$, so

$$\bigcap\{B \in \mathcal{T} | A \subseteq F^{\leftarrow}B\} \subseteq f^{\rightarrow}A.$$

On the other hand, if $f^{\rightarrow}A \not\subseteq \bigcap\{B \in \mathcal{T} | A \subseteq F^{\leftarrow}B\}$ then there exists some $B \in \mathcal{T}$ with $f^{\rightarrow}A \not\subseteq B$ and $A \subseteq F^{\leftarrow}B$. But now $f^{\rightarrow}A \subseteq f^{\rightarrow}(F^{\leftarrow}B) \subseteq B$ by [5, Theorem 2.24(2b)], which is a contradiction. Hence (1) holds, and the proof of (2) is dual and hence omitted. □

These results justify the following definitions.

**Definition 4.3.** Let $\theta : \mathbb{L}_2 \to \mathbb{L}_1$ be a mapping preserving arbitrary meets and joins. We define mappings $\theta^{\rightharpoonup}, \theta^{\rightharpoondown} : \mathbb{L}_1 \to \mathbb{L}_2$ by

$$\theta^{\rightharpoonup}u = \bigwedge\{v \in \mathbb{L}_2 | u \le \theta(v)\}, \quad \theta^{\rightharpoondown}u = \bigvee\{v \in \mathbb{L}_2 | \theta(v) \le u\}$$

for all $u \in \mathbb{L}_1$.

It is trivial to verify that $(f^{\widehat{\theta}})^{\rightarrow}\widehat{\alpha} = \widehat{\theta^{\rightharpoonup}\alpha}$ and $(F^{\widehat{\theta}})^{\rightarrow}\widehat{\alpha} = \widehat{\theta^{\rightharpoondown}\alpha}$. On analogy with $f^{\rightarrow}$, $F^{\rightarrow}$ these mappings preserve arbitrary joins, meets, respectively. Also, $\theta^{\rightharpoonup}$ is a co-adjoint and $\theta^{\rightharpoondown}$ an adjoint of $\theta$.

**Definition 4.4.** Let $(\mathbb{L}_1, \mathcal{U}_1)$, $(\mathbb{L}_2, \mathcal{U}_2)$ be (quasi-) di-uniform Hutton spaces, $\theta : \mathbb{L}_2 \to \mathbb{L}_1$ a mapping that preserves arbitrary joins and meets. Then $\theta$ is said to be $\mathcal{U}_1$–$\mathcal{U}_2$ uniformly bicontinuous provided $(g, h) \in \mathcal{U}_2 \Longrightarrow (\theta \circ g \circ \theta^{\rightharpoondown}, \theta \circ h \circ \theta^{\rightharpoonup}) \in \mathcal{U}_1$.

The following result is now clear from the definitions and the above discussion, and we omit the proof.

**Proposition 4.5.** *With the notation of Definition* 4.4, *$\theta$ is $\mathcal{U}_1$–$\mathcal{U}_2$ uniformly bicontinuous if and only if $(f^{\widehat{\theta}}, F^{\widehat{\theta}})$ is $\mathcal{U}_{\mathbb{L}_1}$–$\mathcal{U}_{\mathbb{L}_2}$ uniformly bicontinuous.*

Denoting by **diQUH** the category of quasi-di-uniform Hutton spaces and morphisms $\theta : (\mathbb{L}_1, \mathcal{U}_1) \to (\mathbb{L}_2, \mathcal{U}_2)$ which are uniformly bicontinuous mappings $\theta : (\mathbb{L}_2, \mathcal{U}_2) \to (\mathbb{L}_1, \mathcal{U}_1)$, we now have a functor $\mathfrak{H}_u$ : **diQUH** → **dfQDiU** defined by

$$\mathfrak{H}_u((\mathbb{L}_1, \mathcal{U}_1) \xrightarrow{\theta} (\mathbb{L}_2, \mathcal{U}_2)) = (M_{\mathbb{L}_1}, \mathcal{M}_{\mathbb{L}_1}, \mathcal{U}_{\mathbb{L}_1}) \xrightarrow{(f^{\widehat{\theta}}, F^{\widehat{\theta}})} (M_{\mathbb{L}_2}, \mathcal{M}_{\mathbb{L}_2}, \mathcal{U}_{\mathbb{L}_2}).$$

In the opposite direction, with a difunctional quasi-uniform texture space $(S, \mathcal{S}, \mathcal{U}_{\mathcal{F}})$ we may associate the quasi-di-uniform Hutton space $(\mathcal{S}, \mathcal{U}_{\mathcal{F}})$. Corresponding to a difunction $(f, F) : (S, \mathcal{S}, \mathcal{U}_{\mathcal{F}}) \to (T, \mathcal{T}, \mathcal{V}_{\mathcal{F}})$ we have the mapping

$\theta_{(f,F)} : \mathcal{T} \to \mathcal{S}$ which preserves arbitrary meets and joins. Note now that $\theta^{\to}_{(f,F)} A = f^{\to} A$, $\theta^{\twoheadrightarrow}_{(f,F)} A = F^{\to} A$, so for $(g,h) \in \mathcal{V}_{\mathcal{F}}$ we have

$$(\theta_{(f,F)} \circ g \circ \theta^{\to}_{(f,F)}, \theta_{(f,F)} \circ h \circ \theta^{\twoheadrightarrow}_{(f,F)}) = (f,F)^{-1}(g,h)$$

which by Definition 2.9 and Definition 4.4 shows that (*f,F*) is uniformly bicontinuous if and only if $\theta_{(f,F)}$ is uniformly bicontinuous. Hence we have the functor $\mathfrak{E}_u$ : **dfQdiU** $\to$ **diQUH** defined by

$$\mathfrak{E}_u((S,\mathcal{S},\mathcal{U}_{\mathcal{F}}) \overset{(f,F)}{\longrightarrow} (T,\mathcal{T},\mathcal{V}_{\mathcal{F}})) = (\mathcal{S},\mathcal{U}_{\mathcal{F}}) \overset{\theta_{(f,F)}}{\longrightarrow} (\mathcal{T},\mathcal{V}_{\mathcal{F}}).$$

Take $(\mathbb{L},\mathcal{U}) \in$ Ob**diQUH** and denote by $e$ the morphism $e : \mathfrak{E}_u(M_{\mathbb{L}},\mathcal{M}_{\mathbb{L}},\mathcal{U}_{\mathbb{L}}) = (\mathcal{M}_{\mathbb{L}},\mathcal{U}_{\mathbb{L}})$ defined by the bijection $\alpha \mapsto \widehat{\alpha}$ from $\mathbb{L}$ to $\mathcal{M}_{\mathbb{L}}$. Then it is trivial to verify that $(M_{\mathbb{L}},\sigma_{\mathbb{L}},\mathcal{U}_{\mathbb{L}},e)$ is a $\mathfrak{E}_u$-costructured arrow with codomain $(\mathbb{L},\mathcal{U})$ which is $\mathfrak{E}_u$-co-universal. Hence $\mathcal{E}_u$ is a co-adjoint, and since $(M_{\mathbb{L}},\mathcal{M}_{\mathbb{L}},\mathcal{U}_{\mathbb{L}}) = \mathfrak{H}_u(\mathbb{L},\mathcal{U})$ we see that $\mathfrak{H}_u$ is the corresponding adjoint. Moreover,

$$\mathfrak{E}_u \circ \mathfrak{H}_u \equiv \mathrm{id}_{\mathbf{diQUH}} \quad \text{and} \quad \mathfrak{H}_e \circ \mathfrak{E}_u \equiv \mathrm{id}_{\mathbf{dfQDiU}},$$

so $\mathfrak{H}_u$ and $\mathfrak{E}_u$ are equivalences between the categories **diQUH** and **dfQDiU**.

Clearly these equivalences restrict to equivalences between **diUH** and **dfDiU** in the obvious way.

The main difference between the definition of (quasi-) di-uniformity on $\mathbb{L}$ and the notion of quasi-uniformity given by Hutton in [11], is that the latter involves single mappings. We now develop a characterization of di-uniformities which resolves this difference.

Let $\mathcal{U}$ be a (quasi-) di-uniformity on $\mathbb{L}$, and consider the following non-empty sets of mappings:

$$\overline{\mathcal{U}} = \{g \in \mathcal{Q} | \exists h \in \mathcal{P}_{\mathbb{L}} \text{ with } (g,h) \in \mathcal{U}\}, \quad \underline{\mathcal{U}} = \{h \in \mathcal{P} | \exists g \in \mathcal{Q}_{\mathbb{L}} \text{ with } (g,h) \in \mathcal{U}\}.$$

We now have

**Lemma 4.6.** *If $\mathcal{U}$ is a di-uniformity on $\mathbb{L}$ and $\overline{\mathcal{U}}, \underline{\mathcal{U}}$ are as defined above*:

U1 $g \in \overline{\mathcal{U}},\ g_1 \in \mathcal{Q}\ with\ g \le g_1 \Longrightarrow g_1 \in \overline{\mathcal{U}}$.
U2 $g_1, g_2 \in \overline{\mathcal{U}} \Longrightarrow g_1 \wedge g_2 \in \overline{\mathcal{U}}$.
U3 $g \in \overline{\mathcal{U}} \Longrightarrow \exists g_1 \in \overline{\mathcal{U}}\ with\ g_1^2 \le g$.
SYM $g \in \overline{\mathcal{U}} \iff \overleftarrow{g} \in \underline{\mathcal{U}}$.
CU1 $h \in \underline{\mathcal{U}},\ h_1 \in \mathcal{P}\ with\ h_1 \le h \Longrightarrow h_1 \in \underline{\mathcal{U}}$.
CU2 $h_1, h_2 \in \underline{\mathcal{U}} \Longrightarrow h_1 \vee h_2 \in \underline{\mathcal{U}}$.
CU3 $h \in \underline{\mathcal{U}} \Longrightarrow \exists h_1 \in \underline{\mathcal{U}}\ with\ h \le h_1^2$.

**Proof.** U1–U3 and CU1–CU3 follow trivially from Definition 4.1(1)–(3). To prove SYM take $g \in \overline{\mathcal{U}}$, whence we have $h$ with $(g,h) \in \mathcal{U}$. Since a direlational uniformity has a base of symmetric direlations by [14, Lemma 3.2], the same is true for di-uniformities on $\mathbb{L}$, so we have $(g_1,h_1) \in \mathcal{U}$ symmetric with $(g_1,h_1) \le (g,h)$. On analogy with the formula in Proposition 2.4 we have $\overleftarrow{g}(\alpha) = \bigvee\{\beta \in \mathbb{L} | g(\beta) \le \alpha\}$ so $g_1 \le g$ implies $\overleftarrow{g} \le \overleftarrow{g}_1$, and by symmetry $\overleftarrow{g}_1 = h_1$, whence $\overleftarrow{g} \in \underline{\mathcal{U}}$, as required. The reverse implication can be established by a similar argument. □

Conversely

**Lemma 4.7.** *Let $\overline{\mathcal{U}} \subseteq \mathcal{Q}$, $\underline{\mathcal{U}} \subseteq \mathcal{P}$ be families of mappings satisfying the conditions of Lemma* 4.6. *Then*

$$\mathcal{U} = \{(g,h) | g \in \overline{\mathcal{U}}, h \in \underline{\mathcal{U}}\}$$

*is a di-uniformity on $\mathbb{L}$ for which $\overline{\mathcal{U}} = \{g \in \mathcal{Q} | \exists h\ with\ (g,h) \in \mathcal{U}\}$ and $\underline{\mathcal{U}} = \{h \in \mathcal{P} | \exists g\ with\ (g,h) \in \mathcal{U}\}$.*

**Proof.** Straightforward. □

The above results show that a di-uniformity $\mathcal{U}$ on $\mathbb{L}$ can be uniquely represented by the sets $\overline{\mathcal{U}}, \underline{\mathcal{U}}$ of mappings satisfying the conditions of Lemma 4.6, and moreover by removing the symmetry condition SYM we have a corresponding representation of a quasi-di-uniformity. Indeed, it is only for quasi-di-uniformities that we actually require both sets since SYM can be used to obtain $\underline{\mathcal{U}}$ from $\overline{\mathcal{U}}$, and conversely.

Let us now look at the morphisms in this representation. The following is immediate, and we omit a detailed proof.

**Proposition 4.8.** $\theta : (\mathbb{L}, \mathcal{U}) \to (\mathbb{M}, \mathcal{V})$ *is a* **diQUH**-*morphism if and only if it is a mapping* $\theta : \mathbb{M} \to \mathbb{L}$ *preserving arbitrary meets and joins that satisfies* $g \in \overline{\mathcal{V}} \Longrightarrow \theta \circ g \circ \theta^{\rightharpoondown} \in \overline{\mathcal{U}}$ *and* $h \in \underline{\mathcal{V}} \Longrightarrow \theta \circ h \circ \theta^{\rightharpoonup} \in \underline{\mathcal{U}}$.

In the case of a **diUH**-morphism one of these conditions suffices, as we now show.

**Lemma 4.9.** *Let* $(\mathbb{L}, \mathcal{U})$, $(\mathbb{M}, \mathcal{V})$ *be di-uniform Hutton spaces,* $\theta : \mathbb{M} \to \mathbb{L}$ *a mapping which preserves arbitrary meets and joins. Then* $g \in \overline{\mathcal{V}} \Longrightarrow \theta \circ g \circ \theta^{\rightharpoondown} \in \overline{\mathcal{U}}$ *if and only if* $h \in \underline{\mathcal{V}} \Longrightarrow \theta \circ h \circ \theta^{\rightharpoonup} \in \underline{\mathcal{U}}$.

**Proof.** By SYM it will be sufficient to show that

$$\overleftarrow{\theta \circ g \circ \theta^{\rightharpoondown}} = \theta \circ \overleftarrow{g} \circ \theta^{\rightharpoonup} \quad \text{and} \quad \overleftarrow{\theta \circ h \circ \theta^{\rightharpoonup}} = \theta \circ \overleftarrow{h} \circ \theta^{\rightharpoondown}.$$

We prove the first equality, leaving the dual second equality to the interested reader. Rather than give a direct proof we verify the corresponding equality:

$$(f, F)^{-1}(\overleftarrow{\varphi}) = \overleftarrow{(f, F)^{-1}(\varphi)}$$

for difunctional quasi-uniform texture spaces. Here $(f, F) : (S, \mathcal{S}, \mathcal{U}_{\mathcal{F}}) \to (T, \mathcal{T}, \mathcal{V}_{\mathcal{F}})$ and $\varphi \in \mathcal{F}^{T}_{\mathcal{R}\mathcal{R}}$. There is no loss of generality in writing $\varphi = \varphi_r$, where $r$ is a relation on $(T, \mathcal{T})$. By definition $\overleftarrow{\varphi_r} = \psi_{r^{\leftarrow}}$, so $(f, F)^{-1}(\overleftarrow{\varphi}) = (f, F)^{-1}(\psi_{r^{\leftarrow}}) = \psi_{(f,F)^{-1}(r^{\leftarrow})}$ by Proposition 2.10. However, $(f, F)^{-1}(r^{\leftarrow}) = ((f, F)^{-1}(r))^{\leftarrow}$ by [14, Proposition 5.5], so $\psi_{(f,F)^{-1}(r^{\leftarrow})} = \overleftarrow{\varphi}_{(f,F)^{-1}(r)} = \overleftarrow{(f, F)^{-1}(\varphi_r)}$ by Proposition 2.10. $\square$

As a result of the above discussion we see that we may represent a di-uniformity either as a non-empty family $\overline{\mathcal{U}} \subseteq \mathcal{Q}$ satisfying U1–U3, and uniform bicontinuity of $\theta$ given by $g \in \overline{\mathcal{V}} \Longrightarrow \theta \circ g \circ \theta^{\rightharpoondown} \in \overline{\mathcal{U}}$, or as a non-empty conjugate family $\underline{\mathcal{U}} \subseteq \mathcal{P}$ satisfying CU1–CU3 and uniform bicontinuity given by $h \in \underline{\mathcal{V}} \Longrightarrow \theta \circ h \circ \theta^{\rightharpoonup} \in \underline{\mathcal{U}}$. Naturally, the symmetry is not apparent in this case, but is easily reinstated when we wish to emphasize the connection with our earlier representations.

In the case of a quasi-di-uniformity $\mathcal{U}$ generally both $\overline{\mathcal{U}}$ and $\underline{\mathcal{U}}$ are required and the two conditions for uniform bicontinuity are no longer equivalent.

It is left to the interested reader to verify that these new representations lead to concretely isomorphic categories.

Finally, let us note that on analogy with Corollary 3.6, if $(\mathbb{L}, \mathcal{U})$ is a di-uniform Hutton space then the uniform ditopology $(\tau, \kappa)$ may be defined in terms of an interior operator and a closure operator by

$$\mathrm{int}(\lambda) = \bigvee\{\alpha \in \mathbb{L} | \exists g \in \overline{\mathcal{U}} \text{ with } g(\alpha) \leq \lambda\},$$
$$\mathrm{cl}(\lambda) = \bigwedge\{\alpha \in \mathbb{L} | \exists h \in \underline{\mathcal{U}} \text{ with } \lambda \leq h(\alpha)\}.$$

We are now in a position to give the promised relation with Hutton uniformities and quasi-uniformities. It is clear that the set $\mathcal{Q}$ of functions on $\mathbb{L}$ defined in [11] is no other that the set $\mathcal{Q} = \mathcal{Q}_{\mathbb{L}}$ considered here, and that a Hutton quasi-uniformity in the sense of [11, Definition 2] is a non-empty subset of $\mathcal{Q}$ satisfying U1–U3 of Lemma 4.6. Hence we have a bijection between the di-uniformities on $\mathbb{L}$ and the Hutton quasi-uniformities on $\mathbb{L}$, and the notion of uniform bicontinuity is seen to coincide with that of uniform continuity for mappings $\theta$ preserving arbitrary meets and joins. Moreover, the uniform topology of a Hutton quasi-uniformity as defined in [11, Definition 4] is precisely the uniform topology $\tau$ defined above.

Now let us consider the case where $\mathbb{L}$ is equipped with an order reversing involution. Then the corresponding Hutton texture has a complementation, and various terms relating to complementation will have their counterparts here. In particular a complemented difunction is easily seen to correspond to a mapping $\theta$ that as well as arbitrary meets and joins preserves also the involution, that is $\theta(b') = \theta(\beta)'$ for all $\beta \in \mathbb{L}$. If $\mathcal{U}$ is a di-uniformity on $\mathbb{L}$ and for $\varphi : \mathbb{L} \to \mathbb{L}$ we define $\varphi' : \mathbb{L} \to \mathbb{L}$ by $\varphi'(\alpha) = (\varphi(\alpha'))'$, $\alpha \in \mathbb{L}$, then $\mathcal{U}' = \{(h', g') | (g, h) \in \mathcal{U}\}$ is also a di-uniformity on $\mathbb{L}$ and $\mathcal{U}$ is called *complemented* if $\mathcal{U} = \mathcal{U}'$ (see [15] for the corresponding concept in a textural setting). We now show that the Hutton uniformities correspond to the complemented di-uniformities.

To see, this recall by [11, Definition 7] that a Hutton quasi-uniformity $\mathcal{D}$ is called a uniformity if $\varphi \in \mathcal{D} \implies \varphi^{-1} \in \mathcal{D}$. Here $\varphi^{-1} : \mathbb{L} \to \mathbb{L}$ is given by

$$\varphi^{-1}(\alpha) = \bigwedge\{\beta \in \mathbb{L} | \varphi(\beta') \le \alpha'\}, \quad \alpha \in \mathbb{L}.$$

It may be noted that when $\theta$ preserves the involution $'$ then $\theta^{-1} = \theta^{\rightharpoonup}$. On the other hand, for $\varphi : \mathbb{L} \to \mathbb{L}$, $\alpha \in \mathbb{L}$,

$$\begin{aligned} \overleftarrow{\varphi'}(\alpha) &= \bigwedge\{\beta \in \mathbb{L} | \alpha \le \varphi'(\beta)\} = \bigwedge\{\beta \in \mathbb{L} | \alpha \le (\varphi(\beta')'\} \\ &= \bigwedge\{\beta \in \mathbb{L} | \varphi(\beta') \le \alpha'\} = \varphi^{-1}(\alpha), \end{aligned}$$

whence $\overleftarrow{\varphi'} = \varphi^{-1}$. If $\mathcal{U}$ is a di-uniformity and $\overline{\mathcal{U}}$ is a Hutton uniformity then $g \in \overline{\mathcal{U}} \implies g' \in \underline{\mathcal{U}}$, and dually $h \in \underline{\mathcal{U}} \implies h' \in \overline{\mathcal{U}}$, so $\{(h', g') | (g, h) \in \mathcal{U}\} \subseteq \mathcal{U}$. Using the fact that the correspondence $\varphi \mapsto \varphi'$ is an involution we likewise obtain $\mathcal{U} \subseteq \{(h', g') | (g, h) \in \mathcal{U}\}$, and so $\mathcal{U}$ is complemented. The reader will note the analogy with the correspondence between classical uniformities and complemented di-uniformities on a discrete texture $(X, \mathcal{P}(X), \pi_X)$ given in [15].


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