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WEAK (C_{11}) MODULES AND ALGEBRAIC TOPOLOGY TYPE EXAMPLES

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1. Introduction. In this note, we provide some counterexamples using the construction technique of trivial extensions for questions below and then investigate whether direct summands of a weak (C_{11})-module are also weak (C_{11}) or not. To this end, affirmative answers are given in special cases. Some results on the endomorphism rings of weak (C_{11})-modules and more examples using algebraic topology to the question [**10**, p. 1821] are also provided.

All rings are associative and have identity elements and all modules are unital right modules. Let R be any ring and M a right R-module. For any submodule K of M the family of submodules N satisfying $K \cap N = 0$ has a maximal member by Zorn's Lemma, which is called *complement of K in M*. A submodule N of M is called a *complement* in M if N is a complement of a submodule of M. It is well known that a submodule is a complement in M if and only if it has no proper essential extensions in M. A module is called a CS-*module*, or *extending*, or it *satisfies* (C_1) provided every complement submodule is a direct summand; equivalently, every submodule is essential in a direct summand of M. Note that semi-simple modules, uniform modules and injective modules are CS. For good sources of references, please see [**3**] or [**6**]. Various generalizations of CS-modules have been studied by some authors see, for example [**4, 8, 10**]. Following Smith [**8**], a module is called a *weak CS-module* if every semi-simple submodule is essential in a direct summand. A module M is called a (C_{11})-*module* if every submodule of M has a complement which is a direct summand of M (see [**10**]). Following [**4**], a module is called a *weak* (C_{11})-*module* if each of its semi-simple submodules has a complement which is a direct summand and denoted (WC_{11}). Note that the following implications hold for a module M.

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$$\begin{array}{ccc} CS & \Longrightarrow & \text{weak}\, CS \\ \Downarrow & & \Downarrow \\ (C_{11}) & \Longrightarrow & (WC_{11}). \end{array}$$

No other implications can be added to this table in general. In particular, [**9**, Example 10] shows that (WC_{11}) does not imply (C_{11}). Recently Zhou [**14**, Example 3] provided an example which makes it clear that there exists a module with (C_{11}) but not weak CS.

A module M is called a CESS-*module* if every complement in M with essential socle is a direct summand of M (see [**8**]). Recall that a CESS-module is a weak CS-module. It is proved in [**8**, Corollary 1.6] that if M is a CESS-module then $M = M_1 \oplus M_2$ for some CS-module M_1 with essential socle and module M_2 with zero socle and asked whether the converse of this result is true or not (see [**8**, Question 1.7]). Among others, Smith's question [**8**, Question 1.7] was answered in the negative by constructing a counterexample in [**14**, Example 1]. Now we ask:

Question 1. Is a direct sum of a module with essential socle and a module with zero socle a (WC_{11})-module?

Question 2. Is a direct sum of a (C_{11})-module with essential socle and a module with zero socle a (C_{11})-module?

Note that these questions are based on the general question, namely, whether being weak (C_{11}), or (C_{11}), is inherited by direct summands or not. In [**11**], the (C_{11}) case of this question has been settled in the negative by providing counterexamples and also investigated in some affirmative cases.

In this paper, we answer the above questions 1 and 2 in the negative by constructing counterexamples and deal with some special cases in which direct summands of a weak (C_{11}) module are also weak (C_{11}). To this end, it is shown that, if $M = M_1 \oplus M_2$ is a weak (C_{11})-module such that $\operatorname{Soc} M_2$ is essential in M_2 and for every direct summand K of M with $K \cap M_2 = 0$, $K \oplus M_2$ is a direct summand of M, then M_1 is a weak (C_{11})-module. In particular, if $M = M_1 \oplus M_2$ is a weak (C_{11})-module such that M_2 is injective with essential socle, then M_1

is a weak (C_{11})-module. Besides, it is obtained that if M is a module satisfying (WC_{11}) and (C_2) with essential socle, then the quotient ring of the endomorphism ring of M over its Jacobson radical is a (von Neumann) regular ring. Further, we give more counterexamples to the question [**10**, p. 1821]. We begin by mentioning a basic result about modules with property (C_{11}) and (WC_{11}).

Lemma 1 (See [**10**, Theorem 2.4] and [**4**, Theorem 2.10]). *Any direct sum of* (C_{11})*-modules* (*respectively,* (WC_{11})*-modules*) *is also a* (C_{11})*-module* (*respectively,* (WC_{11})*-module*).

The following easy proposition shows that the converse of question 1 is true and its proof is given for completeness.

Proposition 2. *Let M be a* (WC_{11})*-module. Then $M = M_1 \oplus M_2$ where M_1 is a submodule of M with essential socle and M_2 a submodule of M with zero socle.*

Proof. Let S denote the socle of M. There exist submodules M_1 and M_2 of M such that $M = M_1 \oplus M_2$, $S \cap M_2 = 0$ and $S \oplus M_2$ is an essential submodule of M. By [**1**, Proposition 9.19], $S = \operatorname{Soc} M = (\operatorname{Soc} M_1) \oplus (\operatorname{Soc} M_2)$. Clearly $\operatorname{Soc} M_2 = 0$ so that $S \leq M_1$. Now $S \oplus M_2$ essential in M implies S essential in M_1, whence the result follows. □

The following example makes it clear that the converse of Proposition 2 is not true in general.

Example 3. Let S be a ring and let V be a $S-S$-bimodule. Assume that S has zero socle and V is semi-simple which is not simple. Let R be the trivial extension of S and the S-module V. Then $R = S \oplus V$ has the following addition and multiplication:

$$(s,a)+(t,b)=(s+t,\, a+b), \quad \text{and} \quad (s,a)(t,b)=(st,\, sb+ta).$$

Let $M_1 = R_R$. Then $\operatorname{Soc} M_1 = 0 \oplus V$ is essential in M_1. Set $I = 0 \oplus V$ and let $M_2 = R/I$. Then $\operatorname{Soc} M_2 = 0$. Now, consider the module $M = M_1 \oplus M_2$. Let N be a simple submodule of M. Then $N = (0 \oplus A) \oplus 0$ for some submodule A of V. Suppose there exists a direct summand L of M such that $N \cap L = 0$ and $N \oplus L$ is essential in M. Now

$$L = \Big\{((s,a),(t,0)+I) : s,t \in R,\ a \in V\Big\}.$$

Then $N \leq L$. But $N \cap L = N = 0$, a contradiction. Hence $L = 0$. However N is not essential in V. It follows that M is not (WC_{11})-module.

Note that if the module V is simple in Example 3 then M is a weak (C_{11})-module by Lemma 1. Now we shall give an example to Question 2. The following example is taken from [**9**, Example 11].

Example 4. An example of Levy [**5**, p. 151, Remark (i)] (see [**9**]) gives a commutative, local ring R with zero socle which is not a (C_{11}) R-module. Let I be the unique maximal ideal of R. Now, let $M_1 = R_R$ and $M_2 = R/I$. Note that $\operatorname{Soc} M_1 = 0$ and $\operatorname{Soc} M_2 = M_2$ which is essential in M_2. Let M be the direct sum $M_1 \oplus M_2$ of R-modules M_1 and M_2. Since M_1 is not a (C_{11})-module and M_2 is simple, then M is not a (C_{11})-module.

Next we deal with when a direct summand of a (WC_{11})-module is a (WC_{11})-module. We first prove an easy result.

Lemma 5. *Let R be a ring and let M be an indecomposable right R-module such that* $\operatorname{Soc} M \neq 0$. *Then M is a* (WC_{11})*-module if and only if M is uniform.*

Proof. The sufficiency is clear. Conversely, suppose that M satisfies (WC_{11}). Thus $\operatorname{Soc} M$ is essential in M. Let $0 \neq X$ be any submodule of M. Then there exists a direct summand L of M such that $\operatorname{Soc} X \cap L = 0$ and $\operatorname{Soc} X \oplus L$ is essential in M. If $L = M$ then $X = 0$, a contradiction. Hence $L = 0$. It follows that X is essential in M. So M is uniform. $\square$

Proposition 6. *Let R be a ring such that the right R-module R is* (WC_{11})*-module and such that every direct summand of a* (WC_{11})*-*

module is a (WC_{11})*-module. Then every indecomposable projective right* R*-module which has a nonzero socle is uniform.*

Proof. Let P be an indecomposable projective right R-module such that $\operatorname{Soc} P \neq 0$. Then there exists a free R-module F such that $F = P \oplus N$ for some submodule N of F. By Lemma 1, F satisfies (WC_{11}) and, by hypothesis, so too does P. Now, by Lemma 5, P is uniform. □

In view of Proposition 6, if R is a right (WC_{11}) R-module such that $\operatorname{Soc} R$ is nonzero and P is any indecomposable projective right R-module of rank $n \geq 2$. Then there exists a free right R-module M which satisfies (WC_{11}) by Lemma 1. Now, P is a direct summand of M and P is not a (WC_{11})-module by Lemma 5. However we do not know so far whether such modules M exist or not.

Lemma 7. *Let a module* $M = M_1 \oplus M_2$ *be a direct sum of submodules* M_1, M_2*. Then the module* M_1 *satisfies* (WC_{11}) *if and only if for every semi-simple submodule* N *of* M_1 *there exists a direct summand* K *of* M *such that* $M_2 \subseteq K$*,* $K \cap N = 0$ *and* $K \oplus N$ *is an essential submodule of* M*.*

Proof. Suppose that M_1 satisfies (WC_{11}). Let N be any semi-simple submodule M_1. There exists a direct summand L of M_1 such that $N \cap L = 0$ and $N \oplus L$ is essential in M_1. Clearly, $L \oplus M_2$ is a direct summand of M, $M_2 \subseteq L \oplus M_2$, $(L \oplus M_2) \cap N = 0$ and $(L \oplus M_2) \oplus N$ is essential in M. Conversely, suppose that M_1 has the stated property. Let H be a semi-simple submodule of M_1. By hypothesis, there exists a direct summand K of M such that $M_2 \subseteq K$, $K \cap H = 0$ and $K \oplus H$ is an essential submodule of M. Now $K = K \cap (M_1 \oplus M_2) = (K \cap M_1) \oplus M_2$, so that $K \cap M_1$ is a direct summand of M, and hence also of M_1, $H \cap (K \cap M_1) = 0$ and $H \oplus (K \cap M_1) = M_1 \cap (H \oplus K)$ which is an essential submodule of M_1. It follows that M_1 is a (WC_{11})-module. □

Theorem 8. *Let a* (WC_{11})*-module* $M = M_1 \oplus M_2$ *be direct sum of submodules* M_1, M_2 *such that,* $\operatorname{Soc} M_2$ *is essential in* M_2 *and for every* *direct summand K of M with $K \cap M_2 = 0$, $K \oplus M_2$ is a direct summand of M. Then M_1 is a (WC_{11})-module.*

Proof. Let N be any semi-simple submodule of M_1. Then $N \oplus \operatorname{Soc} M_2$ is a semi-simple submodule of M. By hypothesis, there exists a direct summand K of M such that $(N \oplus \operatorname{Soc} M_2) \cap K = 0$ and $N \oplus \operatorname{Soc} M_2 \oplus K$ is an essential submodule of M. Since $\operatorname{Soc} M_2$ is essential in M_2 then $N \cap M_2 = 0$ and $N \oplus M_2 \oplus K$ is essential in M. Moreover $M_2 \oplus K$ is a direct summand of M. Now, the result follows by Lemma 7. □

Corollary 9. *Let a (WC_{11})-module $M = M_1 \oplus M_2$ be a direct sum of submodules M_1, M_2 such that,* $\operatorname{Soc} M_2$ *is essential in M_2 and M/M_1 is M_1-injective. Then M_1 is a (WC_{11})-module.*

Proof. By hypothesis, M_2 is M_1-injective. Let L be a direct summand of M such that $L \cap M_2 = 0$. By [**3**, Lemma 7.5] there exists a submodule H of M such that $H \cap M_2 = 0$, $M = H \oplus M_2$ and $L \subseteq H$. Now L is a direct summand of H and hence $L \oplus M_2$ is a direct summand of $M = H \oplus M_2$. By Theorem 8, M_1 is a (WC_{11})-module. □

Corollary 10. *Let a module $M = M_1 \oplus M_2$ be a direct sum of a submodule M_1 and an injective submodule M_2 with essential socle. Then M satisfies (WC_{11}) if and only if M_1 satisfies (WC_{11}).*

Proof. If M satisfies (WC_{11}), then M_1 satisfies (WC_{11}) by Corollary 9. Conversely, if M_1 satisfies (WC_{11}) then M satisfies (WC_{11}) by Lemma 1. □

The next few results concern the endomorphism ring of (WC_{11})-modules. We will use S and $J(S)$ to denote the endomorphism ring of a module M and the Jacobson radical of S, respectively. Further Δ will stand for the ideal $\{\alpha \in S : \ker \alpha \text{ is essential in } M\}$. Recall that a CS-module M is called *continuous* if, for each direct summand N of M and each monomorphism $\varphi : N \longrightarrow M$, the submodule $\varphi(N)$ is also a direct summand of M (see [**6, 3**]). It was proved in [**6**, Proposition 3.5] that if M is continuous, then S/Δ is a (von Neumann) regular ring and $\Delta = J(S)$. This result was generalized to modules with (C_{11})

and (C_2) in [**13**, Theorem 3.3]. Hence, one might conjecture: if M is a (WC_{11})-module with (C_2), then S/Δ is a regular ring and $\Delta = J(S)$. However, the following example eliminates this possibility.

Example 11. Let R be as in Example 4. Let M denote the R-module R. Then M satisfies (WC_{11}) and (C_2). But $J(S) \neq \Delta$.

Proof. First note that R is a commutative local ring. Thus $S/J(S)$ is a (von Neumann) regular ring. Since $\operatorname{Soc} R = 0$, then M satisfies (WC_{11}). By [**9**, Example 11], M also satisfies (C_2). It is straightforward to check that $\Delta \neq J(S)$. □

In contrast to Example 11, we have the following result which was pointed out in the introduction.

Theorem 12. *Let M be a module with essential socle. If M satisfies (WC_{11}) and (C_2), then S/Δ is a regular ring and $\Delta = J(S)$.*

Proof. Let $\alpha \in S$ and let $K = \operatorname{Soc}(ker\alpha)$. By ($WC_{11}$), there exists a direct summand L of M such that L is a complement of K in M. Since $\operatorname{Soc} M$ is essential in M, then $\ker \alpha \cap L = 0$ and hence $\alpha \mid_L$ is a monomorphism. By (C_2), $\alpha(L)$ is a direct summand of M. Hence there exists $\beta \in S$ such that $\beta\alpha = 1 \mid_L$. Then

$$(\alpha - \alpha\beta\alpha)(K \oplus L) = (\alpha - \alpha\beta\alpha)(L) = 0,$$

and so $K \oplus L$ is a submodule of $\ker(\alpha - \alpha\beta\alpha)$. Since $K \oplus L$ is essential in M then $\alpha - \alpha\beta\alpha \in \Delta$. Therefore S/Δ is a regular ring. This also proves that $J(S)$ is contained in Δ. Now, let $f \in \Delta$. Since $\ker f \cap \ker(1-f) = 0$ and $\ker f$ is essential in M, then $\ker(1-f) = 0$. Hence $(1-f)M$ is a direct summand of M by (C_2). However, $(1-f)M$ is an essential submodule of M since $\ker f$ is a submodule of $(1-f)M$. Thus $(1-f)M = M$, and therefore $1-f$ is a unit in S. Hence $f \in J(S)$. It follows that $\Delta = J(S)$. □

Corollary 13. *Let M be a right nonsingular right R-module with essential socle. If M satisfies (WC_{11}) and (C_2), then S is a regular ring.*

Proof. Since M is nonsingular then $\Delta = 0$, by [**7**, Lemma 3.1]. Hence the result follows from Theorem 12. $\square$

Finally we are interested in question [**10**, p. 1821]. It is well known that any direct summand of a CS-module is a CS-module (see [**3**, Lemma 7.1] or [**6**, Proposition 2.7]). In contrast to CS-modules, it was shown that there exists a module M which satisfies (C_{11}) but which has a direct summand which does not satisfy (C_{11}) (see [**11**, Example 4]). We provide more examples in the following. Note first that any indecomposable module satisfying (C_{11}) is uniform.

Proposition 14. *Let F be a field of characteristic zero and n any integer with $n \geq 3$. Let S be the polynomial ring $F[x_1, \dots , x_n]$ in indeterminates $x_1, \dots , x_n$ over F. Let $R = S/Ss$ be the coordinate ring of $(n-1)$-sphere S^{n-1}, where $s = x_1^2 + \cdots + x_n^2 - 1$. If S^{n-1} has nonzero Euler characteristic, then the free R-module $M = \oplus_{i=1}^n R$ satisfies (C_{11}) but M contains a direct summand K which does not satisfy (C_{11}).*

Proof. It is clear that R is a commutative Noetherian domain. The free R-module M satisfies (C_{11}) by Lemma 1. Let $\varphi : M \longrightarrow R$ be the homomorphism defined by $\varphi(a_1 + Ss, \cdots , a_n + Ss) = a_1x_1 + \cdots + a_nx_n + Ss$ for all a_i in S, $1 \leq i \leq n$. Clearly φ is an epimorphism and hence its kernel K is a direct summand of M, i.e., $M = K \oplus K'$ for some submodule $K' \cong R$. Clearly K is not uniform. Note that K is the R-module of regular sections of the tangent bundle of the $(n-1)$-sphere S^{n-1}. Since the Euler characteristic $\chi(S^{n-1}) \neq 0$ it follows that $(n-1)$-sphere cannot have a nonvanishing regular section of its tangent bundle (see [**2**, Corollary VI. 13.3]). Thus K is an indecomposable module. It follows that K does not satisfy (C_{11}). $\square$

Proposition 15. *Let $\mathbf{R}$ be the real field and n any odd integer with $n \geq 3$. Let S be the polynomial ring $\mathbf{R}[x_1, \dots , x_n]$ in indeterminates $x_1, \dots , x_n$ over $\mathbf{R}$. Let R be the ring S/Ss, where $s = x_1^2 + \cdots + x_n^2 - 1$. Let P be the R-module with generators $s_1, \cdots , s_n$ and relation $\sum_{i=1}^n x_is_i = 0$. Then the R-module $P \oplus R$ satisfies (C_{11}) but P does not satisfy (C_{11}).*

Proof. Let $M = P \oplus R$. Then it is clear that M is a free R-module. Note that P is an indecomposable R-module by [**12**, Theorem 3]. Now, by Lemma 1, M is a (C_{11})-module. Since P has uniform dimension $n-1$ then P is not uniform. It follows that P is not a (C_{11})-module. □

The next corollary which is obvious by Proposition 15, or Proposition 14, is Example 4 in [**11**].

Corollary 16. *Let* $\mathbf{R}$ *be the real field and* n *any odd integer with* $n \geq 3$. *Let* S *be the polynomial ring* $\mathbf{R}[x_1, \dots, x_n]$ *in indeterminates* $x_1, \dots, x_n$ *over* $\mathbf{R}$. *Let* R *be the ring* S/Ss, *where* $s = x_1^2 + \cdots + x_n^2 - 1$. *Then the free* R*-module* $M = \oplus_{i=1}^n R$ *satisfies* (C_{11}) *but* M *contains a direct summand* K *which does not satisfy* (C_{11}).

Remarks. (i) If n is 1 or 2 in Proposition 14, or Proposition 15 and Corollary 16, then every direct summand of the module M satisfies (C_{11}) by [**10**, Lemma 4.1].

(ii) If n is any even integer with $n \geq 4$ then the proof of Corollary 16 does not work. For example spheres S^3, S^5, S^7 all have decomposable tangent bundles by the celebrated result of Adams (see [**2**, Corollary VI. 15.16]).

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