PERGAMON

Applied Mathematics Letters 14 (2001) 685–690

Applied Mathematics Letters

www.elsevier.nl/locate/aml

Upper Branch Nonstationary Modes of the Boundary Layer Due to a Rotating Disk

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(Received January 2000; revised and accepted August 2000)

Abstract—A treatment of asymptotic calculation of upper branch nonstationary instability modes is undertaken in the boundary layer flow due to a rotating disk. A numerical spectral solution of the eigenvalue problem shows good agreement with the results of a rational asymptotic approach, based on the extension of the multideck theory of [1].

Keywords—Rotating disk flow, Multi-deck asymptotic theory, Convective instability, Upper branch modes.

1. INTRODUCTION

Rotating-disk flow exhibits a vast diversity of instabilities similar to those of a swept-wing, which have been believed to cause the transition to turbulence. These instability mechanisms may be categorized basically as convective and absolute. Each of these can be provoked by either the inviscid or viscous character of the flow disturbances. The absolute instability feature of the rotating-disk flow has been carried out lately by [2,3]. A direct spatial resonance instability leading to the transient behaviour in the flow characteristics has also been found to occur at a Reynolds number of 445, see [4,5].

Most of the experimental, theoretical, and numerical work has been devoted to the investigation of convective type of instability. Stationary crossflow instability first studied experimentally by [6] (hereafter, referred to as GSW) is an example of convective instability. Hall [1] and Malik [7] investigated theoretically the stability characteristics of the stationary crossflow vortices.

Convective instability is not only induced by the amplification of unstable stationary crossflow disturbances, but it is also due to unstable traveling disturbances. Such instability wave patterns were first detected in the experiment of GSW, and later [8] showed that in a rotating-disk flow, this type of instability occurred much earlier than the stationary one. Bassom [9], applying

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0893-9659/01/$ - see front matter Typeset by $\mathcal{AMS}$-TeX
PII: S0893-9659(01)00029-5

the multideck ideas of [10], studied linear and nonlinear aspects of neutral stability of crossflow vortices.

Recently, experiments on the flow over a rotating disk have been carried out by [11,12] to investigate, in particular, the role of nonstationary disturbances in the route to transition. Jarre [11] found that for the natural transition process over a smooth disk, traveling waves dominate in the early stages of instability. When transition is forced with a roughness element in the experiment of [12], transition was dominated by traveling disturbances with a negative phase speed. Traveling instability mode was also present in the experiment of [13] and they showed that a triad coupling between pairs of traveling eigenmodes and a stationary mode is responsible for the final breakdown to turbulence.

The effect of wall compliance on boundary-layer instability over a rotating disk, recently explored by [14], has made it clear that complete suppression of the absolute instability is possible, removing a major route to transition in the rotating disk. Then there remains the amplification of the perturbations through convective instability. The neutral stability boundaries in this case are of interest and inviscid short wavelength modes are considered in this paper.

The main objective is to study the nonstationary upper branch behaviour of the disturbances. Asymptotic methods based on the triple-deck theory allows the study of nonlinearity, nonparallelism as well as viscous and curvature effects in a natural and rigorous fashion. For this purpose, the asymptotic framework of [1] has been extended to include nonzero frequency waves. An investigation of the second-order eigenvalue problem, allowing viscous and nonstationary effects to appear, reveals the appearance of a phase shift across the critical layer and a wall layer shift. The matching of these shifts generates an eigenrelation which is similar to the one obtained by [1].

This paper is organized as follows. First governing equations and mean flow are given in Section 2. The asymptotic expansion of the neutral stability modes is examined in different regimes of the flow in Section 3. Finally, conclusions follow in Section 4.

2. GOVERNING EQUATIONS AND THE MEAN FLOW

We consider the three dimensional boundary-layer flow of an incompressible fluid on an infinite disk which rotates about its axis with a constant angular velocity Ω. Then, the suitably nondimensionalized Navier-Stokes equations governing the unsteady viscous fluid motion are given by

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}.\nabla)\mathbf{u} + \mathbf{k} \times (\mathbf{k} \times \mathbf{r}) + 2(\mathbf{k} \times \mathbf{u}) = -\nabla p + \frac{1}{\mathrm{Re}} \nabla^2 \mathbf{u}, \qquad \nabla.\mathbf{u} = 0, \tag{1}$$

where $\mathbf{u}$, $\mathbf{k}$, $\mathbf{r}$, and $\mathrm{Re} = R^2$ denote, respectively, velocity vector, normal unit vector, coordinate vector, and Reynolds number.

The mean flow velocities and pressure are given by Von Kármán's exact self-similar solution of (1) for steady flow. The boundary-layer coordinate Z of order O(1) is defined as $Z = zR$, and the self-similar equations take the form

$$(u_B, v_B, w_B, p_B) = \left(rF[Z], rG[Z], \frac{1}{R} H[Z], \frac{1}{R^2} P[Z] \right), \tag{2}$$

where the functions F, G, H, and P satisfy the following ordinary differential equations and boundary conditions:

$$\begin{aligned} &F^2 - (G+1)^2 + F'H - F'' = 0, && 2F(G+1) + G'H - G'' = 0, \\ &P' + H'H - H'' = 0, && 2F + H' = 0, \\ &F = G = H = 0, \text{ at } Z = 0, && F = 0,\ G = -1, \text{ as } Z \to \infty. \end{aligned} \tag{3}$$

3. ASYMPTOTIC ANALYSIS OF NONSTATIONARY UPPER BRANCH MODES

A theoretical approach for the evolution of upper branch stationary modes is presented in [1]. Our intention here is to extend this theory to account for the nonzero frequency waves. Following the work of [1], we define a small parameter $\epsilon = \mathrm{Re}^{-1/6}$. We also assume that disturbances take the form of

$$(U, V, W, P) = (u(z), v(z), w(z), p(z))e^{i/\epsilon^3(\int^r \alpha\, dr + \theta\beta)}e^{-i/\epsilon^2\bar{\omega}t}.$$

As a result of the choice of scalings above, the leading-order problem will be stationary consistent with the analysis of GSW and [9], and the frequency will come into effect in the second-order problem. On the upper branch, we also expand the wave numbers and frequency as

$$(\alpha, \beta, \bar{\omega}) = (\alpha_0, \beta_0, \bar{\omega}_0) + \epsilon(\alpha_1, \beta_1, \bar{\omega}_1) + \cdots.$$

In view of [1], we restrict our attention to neutral disturbances at a local position r. Asymptotic regions and solutions therein are sought in the following.

3.1. Inviscid Region $\zeta = z\epsilon^{-3}(z = \mathrm{O}(\mathrm{Re}^{-1/2}))$

The existence of this inviscid zone of depth $\mathrm{O}(\epsilon^3)$ was shown by GSW. In this region, u, v, w, and p are expanded in the form

$$(u, v, w, p) = (u_0, v_0, w_0, p_0)(\zeta) + \epsilon(u_1, v_1, w_1, p_1)(\zeta) + \cdots.$$

After making substitution of these into equations (1) and equating the terms of order of $\mathrm{O}(\epsilon^{-3})$, the leading-order approximation results in the inviscid Rayleigh equation, see [1]. Defining the effective velocity profile $\bar{U}_B = \alpha_0 rF + \beta_0 G$ and leading-order wave number $\gamma_0^2 = \alpha_0^2 + \beta_0^2/r^2$, the solution is restricted to satisfy $\bar{U}_B$ and $\bar{U}_B''$ to vanish at a nonzero $\zeta = \bar{\zeta}$, so that the singularity is avoided. The eigenvalue problem was solved in [1], and the quantities α_0, β_0, γ_0, and $\bar{\zeta}$ are given therein.

A couple of second-order differential equations results from equating $\mathrm{O}(\epsilon^{-2})$ terms in the inviscid zone, and upon eliminating u_1, v_1, and p_1, we obtain the following nonhomogeneous Rayleigh equation for w_1:

$$\bar{U}_B\left[w_1'' - \gamma_0^2 w_1\right] - \bar{U}_B'' w_1 = 2\bar{U}_B\hat{\alpha}w_0 + \beta_0 r\hat{\epsilon}\left[F'' - F\frac{\bar{U}_B''}{\bar{U}_B}\right]w_0 + \bar{\omega}_0\frac{\bar{U}_B''}{\bar{U}_B}w_0, \tag{4}$$

where $\hat{\alpha} = \alpha_0\alpha_1 + (\beta_0\beta_1/r^2)$ and $\hat{\epsilon} = \alpha_1/\beta_0 - (\alpha_0\beta_1/\beta_0^2)$. Due to the second term on the right-hand side of (4), a Frobenius expansion reveals that a logarithmic singularity appears belonging to w_1 at $\zeta = \bar{\zeta}$. This singularity manifests itself in the form

$$w_1'' \sim \frac{k_c}{\zeta - \bar{\zeta}}w_0(\bar{\zeta}), \qquad \zeta > \bar{\zeta}, \tag{5}$$

where

$$k_c = \left[\beta_0 r\hat{\epsilon}\frac{1}{\bar{U}_B'}\left[F'' - F\frac{\bar{U}_B'''}{\bar{U}_B'}\right] + \omega_0\frac{\bar{U}_B'''}{\bar{U}_B'^2}\right]_{\zeta=\bar{\zeta}}.$$

Such a singularity can be removed by introducing a critical layer at $\zeta = \bar{\zeta}$.

By means of the consideration of the viscous critical layer theory, when the path of the integration is deformed into the complex plane near $\zeta = \bar{\zeta}$ (i.e., continuation below the critical layer), together with (5), the well-known linear phase jump in the inviscid zone is obtained:

$$w_1'\Big|_{\bar{\zeta}^-}^{\bar{\zeta}^+} = i\,\mathrm{sign}\left(\bar{U}_B'(\bar{\zeta})\right)\pi k_c w_0(\bar{\zeta}). \tag{6}$$

3.2. Viscous Sublayer $\xi = z\epsilon^{-4}(z = \mathrm{O}(\mathrm{Re}^{-2/3}))$

It is readily obtained from the leading-order equations in the inviscid zone that

$$\zeta \to 0, \qquad i\left(\alpha_0 u_0 + \frac{\beta_0}{r}\, v_0\right) \to -w_0'(0).$$

Therefore, to satisfy the zero velocity requirement at the wall, a viscous sublayer is required. The thickness of this layer is found to be $\mathrm{O}(\epsilon^4)$, by balancing the convection and diffusion terms in (1). Hall [1] presents the expansion of basic velocity profiles. The various quantities also expand as follows:

$$(u, v, w, p) = (\tilde{u}_0, \tilde{v}_0, 0, 0)(\xi) + \epsilon\,(\tilde{u}_1, \tilde{v}_1, \tilde{w}_0, \tilde{p}_0)(\xi) + \epsilon^2\,(\tilde{u}_2, \tilde{v}_2, \tilde{w}_1, \tilde{p}_1)(\xi) + \cdots .$$

Substituting these into the Navier-Stokes equations (1), and after some manipulations as in [1], for large ξ we find the wall layer shift

$$\tilde{w}_0 \sim w_0'(0)\left[\xi + \frac{Ai'(\xi_0)}{\lambda_0 \kappa}\right],$$

with $\xi_0 = -i\,(\bar{\omega}_0/\lambda_0^2)$, $\lambda_0^3 = i[\alpha_0 r F_0 + \beta_0 G_0]$, and $\kappa = \int_{\xi_0}^{\infty} Ai(\tau)\, d\tau$. On matching with the inviscid zone described earlier, we find that w_1 must satisfy

$$w_1 \to w_0'(0)\,\frac{Ai'(\xi_0)}{\lambda_0 \kappa}, \qquad \zeta \to 0. \tag{7}$$

A solvability condition is a prerequisite for further assessment, which can be rigorously constructed by multiplying (4) with w_0 and integrating from wall to infinity, additionally making use of equations (4), (6), and (7), we obtain the eigenrelation

$$2\hat{\alpha} I_1 + r\hat{\epsilon}[I_2 + a] + \bar{\omega}_0[I_3 + b] = w_0'(0)^2\,\frac{Ai'(\xi_0)}{\lambda_0 \kappa}, \tag{8}$$

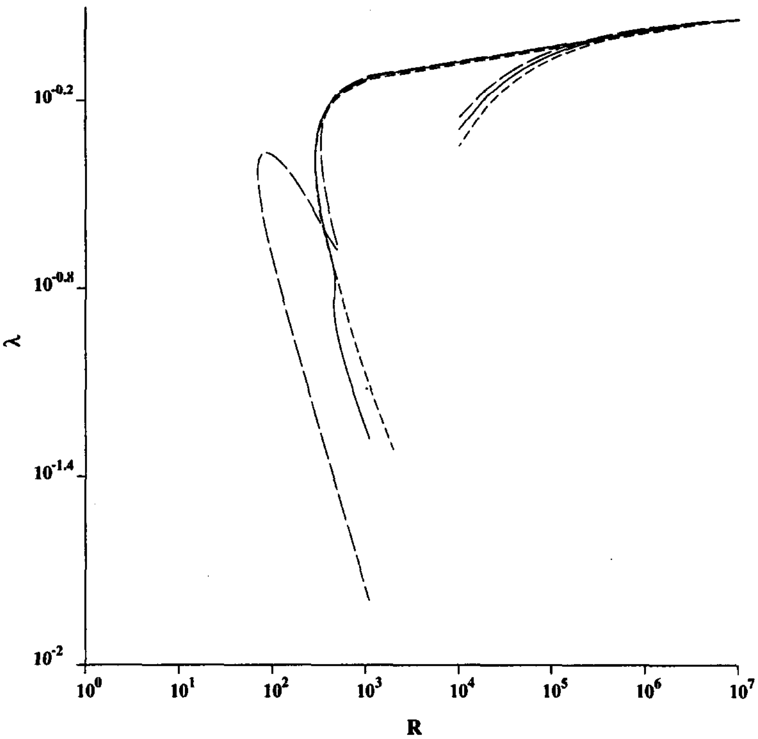
where the quantities I_1, I_2, I_3, a, and b can be found in [4].

Finally, making use of the Reynolds number R based on the boundary-layer thickness and the local azimuthal velocity of the disk, $R = r\,\mathrm{Re}^{1/2}$, the effective wave number $(\alpha^2 + \beta^2/r^2)^{1/2}$ and the wave angle ε in powers of R are given by

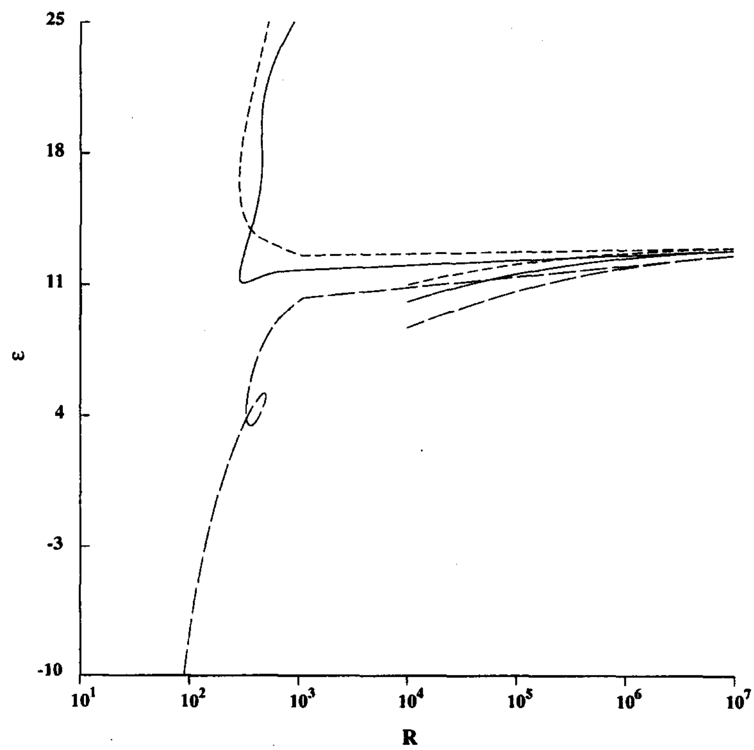
$$\begin{aligned} \gamma = \left(\alpha^2 + \frac{\beta^2}{r^2}\right)^{1/2} &= \gamma_0 + \frac{\hat{\alpha}}{\gamma_0}\, R^{-1/3} + \cdots, \\ \left(\tan^{-1}(\varepsilon)\right) &= \frac{\alpha_0 r}{\beta_0} + \hat{\epsilon} R^{-1/3} + \cdots . \end{aligned} \tag{9}$$

Note that when $\bar{\omega}_0 = 0$, the explicit expressions for the effective wave number and the wave angle outlined in [1] are recovered from the eigenrelation (8). The extra term I_3 comes in due to the consideration of nonzero frequency waves. In this case, equation (8) should be solved numerically to determine the wave number correction $\hat{\alpha}$ and the wave angle correction $\hat{\epsilon}$ required in (9).

Based on these asymptotic findings, comparisons with the numerical calculations are shown in Figure 1 for $\omega = -5$, 0, and 10 ($\bar{\omega} = \omega/R$). It is seen that there is satisfactory agreement between the asymptotic and the numerical results. Moreover, it is also possible to consider $\bar{\omega}_0$ as complex to further investigate temporally growing waves. Furthermore, using the asymptotic relation of the Airy function as $|\bar{\omega}_0| \to \infty$, the right-hand side of (8) can be replaced by $iw_0'(0)^2\bar{\omega}_0/\lambda_0^3$, which then gives explicit expressions for the corrections $\hat{\alpha}$ and $\hat{\epsilon}$.



(a)



(b)

Figure 1. A comparison of the numerical and asymptotic calculations of the stationary — and nonstationary - - ($\omega = -5$) and – – ($\omega = 10$) waves, in (R, λ) and (R, ε) planes. The long curves show numerical results and the shorter lines asymptotic ones.

The physical importance of the nonstationary disturbance waves, some of which are demonstrated in Figure 1, is that amplification of such single waves or wave packets is believed to be one of the main sources causing transition to turbulence in the rotating-disk flow, as also pointed out in the earlier experiment of GSW as well as in the recent experiments of [11–13]. This, in turn, will shed light on the role of the traveling disturbances in the route to transition to turbulence in several fluid dynamics flows of, in particular, engineering and aeronautical interest, such as the flow over aircraft wings.

4. CONCLUSIONS

The upper branch neutral stability of three-dimensional disturbances imposed on Von Kármán's boundary-layer profile has been investigated asymptotically, in particular, for the nonstationary crossflow disturbances. The multideck theory of [1] has been extended to include the nonzero frequency waves for the upper branch instability modes, and an eigenrelation has been obtained which involves the correction terms for the wave number and wave angle. The wave number and frequencies calculated from this eigenrelation have been found to compare well with the numerical results. It has been found that at very large Reynolds numbers, the upper branch for all waves tends asymptotically to a finite value.

As a further work, the asymptotic work of [1] could also be extended to study the nonzero frequency waves developing along the lower branch of the neutral curve, and this is currently under consideration.

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