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An alternative perspective on injectivity of modules

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Given modules M and N, M is said to be N-*subinjective* if for every extension K of N and every homomorphism $\varphi : N \to M$ there exists a homomorphism $\phi : K \to M$ such that $\phi|_N = \varphi$. For a module M, *the subinjectivity domain of* M is defined to be the collection of all modules N such that M is N-subinjective. As an opposite to injectivity, a module M is said to be indigent if its subinjectivity domain is smallest possible, namely, consisting of exactly the injective modules. Properties of subinjectivity domains and of indigent modules are studied. In particular, the existence of indigent modules is determined for some families of rings including the ring of integers and Artinian serial rings. It is also shown that some rings (e.g. Artinian chain rings) have *no middle class* in the sense that all modules are either injective or indigent. For various classes of modules (such as semisimple, singular and projective), necessary and sufficient conditions for the existence of indigent modules of those types are studied. Indigent modules are analog to the so-called *poor modules*, an opposite of injectivity (in terms of injectivity domains) recently studied in papers by Alahmadi, Alkan and López-Permouth and by Er, López-Permouth and Sökmez. Relations between poor and indigent modules are also investigated here.

1. Introduction and preliminaries

The purpose of this paper is to initiate the study of an alternative perspective on the analysis of the injectivity of a module. In contrast to the well-known notion of relative injectivity, we introduce

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the notion of subinjectivity. Namely, a module M is said to be A-subinjective (or subinjective relative to A) if for every extension B of A and every homomorphism $\varphi : A \to M$ there exists a homomorphism $\phi : B \to M$ such that $\phi|_A = \varphi$. For every module M, the subinjectivity domain of M consists of $\{A \mid M$ is subinjective relative to $A\}$. In the same way that a module is injective if and only if its injectivity domain consists of the entire class *Mod-R*, it is clear that a module is injective also if and only if its subinjectivity domain equals *Mod-R*. An interesting situation arises, however, when one studies modules which are not injective. While there are many questions one may consider regarding this new perspective on injectivity, as a first project, we focus on this paper in those modules which are the least injective with respect to their subinjectivity domains. As subinjectivity domains clearly include all injective modules, a reasonable opposite to injectivity in this context is obtained by considering modules whose subinjectivity domain consists of only injective modules. We initiate the study of those modules here and we refer to them as *indigent modules*.

Indigent modules are the subinjectivity domain analog of "poor modules". The notion of a poor module was introduced as an opposite to injectivity by Alahmadi, Alkan and López-Permouth (see [1]). Poor modules have been studied further in [5] where Er, López-Permouth and Sökmez prove that every ring has a poor module and characterize those rings having semisimple poor modules. Also in the same paper, the structure of rings over which every module is poor or injective is investigated.

In this paper, we study the properties of subinjectivity domains and of indigent modules. We also establish connections between indigent and poor modules. We show that Artinian serial rings have an indigent module. Moreover, if R is an Artinian serial ring with $J(R)^2 = 0$, then R has a semisimple indigent module. Also, we are able to show that the ring of integers $\mathbb{Z}$ has a semisimple indigent module, whereas no simple $\mathbb{Z}$-module is indigent. In addition, we study indigent modules versus poor modules. We observe that indigent modules are poor over an Artinian serial ring R with $J(R)^2 = 0$, but indigent modules need not be poor over an Artinian serial ring even though a simple indigent module is poor over this ring. In fact, all non-injective modules over an Artinian chain ring are indigent. But this is not the case for poor modules, i.e., there are examples of Artinian chain rings which have modules that are neither injective nor poor. However, in some situations poor and indigent modules coincide. For instance, if R is a non-semisimple *QF*-ring with homogeneous right socle and $J(R)^2 = 0$, then poor and indigent modules coincide.

Throughout this paper, R will be an associative ring with identity and modules will be unital right R-modules. *Mod-R* will denote the category of all right R-modules over a ring R. If M is an R-module, then $E(M)$, $Rad(M)$, $Z(M)$ and $Soc(M)$ will respectively denote the injective hull, Jacobson radical, the singular submodule and socle of M. $J(R)$ will stand for the Jacobson radical of R. We will use the notations $\leqslant$, $\leqslant_e$ and $\leqslant^{\oplus}$ in order to indicate submodules, essential submodules and direct summands, respectively.

Recall that a module is said to be uniserial if the lattice of its submodules is linearly ordered under set inclusion. A ring R is said to be a right chain ring if it is a uniserial module as a right module over itself. A left chain ring is defined similarly. A ring R is a chain ring if it is both a right and a left chain ring. A serial module is a module that is a direct sum of uniserial modules. A ring R is called right (left) serial if it is a serial module as a right (left) module over itself. If both conditions hold R is a serial ring. Note that every (right) chain ring is a (right) serial ring.

A ring R is called *right V-ring* if simple right R-modules are injective. The notion of right V-rings may be generalized to that of *right GV-rings (generalized V-rings)*, in which every simple module is either injective or projective. Right *GV*-rings were introduced in [9].

A module M is called *quasi-injective* if it is M-injective. Any quasi-injective module M satisfies the following two conditions (see [10, Proposition 2.1]):

($C1$) For every submodule N of M there exists $M_1 \leqslant^{\oplus} M$ with $N \leqslant_e M_1$.
($C2$) For any summand M' of M, every exact sequence $0 \to M' \to M$ splits.

Note that modules which satisfy ($C1$) have also been called *extending*. If a module M satisfies ($C2$), then it satisfies the following condition (see [10, Proposition 2.2]):

($C3$) If M_1 and M_2 are summands of M with $M_1 \cap M_2 = 0$, then $M_1 \oplus M_2$ is a summand of M.

Recall from [10] that a module M is called *continuous (quasi-continuous)* if it satisfies $(C1)$ and $(C2)$ ($(C1)$ and $(C3)$). Hence the following hierarchy exists:

$$\text{Injective} \quad \Rightarrow \quad \text{quasi-injective} \quad \Rightarrow \quad \text{continuous} \quad \Rightarrow \quad \text{quasi-continuous.}$$

For additional concepts and results not mentioned here, we refer the reader to [2,6,7,10].

2. The notion of subinjectivity and the subinjectivity domain of a module

Definition 2.1. Given modules M and N, we say that M is *N-subinjective* if for every module K with $N \leqslant K$ and every homomorphism $\varphi : N \to M$ there exists a homomorphism $\phi : K \to M$ such that $\phi|_N = \varphi$. The *subinjectivity domain* of a module M, $\underline{\mathfrak{In}}^{-1}(M)$, is defined to be the collection of all modules N such that M is N-subinjective.

Our next lemma shows for M to be N-subinjective, one only needs to extend maps to $E(N)$.

Lemma 2.2. *The following statements are equivalent for any modules M and N:*

(1) *M is N-subinjective.*
(2) *For each $\varphi : N \to M$ and for every module K with $N \leqslant_e K$ there exists a homomorphism $\phi : K \to M$ such that $\phi|_N = \varphi$.*
(3) *For each $\varphi : N \to M$ there exists a homomorphism $\phi : E(N) \to M$ such that $\phi|_N = \varphi$.*
(4) *For each $\varphi : N \to M$ there exists an injective extension E of N and a homomorphism $\phi : E \to M$ such that $\phi|_N = \varphi$.*

Proof. The implications (1) ⇒ (2) ⇒ (3) ⇒ (4) are obvious. To show (4) ⇒ (1), let $N \subseteq N'$ and $\varphi : N \to M$. By assumption, there exists an injective extension E of N and a homomorphism $\phi : E \to M$ such that $\phi|_N = \varphi$. Since E is injective, there exists a $\psi : N' \to E$ such that $\psi|_N = i$, where $i : N \to E$ is the inclusion. Then we get that $(\phi\psi)|_N = \varphi$. This gives that $N \in \underline{\mathfrak{In}}^{-1}(M)$. □

Applying Lemma 2.2(3) to the identity $M \to M$, one sees that a module M is M-subinjective if and only if it is injective.

It is clear that injective modules are contained in the subinjectivity domain of any module, and that a module is injective if and only if its subinjectivity domain consists of all modules in *Mod-R*. Moreover, we have the following fact.

Proposition 2.3. $\bigcap_{M \in \textit{Mod-}R} \underline{\mathfrak{In}}^{-1}(M) = \{A \in \textit{Mod-}R \mid A \text{ is injective}\}$.

Proof. Let $N \in \bigcap_{M \in \textit{Mod-}R} \underline{\mathfrak{In}}^{-1}(M)$. Then $N \in \underline{\mathfrak{In}}^{-1}(N)$ which means that N is injective. □

According to well-known Baer's Criterion, an R-module is injective if it is injective relative to R. However, a module need not be injective if it is R-subinjective. For instance, every module over a self-injective ring R is R-subinjective.

Proposition 2.4. *The following properties hold for a module N:*

(1) *$\prod_{i \in I} M_i$ is N-subinjective if and only if each M_i is N-subinjective.*
(2) *If each M_i is N-subinjective for $i = 1, \dots, n$, then so is $\bigoplus_{i=1}^{n} M_i$.*
(3) *Every direct summand of an N-subinjective module is an N-subinjective module. Conversely, if N is a finitely generated module and M_i, $i \in I$ is a family of N-subinjective modules indexed in an arbitrary index set I, then $\bigoplus_{i \in I} M_i$ is an N-subinjective module.*

Proof. (1) Suppose that M_i is N-subinjective for each $i \in I$. Consider a homomorphism $\varphi : N \to \prod_{i\in I} M_i$. Let $\pi_i : \prod_{i\in I} M_i \to M_i$ be the canonical epimorphism for each $i \in I$. Then there exists a $\phi_i : E(N) \to M_i$ such that $\phi_i|_N = \pi_i\varphi$ for each $i \in I$. Define an R-homomorphism $\psi : E(N) \to \prod_{i\in I} M_i$ via $x \mapsto (\phi_i(x))$. Then $\psi|_N = \varphi$.

For the converse, let $i \in I$ and $\varphi : N \to M_i$. There exists a $\phi : E(N) \to \prod_{i\in I} M_i$ such that $\phi|_N = e_i\varphi$, where e_i is the inclusion $M_i \to \prod_{i\in I} M_i$. Let $\pi_i : \prod_{i\in I} M_i \to M_i$ be the canonical epimorphism. Then $(\pi_i\phi)|_N = \varphi$. Hence, $N \in \bigcap_{i\in I} \underline{\mathfrak{In}}^{-1}(M_i)$.

The proofs of (2) and (3) are similar to the proof of (1). □

The subinjectivity domain $\underline{\mathfrak{In}}^{-1}(M)$ of M need not be closed under submodules. For instance, let M be a non-injective module. Then M is not M-subinjective by Proposition 2.5(5) below, but it is subinjective relative to its injective hull. Notice that $\underline{\mathfrak{In}}^{-1}(M)$ is closed under submodules if and only if M is injective, if and only if $\underline{\mathfrak{In}}^{-1}(M)$ is contained in the injectivity domain of M.

Likewise, in light of Proposition 2.5(1), we cannot expect that if M is N-subinjective and $N \subset K$ then M is K-subinjective. Simply take a non-injective module M and consider K to be the sum $N \oplus N^*$ for some N^* such that M is not N^*-subinjective. On the other hand, one may expect that the result holds true if one assumes that K is an essential extension of N. We do not know if this is true in general but we can prove it for the special case when M is non-singular (see part (4) of the following proposition).

Proposition 2.5. *The following properties hold for any ring R and R-modules N and M:*

(1) *If $N = \bigoplus_{i=1}^n N_i$, then M is N-subinjective if and only if M is N_i-subinjective for each $i = 1, \dots, n$.*
(2) *If R is right Noetherian and I is any index set, then M is $\bigoplus_{i\in I} N_i$-subinjective if and only if M is N_i-subinjective for each $i \in I$.*
(3) *If R is a right hereditary right Noetherian ring and M is N-subinjective, then M is N/K-subinjective for any submodule K of N.*
(4) *If M is a non-singular N-subinjective module, then M is K-subinjective for any essential extension K of N.*
(5) *If $N \leqslant M$ and M is N-subinjective, then $E(N) \leqslant M$. In particular, M is M-subinjective if and only if M is injective.*

Proof. (1) Let $\varphi : N_i \to M$, and consider the canonical epimorphism $\pi : N \to N_i$. Since $N \in \underline{\mathfrak{In}}^{-1}(M)$, there exists a $\phi : E(N_i) \oplus E(\bigoplus_{i\neq j} N_j) \to M$ such that $\phi|_N = \varphi\pi$. Then $\psi = \phi|_{E(N_i)} : E(N_i) \to M$, and hence $\psi|_{N_i} = \varphi$. Now let $\varphi : N \to M$. Then there exists $\psi_i : E(N_i) \to M$ such that $\psi_i|_{N_i} = \varphi\pi_i$ for each $i = 1, \dots, n$. Define $\psi : \bigoplus_{i=1}^n E(N_i) \to M$, $x_1 + \cdots + x_n \mapsto \psi_1(x_1) + \cdots + \psi(x_n)$. Hence, we get that $\psi|_N = \varphi$.

(2) Since R is right Noetherian, $E(N) = \bigoplus_{i\in I} E(N_i)$. The rest of the proof is similar to that of (1).

(3) Since R is right Noetherian, we have a decomposition $M = M_1 \oplus M_2$, where M_1 is an injective module and M_2 is a reduced module, i.e., a module which does not have non-zero injective submodules. Then $\underline{\mathfrak{In}}^{-1}(M) = \underline{\mathfrak{In}}^{-1}(M_1) \cap \underline{\mathfrak{In}}^{-1}(M_2)$ by Proposition 2.4(1). But since M_1 is injective, its subinjectivity domain consists of all R-modules. Therefore, $\underline{\mathfrak{In}}^{-1}(M) = \underline{\mathfrak{In}}^{-1}(M_2)$. On the other hand, R being right hereditary implies that $\underline{\mathfrak{In}}^{-1}(M_2) = \{N \in \mathit{Mod}\text{-}R \mid \mathit{Hom}_R(N, M_2) = 0\}$. It is easy to see that this set is closed under taking homomorphic images.

(4) Let $f : K \to M$ be any homomorphism. Since M is N-subinjective, there exists $g : E(N) \to M$ such that $g|_N = f|_N$. Then $N \subseteq \mathit{Ker}(g - f)$. Because N is an essential submodule of K, $\mathit{Ker}(g - f)$ is essential in K, too. Therefore, $K/\mathit{Ker}(g - f)$ is singular. On the other hand, $K/\mathit{Ker}(g - f)$ is isomorphic to a submodule of the non-singular module M. Hence, $K = \mathit{Ker}(g - f)$ which means that $g|_K = f|_K$.

(5) Since N is essential in $E(N)$, $E(N)$ can be embedded into M because of N-subinjectivity assumption. □

Example 2.6. The condition that 'R is right Noetherian' in Proposition 2.5(2) is not superfluous. If R is a non-Noetherian ring, then there exist injective modules M_i such that $M = \bigoplus_{i\in I} M_i$ is not injective. Then M is M_i-subinjective for each $i \in I$, whereas M is not M-subinjective. Also, the M_i's are M-subinjective. So the statement (2) of Proposition 2.4 need not be true for arbitrary index sets.

Example 2.7. The condition that 'R is right hereditary' in Proposition 2.5(3) is not superfluous. Let $R = \mathbb{Z}/8\mathbb{Z}$. Then $2R$ is R-subinjective but it is not subinjective relative to any homomorphic image of R (see Corollary 5.3 below).

A submodule $N \leqslant M$ is *pure* in M in case $N \cap MI = NI$ for each left ideal I of R (see [2, Exercise 11, p. 232]).

Proposition 2.8. (1) *Let $R = R_1 \oplus R_2$ be a ring decomposition. Then M is N-subinjective in Mod-R if and only if MR_i is NR_i-subinjective in Mod-R_i for each $i = 1, 2$.*

(2) *Let I be an ideal of a ring R, and let M and N be R/I-modules. If M is N-subinjective as an R/I-module, then it is N-subinjective as an R-module. The converse holds if N is a pure submodule of $E(N)$.*

Proof. (1) By assumption, we have $K = KR_1 \oplus KR_2$ for any R-module K. Now assume that M is N-subinjective. Let $f_i : NR_i \to MR_i$ be an R_i-homomorphism. We can define an R-homomorphism $f' : N \to M$, $n_1r_1 + n_2r_2 \mapsto f_i(n_ir_i)$, where $n_1, n_2 \in N$, $r_i \in R_i$ for $i = 1, 2$. Then there exists $g : E(NR_1) \oplus E(NR_2) \to M$ such that $g|_N = f'$. Hence, the result follows. For the converse, note that $E(N) \hookrightarrow E(NR_1) \oplus E(NR_2)$ since $E(NR_1) \oplus E(NR_2)$ is an injective R-module.

(2) Let $E_{R/I}(N)$ be the injective hull of $N_{R/I}$. Since $E_{R/I}(N)$ is also an injective R-module, the result follows from Lemma 2.2(4). For the reverse implication, let N be pure in $E(N)$. Then N being both pure and essential in $E(N)$ implies that $E(N)$ has an R/I-module structure. $\square$

Projective modules exhibit a very interesting behaviour when they appear in subinjectivity domains. Namely, if a projective module is in the subinjectivity domain of a module then it also appears in the subinjectivity domains of all quotients of that module. That is the subject of our next proposition.

Proposition 2.9. *Consider the following statements for a module N:*

(1) *N is projective.*
(2) *Every homomorphic image of an N-subinjective module is N-subinjective.*
(3) *Every homomorphic image of an injective module is N-subinjective.*

Then (1) $\Rightarrow$ (2) $\Rightarrow$ (3), *and* (3) $\Rightarrow$ (1) *if the injective hull $E(N)$ of N is projective.*

Proof. (1) $\Rightarrow$ (2) Let M be an N-subinjective module. Let $K \leqslant M$ and let $f : N \to M/K$ be a homomorphism. Since N is projective, there exists a homomorphism $g : N \to M$ such that $\pi g = f$, where $\pi : M \to M/K$ is the canonical epimorphism. But N-subinjectivity of M implies that g can be extended to a homomorphism $h : E(N) \to M$. It follows that the homomorphism $\pi h : E(N) \to M/K$ extends f. (2) $\Rightarrow$ (3) is obvious. For (3) $\Rightarrow$ (1) assume that $E(N)$ is projective. Let M and K be modules such that $K \leqslant M$, and let $f : N \to M/K$. Then we have $if : N \to E(M)/K$, where $i : M/K \to E(M)/K$ is the inclusion. By hypothesis, $E(M)/K$ is N-subinjective, so there exists $g : E(N) \to E(M)/K$ which extends if. But $E(N)$ is projective. Therefore, there is a homomorphism $h : E(N) \to E(M)$ such that $\pi' h = g$, where $\pi' : E(M) \to E(M)/K$ is the canonical epimorphism. Hence, if we consider $hi : N \to M$, then $\pi hi = f$, where $\pi : M \to M/N$ is the canonical epimorphism. $\square$

If F is free and M is F-subinjective, then so is every homomorphic image of M by Proposition 2.9.

3. Modules whose subinjectivity domain consists of only injective modules

Since a module is injective if and only if it is N-subinjective for all N, it makes sense to wonder about the extreme opposite: What are the modules which are subinjective with respect to the smallest possible collection of modules? It is clear that such a smallest collection would have to consist precisely of the injective modules. That is the motivation for our next definition which was inspired by another "opposite" of injectivity, the poor modules studied in [1] and [5].

Definition 3.1. We will call a module M *indigent* in case $\underline{\mathfrak{In}}^{-1}(M) = \{A \in \text{Mod-}R \mid A \text{ is injective}\}$.

Considering that the notion of indigent modules is formally so similar to that of poor modules, one would expect that many results in this theory will echo those of the other one. That is indeed the case but differences are also abundant and interesting. We start with two results that are very close to their "poor" counterparts.

Proposition 3.2. *If a module M is indigent, then $M \oplus N$ is indigent for any module N.*

Proof. Let $A \in \underline{\mathfrak{In}}^{-1}(M \oplus N)$. Then by Proposition 2.4(3), $A \in \underline{\mathfrak{In}}^{-1}(M)$. Since M is indigent, A is injective. □

Proposition 3.3. *For an arbitrary ring R, the following conditions are equivalent*:

(1) *R is semisimple Artinian.*
(2) *Every* (*non-zero*) *R-module is indigent.*
(3) *There exists an injective indigent module.*
(4) $\{0\}$ *is an indigent module.*
(5) *R has an indigent module and non-zero direct summands of indigent modules are indigent.*
(6) *R has an indigent module and non-zero factors of indigent modules are indigent.*

Considering that quasi-injective, continuous and quasi-continuous modules are all generalizations of injectivity, one may be tempted to think that the idea of considering modules whose domain of subinjectivity is contained in one of those classes may be yielding more than just the indigent modules. Our next proposition shows that this is not the case.

First note that, according to [10, Proposition 2.10], if $M_1 \oplus M_2$ is quasi-continuous, then M_1 and M_2 are relatively injective (i.e., M_1 is M_2-injective and M_2 is M_1-injective).

Proposition 3.4. *The following conditions are equivalent for a module M*:

(1) *M is indigent.*
(2) $\underline{\mathfrak{In}}^{-1}(M) \subseteq \{N \in \text{Mod-}R \mid N \text{ is quasi-injective}\}$.
(3) $\underline{\mathfrak{In}}^{-1}(M) \subseteq \{N \in \text{Mod-}R \mid N \text{ is continuous}\}$.
(4) $\underline{\mathfrak{In}}^{-1}(M) \subseteq \{N \in \text{Mod-}R \mid N \text{ is quasi-continuous}\}$.

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) Obvious. For (4) $\Rightarrow$ (1), let M be N-subinjective. Since M is always $E(N)$-subinjective, we have that M is $N \oplus E(N)$-subinjective by Proposition 2.5(1). By assumption, $N \oplus E(N)$ is quasi-continuous. But then N is $E(N)$-injective and hence $N = E(N)$ is injective by [10, Proposition 2.10]. □

Certainly, the first problem that comes to mind with the introduction of the notion of indigent modules is whether such modules exist over all rings. We have not been able to answer this question entirely but we can guarantee so far that this is indeed the case for various rings, including the ring of integers and the so-called $\sum$-cyclic rings. In [5], the existence of poor modules is proven for arbitrary rings and, indeed, two generic constructions for poor modules over arbitrary rings are given. We attempt similar constructions here. The reason why we are focusing on Noetherian rings is because of an inherent difficulty in the study of subinjectivity. Namely, we have not been able to identify a manageable subclass that characterizes subinjectivity domains. In other words, while injectivity domains are characterized by the cyclic modules they contain (see [10]), we do not have the benefit of any similar characterization for subinjectivity domains. Due to this difficulty, our attempts to prove that a suitable module $\mathfrak{M}$ is indigent have been based on using the type of additional information one has about modules over Noetherian rings. For the remainder of this section, let $\mathfrak{M} = \bigoplus_{U \in \mathfrak{B}} U$, where $\mathfrak{B}$ is a complete set of non-injective uniform modules, and let $\mathfrak{N} = \bigoplus_{N \in \Gamma} N$, where Γ is any complete set of representatives of cyclic modules. We suspect that $\mathfrak{M}$ is indigent, at least over all Noetherian rings, but we have not been able to prove it in general. We do know that the construction yields an indigent module when the ring is Artinian serial (Theorem 3.5). The second construction is shown to yield indigent modules over all right $\sum$-cyclic rings (Theorem 3.8). In particular, this second construction is therefore also successful for Artinian serial rings (Corollary 3.9). Interestingly, $\mathfrak{M}$ and $\mathfrak{N}$ are indigent over the ring of integers (see Corollary 4.5).

Theorem 3.5. *Let R be a right Noetherian ring. Then $\underline{\mathfrak{In}}^{-1}(\mathfrak{M})$ does not contain non-injective extending modules. In particular, $\mathfrak{M}$ is indigent when R is Artinian serial.*

Proof. First we will show that if U is a non-injective uniform module, then U does not belong to $\underline{\mathfrak{In}}^{-1}(\mathfrak{M})$. For if $\mathfrak{M}$ is U-subinjective, then U is U-subinjective by Proposition 2.4(3). But since U is non-injective, this is a contradiction. Now let N be a non-injective extending module. R being right Noetherian implies that N is a direct sum of uniform modules [10, Theorem 2.19]. Because R is right Noetherian and N is non-injective, N has a non-injective uniform summand U. But $U \notin \underline{\mathfrak{In}}^{-1}(\mathfrak{M})$. By Proposition 2.5(1), $N \notin \underline{\mathfrak{In}}^{-1}(\mathfrak{M})$.

If R is a ring whose right modules are direct sums of extending modules, then R is an Artinian ring by [4, Theorem 1]. Since a non-injective R-module contains a non-injective extending summand, $\mathfrak{M}$ is an indigent R-module. Now assume that R is an Artinian serial ring. It follows from [4] that every R-module is a direct sum of extending modules. Hence, $\mathfrak{M}$ is an indigent module. □

We suspect that $\mathfrak{M}$ is indigent over all Noetherian rings but we have not been able to prove it. The aim is to show that if $\mathfrak{M}$ is N-subinjective then N is injective. The next proposition shows that this is indeed the case for $N = R$ when R is assumed to be hereditary and prime.

Proposition 3.6. *If $\mathfrak{M}$ is R-subinjective over a hereditary Noetherian prime ring, then R is semisimple Artinian.*

Proof. Let P be a finitely generated projective R-module. Assume that $\mathfrak{M}$ is P-subinjective. But P is isomorphic to a direct sum of uniform right ideals of R. Then we obtain that P is injective. Therefore, if $\mathfrak{M}$ is R-subinjective, then R is semisimple Artinian. □

We do not know if $\mathfrak{M}$ is indigent over a *QF*-ring in general but we have the following observation.

Proposition 3.7. *Let R be a QF-ring. Assume that every singular module is extending. Then $\mathfrak{M}$ is an indigent module.*

Proof. Let N be a non-injective module. By [11, Theorem 7], N is a direct sum of an injective and singular module. So N has a non-injective singular summand S which is extending by assumption. It follows from Lemma 3.5 that $S \notin \underline{\mathfrak{In}}^{-1}(\mathfrak{M})$. Hence, $N \notin \underline{\mathfrak{In}}^{-1}(\mathfrak{M})$. □

Following Faith [7], we say that a ring R is (right) $\sum$-cyclic if every (right) R-module decomposes into a direct sum of cyclic modules. Note that a right $\sum$-cyclic ring has to be right Artinian by [7, Theorem 20.23].

Theorem 3.8. *$\mathfrak{N}$ is an indigent module over a (right) $\sum$-cyclic ring R.*

Proof. Let $\mathfrak{N}$ be an A-subinjective module. Since R is (right) $\sum$-cyclic, $A = \bigoplus_{i\in I} A_i$, where each A_i is cyclic. Then $\mathfrak{N}$ is A_i-subinjective by Proposition 2.5(1). But it follows from Proposition 2.4(3) that each A_i is A_i-subinjective which implies that A_i is injective. Since R is (right) Noetherian, A is injective, too. □

Corollary 3.9. *$\mathfrak{N}$ is an indigent module over an Artinian serial ring R.*

Proof. Because Artinian serial rings are $\sum$-cyclic (see [6, Theorem 5.6]), the result follows from Theorem 3.8. $\square$

4. Indigent modules of specific types

A second problem to be considered, a variation of the question of existence of indigent modules, is that of the existence of modules of specific types. For which rings do there exist, say, semisimple, singular, or projective modules which are indigent? This section deals with such questions and offers some necessary and some sufficient conditions for the existence or non-existence of various types of indigent modules. Of course, a byproduct of this enquiry is likely to be an increase in the family of rings which are guaranteed to have indigent modules. Such is the case as, for instance, we can prove that the integers do have indigent modules (Corollary 4.5). The existence of semisimple poor modules was fully characterized in [5] and the existence of singular and projective poor modules were studied in [1].

Lemma 4.1. *Let S be a module with $Rad(S) = 0$ and let N be a module such that $Rad(N) = N$. Then S is N-subinjective.*

Proof. $Hom(N, S) = 0$ since N doesn't have a maximal submodule and $Rad(S) = 0$. $\square$

We obtain the following result by the previous lemma:

Proposition 4.2. *If a ring R has an indigent module with a zero Jacobson radical, then every module N with $Rad(N) = N$ is injective.*

By Proposition 4.2, we conclude that if a ring R has a semisimple indigent module, then every module N with $Rad(N) = N$ is injective. But the converse of this fact need not be true.

Example 4.3. If R is a right Noetherian right V-ring which is not semisimple Artinian, then R doesn't have a semisimple indigent module.

Proposition 4.4. *Let R be any ring. Suppose that an R-module N is injective if and only if $Rad(N) = N$. Then R has a semisimple indigent module.*

Proof. Let $S = \bigoplus_{i \in I} S_i$, where I is any complete set of representatives of simple modules. Consider a non-injective module N. By assumption, N has a maximal submodule, and hence N has a simple factor S_i for some $i \in I$. But then S_i cannot be N-subinjective since $Hom(E(N), S_i) = 0$. Hence, S is not N-subinjective. $\square$

Corollary 4.5. *The ring of integers $\mathbb{Z}$ has a semisimple indigent module.*

Proof. By Proposition 4.4, $\bigoplus_{p \text{ is prime}} \mathbb{Z}_p$ is an indigent $\mathbb{Z}$-module. $\square$

Theorem 4.6. *Let R be an Artinian serial ring with $J(R)^2 = 0$. Then R has a semisimple indigent module.*

Proof. Since R is Artinian, $R/J(R)$ is semisimple. Let $R/J(R) = \bigoplus_{i=1}^{n} S_i$, and assume that $R/J(R)$ is K-subinjective. Then, by Proposition 2.4(3), each S_i is K-subinjective. By our assumption, $K = K_1 \oplus K_2$, where K_1 is semisimple and K_2 is injective. Let S be a simple direct summand of K_1. Then there exists $j \in I$ such that $S \cong S_j$. But $M_j \cong S \in \underline{\mathfrak{In}}^{-1}(M_j)$ which means that M_j is injective. Hence, K_1 is injective, and so K is injective, too. $\square$

Example 4.7. The ring $R=\mathbb{Z}/8\mathbb{Z}$ is an Artinian serial ring which has a simple indigent module but $J(R)^2\neq 0$.

Proposition 4.8. *QF serial rings have a singular indigent module.*

Proof. By [11, Theorem 7], every module is a direct sum of a singular and an injective module over *QF*-rings. The result follows from this fact and Proposition 2.4(3). □

The next two results are companion propositions to [1, Theorem 6.2] and [1, Theorem 6.3], respectively.

Proposition 4.9. *Let $R=R_1\oplus R_2$ be a ring decomposition. If M is an indigent R-module, then MR_i is an indigent R_i-module for each $i=1,2$. However, if R_i is not self-injective, then MR_j is not an indigent R-module, where $i\neq j$.*

Proof. Let K be an R_i-module and let MR_i be K-subinjective. By Proposition 2.5(1), M is K-subinjective, too. Since M is indigent, K is an injective R-module. Hence, $KR_i=K$ is an injective R_i-module. Therefore, MR_i is an indigent R_i-module.

Now suppose that R_2 is not self-injective. But for any injective R_1-module K, MR_1 is $K\oplus R_2$-subinjective. Since R_2 is non-injective, MR_1 is not indigent as an R-module. □

Proposition 4.10. *Let R be a ring that decomposes as a direct sum $R_1\oplus S_1$ of rings $R_1\cong R$ and S_1, which is not self-injective. Then an indigent R-module is not Artinian.*

Proof. Let M be an indigent R-module. By Proposition 4.9, $M_1=MR_1$ is an indigent R_1-module, but is not an indigent R-module. Hence, M_1 is a non-zero proper R-submodule of M. As $R_1\cong R$, we have $R_1=R_2\oplus S_2$, where $R_2\cong R_1$ and S_2 is not self-injective. Repeating the same argument, we get that $M_2=M_1R_2$ is an indigent R_2-module and a non-zero proper R-submodule of M_1. Hence, we obtain an infinite chain of R-submodules $\{M_n\}$ of M. Thus, M is not Artinian. □

Recall that a ring is called *SI* if every singular module is injective (see [8]).

Proposition 4.11. *If a ring R has a non-singular indigent right module, then R is a right SI-ring.*

Proof. Let M be a non-singular indigent module and N be a singular module. Since any homomorphism from N to M is zero, M is N-subinjective. Hence, N is injective. □

Corollary 4.12. *Let R be a ring that is not a right SI-ring. Assume that every right R-module is either injective or indigent. Then every non-singular right R-module is semisimple.*

Proof. By Proposition 4.11, every non-singular module is injective, and hence semisimple. □

A module is said to be semiartinian in case every homomorphic image of it has an essential socle. A ring R is called right semiartinian if it is semiartinian as a right R-module.

Proposition 4.13. *If every non-zero cyclic right R-module is indigent, then R is right semiartinian.*

Proof. Assume that R is not right semiartinian. Then there exists a right ideal I of R such that $Soc(R/I)=0$. By hypothesis, R/I is indigent. On the other hand, R/I is S-subinjective for a simple right R-module S. So S is injective. S being both injective and indigent implies that R is semisimple Artinian by Proposition 3.3. But this leads to a contradiction. □

Example 4.14. The converse of Proposition 4.13 need not be true. If we let $R = \mathbb{Z}/8\mathbb{Z}$, then R is not an indigent R-module because R is self-injective; but R is an Artinian ring.

Proposition 4.15. *Assume that every non-zero cyclic right R-module is indigent. Then R is semisimple Artinian or $Z(M) \leqslant_e M$ for any right R-module M (or equivalently $Z(R_R) \leqslant_e R$).*

Proof. Suppose that $Z(M)$ is not an essential submodule of M. Then there exists a non-zero $x \in M$ such that $xR \cap Z(M) = 0$. So xR is non-singular, and by hypothesis, xR is indigent. Then R is a right *SI*-ring by Proposition 4.11. If S is a simple singular right R-module, then it is both injective and indigent. This means that R is semisimple Artinian by Proposition 3.3. $\square$

Proposition 4.16. *Suppose that all simple modules are indigent over a non-semisimple ring R. Then R has only one simple singular module up to isomorphism.*

Proof. Suppose a ring R satisfies the given condition. If there exist non-isomorphic simple modules S_1 and S_2, then we have $Hom_R(S_1, S_2) = 0$. But then S_2 is S_1-subinjective, whence S_1 is injective. Since S_1 is both injective and indigent, R is a semisimple Artinian ring. This contradiction shows that $S_1 \cong S_2$. Since R is not semisimple Artinian, the unique simple module is singular. $\square$

Lemma 4.17. *Let N be an A-injective module for an injective module A. Then submodules of A are contained in the subinjectivity domain of N. Moreover, if N is indigent, then A is semisimple.*

Proof. Let $B \leqslant A$. Since A is injective, $E(B) \leqslant A$. Hence, the A-injectivity of N implies that $B \in \underline{\mathfrak{In}}^{-1}(N)$. $\square$

Corollary 4.18. *No abelian p-group is indigent.*

Two modules are called *orthogonal* if they have no non-zero isomorphic submodules (see [10]).

Theorem 4.19. *Let M be a projective semisimple indigent module. Then any semisimple module B orthogonal to M is injective.*

Proof. First we will show that $Hom(X, M) = 0$ for any submodule X of $E(B)$. Let X be a submodule of $E(B)$ and let $f : X \to M$ be a homomorphism. Then $f(X) \leqslant^{\oplus} M$, and so $f(X)$ is projective. This gives that $Ker(f) \leqslant^{\oplus} X$. Hence, $X = Y \oplus Ker(f)$ and $Y \cong f(X)$. If $f(X \cap B) \neq 0$, then $X \cap B \cong f(X \cap B) \oplus (Ker(f) \cap (X \cap B))$. But this contradicts the hypothesis that B and M are orthogonal. So we have that $f(X \cap B) = 0$ which implies that $f(X) = 0$. Hence, M is $E(B)$-injective. Since M is indigent and $E(B)$ is injective, $E(B)$ is semisimple by Lemma 4.17. Thus, $B = E(B)$. $\square$

The results in the remainder of this section all have counterparts in the theory of poor modules. The poor module versions of these results appear in Section 4 of [1]. The proofs from that reference carry through almost exactly. For that reason, we will list the results here but without proofs.

Corollary 4.20. *Let R be a ring that has a simple projective indigent module. Then R is a GV-ring.*

Corollary 4.21. *Let R be a ring which is not semisimple Artinian. If there exists a simple projective indigent module, then the following hold*:

(1) *Every direct sum of simple injective modules is injective.*
(2) *Every simple R-module is either injective or indigent.*

Recall that a ring R is called *right Kasch* if every simple R-module is embedded in R.

Theorem 4.22. *If R is a right Kasch ring which has a non-zero simple projective indigent module, then R is semisimple Artinian.*

Theorem 4.23. *Let R be a semiperfect ring. If R has a simple projective indigent module, then $R = R_1 \oplus R_2$, as a ring direct sum, where R_1 is semisimple Artinian and R_2 is semiperfect with projective indigent homogeneous socle.*

Corollary 4.24. *If there is a projective simple indigent module M, then $Soc(R)$ is projective. Indeed, the socle of any projective R-module under this hypothesis is projective.*

Theorem 4.25. *A semiprime ring with a finite right uniform dimension and a projective simple indigent module is semisimple Artinian.*

5. Poor and indigent modules

Considering that the notions of poor and indigent modules are defined similarly, one would expect them to coincide frequently. Questions about when poor modules are indigent and conversely are the subject of this last section. We will show examples that indigent modules are not always poor. We do not know at this moment any examples of poor modules which are not indigent.

We show above that $M = \bigoplus_{N \in \Gamma} N$, where Γ is any complete set of representatives of cyclic modules, is an indigent module over an Artinian serial ring. On the other hand, M is a poor module, too (see [5, Proposition 2]). But, as the next theorem shows, indigent modules need not be poor over an Artinian serial ring.

Theorem 5.1. *Let R be an Artinian chain ring. Then non-injective R-modules are indigent.*

Proof. Since R is an Artinian chain ring, every R-module is a direct sum of cyclic uniserial modules. Then it is enough to consider cyclic modules by Propositions 2.4 and 2.5. Because R is an Artinian chain ring, the (right) ideals of R are zero and the powers $J(R)^n$ of $J(R)$. Moreover, if $p \in J(R) \setminus J(R)^2$, then $J(R)^n = p^n R$ for every $n \geqslant 0$ (see [6, p. 115]). But R is Artinian, so lattice of its (right) ideals is finite. Hence, we have the following chain for some positive integer n:

$$R \supset pR \supset p^2 R \supset \cdots \supset p^n R \supset 0.$$

Therefore, it is enough to show that $p^n R$ is indigent for every non-zero n. By [6, Lemma 5.4], $E(p^n R) = R$ for every n.

Take $X = p^n R$, where $n \neq 0$. If $m \geqslant n$, then X is not $p^m R$-subinjective. Otherwise we get $R \subseteq X$ which is a contradiction. Now suppose that $0 \neq m < n$. Consider the homomorphism $f : p^m R \to X$, $f(p^m) = p^n$. If f was extended by a homomorphism $g : R \to X$, then we would have $g(p^m) = f(p^m)$ which means that $g(1)p^m = p^n$. But $g(1) \in X$, so we get $p^n \in p^{(m+n)} R$, a contradiction. Hence, X is not $p^m R$-subinjective for $0 \neq m < n$. Thus, X is subinjective relative to only injective modules. $\square$

Example 5.2. The converse of Theorem 5.1 is not true in general. The ring $R = \left[\begin{smallmatrix} F & F \\ 0 & F \end{smallmatrix}\right]$ is not a chain ring but non-injective R-modules are indigent.

Corollary 5.3. *Let p be a prime and n be a positive integer. A non-injective module over the ring $\mathbb{Z}/p^n\mathbb{Z}$ is indigent.*

Example 5.4. Let $R = \mathbb{Z}/8\mathbb{Z}$. Since $2R$ is a quasi-injective R-module, it is not poor. But by Corollary 5.3, $2R$ is indigent.

Theorem 5.5. *Let R be an Artinian serial ring with $J(R)^2 = 0$. Then indigent modules are poor.*

Proof. Let N be an indigent R-module. Suppose that N is A-injective. By [3, Theorem 7], we have a decomposition $A = A_1 \oplus A_2$, where A_1 is semisimple and A_2 is injective. Then N is A_2-injective. By Lemma 4.17, A_2 is semisimple. Thus, N is a poor module. □

Lemma 5.6. *Let N be a non-singular module and A be a semisimple module such that $A \in \underline{\mathfrak{In}}^{-1}(N)$. If $A \leqslant B \leqslant E(A)$, then N is B-injective. Moreover, if N is poor, then A is injective.*

Proof. Let T be an essential submodule of B and $f : T \to B$ be a homomorphism. Since A is semisimple, it is contained in T. It follows that $f|_A$ can be extended to a homomorphism $g : E(A) \to N$ because $A \in \underline{\mathfrak{In}}^{-1}(N)$. We will show that $g|_T = f$. Let $h = g - f$. Because $h(A) = 0$ we have that $A \subseteq Ker(h)$. Since $A \leqslant_e T$, we get that $T/Ker(h) \cong h(T)$ is singular. But N is non-singular so that $T = Ker(h)$. □

Theorem 5.7. *Let R be an Artinian serial ring with $J(R)^2 = 0$. Then non-singular poor modules are indigent.*

Proof. Let N be a non-singular poor module and let N be A-subinjective. Then we have a decomposition $A = A_1 \oplus A_2$, where A_1 is semisimple and A_2 is injective. By Proposition 2.5(1), N is A_1-subinjective. It follows from Lemma 5.6 that A_1 is injective. □

Corollary 5.8. *Let M be a semisimple projective module over an Artinian serial ring R with $J(R)^2 = 0$. Then M is poor if and only if M is indigent.*

Theorem 5.9. *Let $R = \prod_{i=1}^{n} R_i$, where R_i is a non-semisimple Artinian serial ring with $J(R_i)^2 = 0$ and a unique non-injective simple for each $i = 1, \dots, n$. Then an R-module is poor if and only if it is indigent.*

Proof. The sufficiency follows from Theorem 5.5. For the necessity, let M be a poor R-module. Suppose S_i is the unique non-injective simple of R_i for each $i = 1, \dots, n$. We have $M = \bigoplus_{i=1}^{n} MR_i$. Then $S_i \leqslant^{\oplus} MR_i$ for $i = 1, \dots, n$. For if there exists an i such that S_i is not a direct summand of MR_i, then MR_i is an injective R_i-module. But then M is $R_i \oplus (\bigoplus_{i \neq j} S_j)$-injective, which is a contradiction.

Now suppose that M is N-subinjective, where N is a non-injective module. Then there exists an i such that NR_i has a non-injective simple summand which is isomorphic to S_i. By Proposition 2.8(1), we obtain S_i is S_i-subinjective, which is a contradiction. Therefore, N is injective. Thus, M is indigent. □

Corollary 5.10. *Let R be a non-semisimple QF-ring with homogeneous right socle and $J(R)^2 = 0$. Then poor and indigent modules coincide. In addition, every R-module is either indigent or injective.*

Proof. Since R is non-semisimple and has a homogeneous socle, R is an Artinian serial ring with a unique simple module up to isomorphism. This simple module is non-injective because R is non-semisimple. Hence, the result follows from Proposition 5.9. □

Example 5.11. Let $R = \left[\begin{smallmatrix} F & F \\ 0 & F \end{smallmatrix}\right]$. R is not a *QF*-ring but poor R-modules coincide with indigent R-modules.

Proposition 5.12. *A simple indigent module of an Artinian serial ring is poor.*

Proof. Let R be an Artinian serial ring and let S be a simple indigent R-module. Suppose that S is T-injective. Since R is Artinian serial, without loss of generality, we can assume that T is cyclic uniserial. Since $T/TJ(R)$ is both semisimple and uniserial, it is simple. It follows from [6, Lemma 1.14] that T is a local module. So $Rad(T)$ is a maximal submodule of T. Hence, we get that $Hom(Rad(T), S) = 0$. This obviously gives us that S is $Rad(T)$-subinjective whence $Rad(T)$ is injective. If $Rad(T) \neq 0$, then it is both an essential submodule and a direct summand of T. But this is a contradiction since $Rad(T) \neq T$. Hence, $Rad(T) = 0$, and so we get that T is simple. □

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