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The prime dicompletion of a di-uniformity on a plain texture

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ABSTRACT

Working within a plain texture $(S,\mathcal{S})$, the authors construct a completion of a dicovering uniformity υ on $(S,\mathcal{S})$ in terms of prime $\mathcal{S}$-filters. In case υ is separated, a separated completion is then obtained using the T_0-quotient, and it is shown that this construction produces a reflector. For a totally bounded di-uniformity it is verified that these constructions lead to dicompactifications of the uniform ditopology. A condition is given under which complementation is preserved on passing to these completions, and an example on the real texture $(\mathbb{R},\mathcal{R},\rho)$ is presented.

1. Introduction

Di-uniformities on a texture were introduced in [10], and the effect of a complementation on the base texture, and the relation with classical quasi-uniformities and uniformities were discussed in [11]. Relations with a textural analog of quasi-proximities appear in [13], while in [12] notions of completeness and total boundedness are given for di-uniformities on a texture. Our aim in this paper is to construct a dicompletion of a di-uniformity.

There are considerable difficulties involved in constructing a dicompletion for a di-uniformity on a general texture, and this case is currently open. Here we confine our attention to di-uniformities on a plain texture. A plain texture is one for which the texturing is closed under arbitrary unions. This is a proper, but still quite extensive subclass of the class of textures, and includes the complemented discrete textures $(X,\mathcal{P}(X),\pi_X)$, which are the basis of a representation of classical quasi-uniformities and uniformities [11], and such important complemented textures as the unit interval texture $(\mathbb{I},\mathcal{I},\iota)$ [10], and the real texture $(\mathbb{R},\mathcal{R},\rho)$ defined by $\mathcal{R}=\{(-\infty,r) \mid r\in\mathbb{R}\}\cup\{(-\infty,r] \mid r\in\mathbb{R}\}\cup\{\emptyset,\mathbb{R}\}$, and $\rho(-\infty,r)=(-\infty,-r]$, $\rho(-\infty,r]=(-\infty,-r)$, $r\in\mathbb{R}$, $\rho(\mathbb{R})=\emptyset$, $\rho(\emptyset)=\mathbb{R}$.

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There is a considerable simplification introduced in considering plain textures. For example, in terms of the p-sets and q-sets we have $P_s \not\subseteq Q_s$ for each $s \in S$. Hence, for $A \in \mathcal{S}$, $s \in A$, $P_s \subseteq A$ and $A \not\subseteq Q_s$ are equivalent to one another. Another important consideration is that a difunction (f, F) between plain textures $(S, \mathcal{S})$ and $(T, \mathcal{T})$ may be represented by a point function $\varphi : S \to T$ for which

(a) $P_u \subseteq P_s \Longrightarrow P_{\varphi(u)} \subseteq P_{\varphi(s)}$

and $f^{\leftarrow}B = F^{\leftarrow}B = \varphi^{-1}[B]$ for all $B \in \mathcal{T}$. Moreover, every point function $\varphi : S \to T$ satisfying (a) gives rise to a difunction $(f_\varphi, F_\varphi) : (S, \mathcal{S}) \to (T, \mathcal{T})$ through the equalities $f_\varphi = \bigcup\{\overline{P}_{(s,\varphi(s))} \mid s \in S\}$, $F_\varphi = \bigcap\{\overline{Q}_{(s,\varphi(s))} \mid s \in S\}$. See the discussion in [6]. Difunctions are not, however, completely redundant even in this context, for although φ is surjective if and only if (f_φ, F_φ) is surjective, the injectivity of φ need not imply that of (f_φ, F_φ), so the injectivity of (f_φ, F_φ) carries more information about the textures. It is for this reason that the injectivity of the embedding difunction for the completions described below has been established in each case.

Textures first arose in connection with the representation of Hutton algebras and lattices of $\mathbb{L}$-fuzzy sets in a point-based setting [4,5], and have subsequently proved to be a fruitful setting for the investigation of complement-free concepts in mathematics. They also have potential importance in providing economic computational models of important mathematical spaces, and emphasis has been placed on the development of concepts such as direlation, difunction, dicover and difilter, which enable the expression of powerful results within a very minimal structure. Thus, for example, the statement that every open, coclosed dicover of the unit interval texture under its usual ditopology has a finite, cofinite sub-dicover may be seen to be equivalent to the compactness of $\mathbb{I}$ under its usual topology, even though the texturing $\mathcal{I}$ and the usual ditopology involve only the sets $[0, r)$, $[0, r]$, $r \in \mathbb{I}$.

The reader may refer to [3–8] for background material and motivation on textures and ditopological texture spaces. In particular, [6] contains most of the basic material we will need on textures, and [7] that on ditopological texture spaces, while [8] discusses separation. Constant reference will be made to [10] for definitions and results relating to di-uniformities, none which will be repeated here. On the other hand, for the benefit of the reader we will briefly recall the necessary definitions and results relating to completeness and complementation from [12,11], respectively, as they are needed in the text. Finally terms from lattice theory not defined here are as given in [9], while we will follow [1] for terms from category theory. In particular, if **A** is a category, Ob **A** will denote the class of objects and Mor **A** the class of morphisms of **A**. We will sometimes use the notation $\mathbf{A}(A_1, A_2)$ for the set of **A** morphisms from A_1 to A_2.

2. Prime $\mathcal{S}$-filters and the prime dicompletion

Let $(S, \mathcal{S})$ be a texture. Then a difilter [12] on $(S, \mathcal{S})$ is a product $\mathcal{F} \times \mathcal{G}$ of an $\mathcal{S}$-filter $\mathcal{F}$ and an $\mathcal{S}$-cofilter $\mathcal{G}$. The difilter $\mathcal{F} \times \mathcal{G}$ is called regular if $\mathcal{F} \cap \mathcal{G} = \emptyset$, which is equivalent to requiring $A \not\subseteq B$ for all $A \in \mathcal{F}$, $B \in \mathcal{G}$.

By Zorn's lemma every regular difilter is contained in a maximal regular difilter. It is shown in [12] that $\mathcal{F} \times \mathcal{G}$ is a maximal regular difilter if and only if $\mathcal{F} \cup \mathcal{G} = \mathcal{S}$, and that $\mathcal{F}$ and $\mathcal{G}$ are then prime, where the $\mathcal{S}$-filter $\mathcal{F}$ is prime if $A, B \in \mathcal{S}$, $A \cup B \in \mathcal{F} \Longrightarrow A \in \mathcal{F}$ or $B \in \mathcal{F}$, while $\mathcal{G}$ is prime if $A \cap B \in \mathcal{G} \Longrightarrow A \in \mathcal{G}$ or $B \in \mathcal{G}$. It follows that the mapping

$$\mathcal{F} \mapsto \mathcal{F} \times (\mathcal{S} \setminus \mathcal{F})$$

is a one-to-one onto correspondence between the prime $\mathcal{S}$ filters on S and the maximal regular difilters on $(S, \mathcal{S})$.

Example 2.1. For $s \in S$ let $\mathcal{P}_s = \{A \in \mathcal{S} \mid P_s \subseteq A\}$, $\mathcal{Q}_s = \{B \in \mathcal{S} \mid P_s \not\subseteq B\}$. Clearly $\mathcal{P}_s \times \mathcal{Q}_s$ is a difilter, and $\mathcal{P}_s \cap \mathcal{Q}_s = \emptyset$, $\mathcal{P}_s \cup \mathcal{Q}_s = \mathcal{S}$ so it is maximal regular. In particular $\mathcal{P}_s$, $\mathcal{Q}_s$ are prime.

Now let υ be a dicovering uniformity on $(S, \mathcal{S})$. A difilter $\mathcal{F} \times \mathcal{G}$ on $(S, \mathcal{S})$ is called υ-Cauchy [12] if $\mathcal{C} \cap (\mathcal{F} \times \mathcal{G}) \neq \emptyset$ for all $\mathcal{C} \in \upsilon$. If $\mathcal{F}$ is a prime $\mathcal{S}$-filter then by extension we will refer to $\mathcal{F}$ as *Cauchy* if the corresponding maximal regular difilter $\mathcal{F} \times (\mathcal{S} \setminus \mathcal{F})$ is υ-Cauchy.

For the remainder of this paper we will let $(S, \mathcal{S})$ be a plain texture, and υ a dicovering uniformity on $(S, \mathcal{S})$ with uniform ditopology $(\tau_\upsilon, \kappa_\upsilon) = (\tau, \kappa)$. Set

$$\widetilde{S} = \{\mathcal{F} \mid \mathcal{F} \text{ is a Cauchy prime } \mathcal{S}\text{-filter on } (S, \mathcal{S})\}.$$

Since $(S, \mathcal{S})$ is plain, the family $\eta(s)$ of τ-neighborhoods of $s \in S$ is given by $\eta(s) = \{N \in \mathcal{S} \mid \exists G \in \tau \text{ with } P_s \subseteq G \subseteq N\}$, and so is an $\mathcal{S}$-filter satisfying $\eta(s) \subseteq \mathcal{P}_s$. Likewise the set $\mu(s) = \{M \in \mathcal{S} \mid \exists K \in \kappa \text{ with } M \subseteq K \subseteq Q_s\}$ of κ-coneighborhoods of s is an $\mathcal{S}$-cofilter satisfying $\mu(s) \subseteq \mathcal{Q}_s$. It follows by [12, Proposition 2.8] that the difilter $(\mathcal{P}_s, \mathcal{Q}_s)$ is diconvergent, whence it is υ-Cauchy by [12, Proposition 3.2]. This verifies that $\mathcal{P}_s \in \widetilde{S}$, and so $\epsilon(s) = \mathcal{P}_s$ defines a mapping $\epsilon : S \to \widetilde{S}$.

We wish to define a plain texturing of $\widetilde{S}$. To this end, for $A \in \mathcal{S}$ let $\widetilde{A} = \{\mathcal{F} \in \widetilde{S} \mid A \in \mathcal{F}\}$. Note that, since $S \in \mathcal{F}$ for all $\mathcal{S}$-filters $\mathcal{F}$, this notation is consistent for $A = S$.

Lemma 2.2. *Take $A, B \in \mathcal{S}$. Then*:

(1) $A \subseteq B \Longleftrightarrow \widetilde{A} \subseteq \widetilde{B}$.
(2) $\widetilde{A \cap B} = \widetilde{A} \cap \widetilde{B}$ *and* $\widetilde{A \cup B} = \widetilde{A} \cup \widetilde{B}$.

Proof. (1) For $A \subseteq B$ we clearly have $\widetilde{A} \subseteq \widetilde{B}$. On the other hand, if $A \not\subseteq B$ we have $s \in A \setminus B$, whence $\mathcal{P}_s \in \widetilde{A} \setminus \widetilde{B}$ and we deduce $\widetilde{A} \not\subseteq \widetilde{B}$.

(2) By (1) we certainly have $\widetilde{A \cap B} \subseteq \widetilde{A} \cap \widetilde{B}$. On the other hand $\mathcal{F} \in \widetilde{A} \cap \widetilde{B} \Longrightarrow A, B \in \mathcal{F} \Longrightarrow A \cap B \in \mathcal{F}$ since $\mathcal{F}$ is an $\mathcal{S}$-filter, so $\mathcal{F} \in \widetilde{A \cap B}$, as required. The proof of the second equality is similar, using the fact that $\mathcal{F}$ is prime. □

Now let $\widetilde{\mathcal{S}}$ be the set of arbitrary unions of arbitrary intersections of sets of the form $\widetilde{A}$, $A \in \mathcal{S}$. We have:

Lemma 2.3. *Let* $(S, \mathcal{S})$ *be a plain texture and* $(\widetilde{S}, \widetilde{\mathcal{S}})$ *as defined above. Then*:

(1) $(\widetilde{S}, \widetilde{\mathcal{S}})$ *is a plain texture.*
(2) *The p-sets and q-sets for* $(\widetilde{S}, \widetilde{\mathcal{S}})$ *are given by*

$$P_{\mathcal{F}} = \{\mathcal{H} \in \widetilde{S} \mid \mathcal{F} \subseteq \mathcal{H}\}, \qquad Q_{\mathcal{F}} = \{\mathcal{K} \in \widetilde{S} \mid \mathcal{K} \not\subseteq \mathcal{F}\}$$

for $\mathcal{F} \in \widetilde{S}$.
(3) *The equalities*

$$e = \bigcup\{\overline{P}_{(s,\epsilon(s))} \mid s \in S\}, \qquad E = \bigcap\{\overline{Q}_{(s,\epsilon(s))} \mid s \in S\}$$

define an injective difunction $(e, E) : (S, \mathcal{S}) \to (\widetilde{S}, \widetilde{\mathcal{S}})$.
(4) *The mapping* $\epsilon : S \to \widetilde{S}$ *is a textural isomorphism between* $(S, \mathcal{S})$ *and the restriction* $(\epsilon(S), \widetilde{\mathcal{S}}|_{\epsilon(S)})$ *of* $(\widetilde{S}, \widetilde{\mathcal{S}})$ *to* $\epsilon(S)$.

Proof. (1) By definition $\widetilde{\mathcal{S}}$ is closed under arbitrary unions, while by the complete distributivity of $(\mathcal{P}(\widetilde{S}), \subseteq)$ it is also closed under arbitrary intersections. In particular $\widetilde{\mathcal{S}}$ is a complete lattice which is completely distributive since $(\mathcal{P}(\widetilde{S}), \subseteq)$ is. It remains to show that $\widetilde{\mathcal{S}}$ separates the points of $\widetilde{S}$. Take $\mathcal{F}_1, \mathcal{F}_2 \in \widetilde{S}$ with $\mathcal{F}_1 \neq \mathcal{F}_2$. Then $\mathcal{F}_1 \not\subseteq \mathcal{F}_2$ or $\mathcal{F}_2 \not\subseteq \mathcal{F}_1$. Suppose the former and take $A \in \mathcal{F}_1$ with $A \notin \mathcal{F}_2$. Then $\widetilde{A} \in \widetilde{\mathcal{S}}$, $\mathcal{F}_1 \in \widetilde{A}$ and $\mathcal{F}_2 \notin \widetilde{A}$. Likewise the second case leads to $\widetilde{B} \in \widetilde{\mathcal{S}}$ with $\mathcal{F}_2 \in \widetilde{B}$, $\mathcal{F}_1 \notin \widetilde{B}$. This completes the proof that $(\widetilde{S}, \widetilde{\mathcal{S}})$ is a plain texture.

(2) To prove the formula for $P_{\mathcal{F}}$ first take $\mathcal{H} \in P_{\mathcal{F}}$. Then $A \in \mathcal{F} \Longrightarrow \mathcal{F} \in \widetilde{A} \Longrightarrow \mathcal{H} \in P_{\mathcal{F}} \subseteq \widetilde{A} \Longrightarrow A \in \mathcal{H}$. Hence $\mathcal{F} \subseteq \mathcal{H}$. On the other hand, for $\mathcal{H} \in \widetilde{S}$ suppose $\mathcal{F} \subseteq \mathcal{H}$. Then for any $A \in \mathcal{S}$ we have $\mathcal{F} \in \widetilde{A} \Longrightarrow \mathcal{H} \in \widetilde{A}$, so by the way the elements of $\widetilde{\mathcal{S}}$ are formed we see that $\mathcal{F} \in C \Longrightarrow \mathcal{H} \in C$ for all $C \in \widetilde{\mathcal{S}}$. But $\mathcal{F} \in P_{\mathcal{F}} \in \widetilde{\mathcal{S}}$, so $\mathcal{H} \in P_{\mathcal{F}}$ as required.

The equality for $Q_{\mathcal{F}}$ follows from the definition $Q_{\mathcal{F}} = \bigcup\{P_{\mathcal{H}} \mid \mathcal{F} \notin P_{\mathcal{H}}\}$ and the formula for $P_{\mathcal{H}}$.

(3) In order to show that (e, E) is a difunction we need only verify condition (a) mentioned in the introduction. Hence take $s, s' \in S$ with $P_s \not\subseteq Q_{s'}$, that is $P_{s'} \subseteq P_s$. We must verify $P_{\epsilon(s)} \not\subseteq Q_{\epsilon(s')}$, that is $P_{\mathcal{P}_{s'}} \subseteq P_{\mathcal{P}_s}$. However, by the above, $\mathcal{H} \in P_{\mathcal{P}_{s'}} \Longrightarrow \mathcal{P}_{s'} \subseteq \mathcal{H} \Longrightarrow \mathcal{P}_s \subseteq \mathcal{H} \Longrightarrow \mathcal{H} \in P_{\mathcal{P}_s}$ since $\mathcal{P}_s \subseteq \mathcal{P}_{s'}$, which gives the required result.

To show that (e, E) is injective take $s, s' \in S$ and $\mathcal{T} \in \widetilde{S}$ with $e \not\subseteq \overline{Q}_{(s,\mathcal{T})}$, $\overline{P}_{(s',\mathcal{T})} \not\subseteq E$. By the definition of (e, E) this leads to $P_{\epsilon(s)} \not\subseteq Q_{\mathcal{T}}$ and $P_{\mathcal{T}} \not\subseteq Q_{\epsilon(s')}$, so $P_{\epsilon(s)} \not\subseteq Q_{\epsilon(s')}$ and hence $P_{\epsilon(s')} \subseteq P_{\epsilon(s)}$. We must establish $P_s \not\subseteq Q_{s'}$, which is equivalent to $P_{s'} \subseteq P_s$ as $(S, \mathcal{S})$ is plain. Take $z \in P_{s'}$. Then $\mathcal{P}_{s'} \subseteq \mathcal{P}_z$ since $A \in \mathcal{P}_{s'} \Longrightarrow z \in P_{s'} \subseteq A \Longrightarrow A \in \mathcal{P}_z$. By (2) this now gives $\mathcal{P}_z \in P_{\mathcal{P}_{s'}} = P_{\epsilon(s')} \subseteq P_{\epsilon(s)} = P_{\mathcal{P}_s}$, so $\mathcal{P}_s \subseteq \mathcal{P}_z$ again by (2). Since $s \in P_s \in \mathcal{S}$ we now have $P_s \in \mathcal{P}_z$, so $z \in P_s$ and we have proved $P_{s'} \subseteq P_s$. Hence (e, E) is injective.

(4) Since $(\widetilde{S}, \widetilde{\mathcal{S}})$ is plain it is clear that $(\epsilon(S), \widetilde{\mathcal{S}}|_{\epsilon(S)})$ is a plain texture. Since $\epsilon : S \to \epsilon(S)$ is onto, the corresponding difunction $(f_\epsilon, F_\epsilon) : (S, \mathcal{S}) \to (\epsilon(S), \widetilde{\mathcal{S}}|_{\epsilon(S)})$ is surjective by [2, Lemma 2.7]. It is also injective. This may be proved directly, or easily deduced from the injectivity of (e, E). By [6, Proposition 3.14(5)] we see (f_ϵ, F_ϵ) is an isomorphism in the category **dfPTex**. But **dfTex** is isomorphic to **fPTex** by [6, Theorem 3.10], whence ϵ is an isomorphism in **fPTex**. This shows ϵ is a textural isomorphism between $(S, \mathcal{S})$ and $(\epsilon(S), \widetilde{\mathcal{S}}|_{\epsilon(S)})$ by [6, Proposition 3.15]. □

Lemma 2.4. *For* $\mathcal{C} \in \upsilon$ *define* $\widetilde{\mathcal{C}} = \{(\widetilde{A}, \widetilde{B}) \mid A\, \mathcal{C}\, B\}$. *Then*:

(1) $\widetilde{\mathcal{C}}$ *is a dicover of* $(\widetilde{S}, \widetilde{\mathcal{S}})$.
(2) *For* $\mathcal{D}, \mathcal{C} \in \upsilon$ *we have* $\mathcal{D} \prec_{(\star)} \mathcal{C} \Longrightarrow \widetilde{\mathcal{D}} \prec_{(\star)} \widetilde{\mathcal{C}}$.
(3) *For* $\mathcal{E}, \mathcal{D}, \mathcal{C} \in \upsilon$, $\mathcal{E}$ *anchored, we have* $\mathcal{E} \prec_{(\Delta)} \mathcal{D} \prec_{(\star)} \mathcal{C} \Longrightarrow \widetilde{\mathcal{E}} \prec \widetilde{\mathcal{D}}^{\Delta} \prec \widetilde{\mathcal{C}}$.

Proof. (1) Set $\mathcal{C} = \{(A_j, B_j) \mid j \in J\}$, and let J_1, J_2 be a partition of J. Suppose that $\bigcap_{j \in J_1} \widetilde{B}_j \not\subseteq \bigcup_{j \in J_2} \widetilde{A}_j$ and take $\mathcal{F} \in \bigcap_{j \in J_1} \widetilde{B}_j$, $\mathcal{F} \notin \bigcup_{j \in J_2} \widetilde{A}_j$. By definition $\mathcal{F} \times (\mathcal{S} \setminus \mathcal{F})$ is a Cauchy difilter and so $\mathcal{C} \cap (\mathcal{F} \times (\mathcal{S} \setminus \mathcal{F})) \neq \emptyset$. Hence there exists

$j \in J$ with $A_j \in \mathcal{F}$, $B_j \notin \mathcal{F}$, that is $\mathcal{F} \in \widetilde{A}_j$ and $\mathcal{F} \notin \widetilde{B}_j$. But either $j \in J_1$, which contradicts $\mathcal{F} \in \bigcap_{j \in J_1} \widetilde{B}_j$, or $j \in J_2$ which contradicts $\mathcal{F} \notin \bigcup_{j \in J_2} \widetilde{A}_j$. Hence $\bigcap_{j \in J_1} \widetilde{B}_j \subseteq \bigcup_{j \in J_2} \widetilde{A}_j$, which establishes that $\widetilde{\mathcal{C}}$ is a dicover.

(2) If $\mathrm{St}(C, \mathcal{D}) \subseteq A$ then $\mathrm{St}(\widetilde{C}, \widetilde{\mathcal{D}}) = \bigcup\{\widetilde{C'} \mid \exists D',\ C' \mathcal{D} D',\ \text{with } \widetilde{C} \not\subseteq \widetilde{D'}\} \subseteq \widetilde{A}$ since $\widetilde{C} \not\subseteq \widetilde{D'} \iff C \not\subseteq D'$ by Lemma 2.2, whence $C' \subseteq A$ and so $\widetilde{C'} \subseteq \widetilde{A}$, again by Lemma 2.2. In just the same way $B \subseteq \mathrm{CSt}(D, \mathcal{D}) \Longrightarrow \widetilde{B} \subseteq \mathrm{CSt}(\widetilde{B}, \widetilde{\mathcal{D}})$, whence (2) follows at once.

(3) Take $E \mathcal{E} F$. Since $\mathcal{E}$ is anchored, [10, Definition 2.1(2)] clearly implies $\mathcal{E} \prec \mathcal{E}^{\Delta}$, so there exists $s \in S$ with $E \subseteq \mathrm{St}(P_s, \mathcal{E})$, $\mathrm{CSt}(Q_s, \mathcal{E}) \subseteq F$. On the other hand, $\mathcal{E} \prec_{(\Delta)} \mathcal{D}$ gives $C \mathcal{D} D$ with $\mathrm{St}(P_s, \mathcal{E}) \subseteq C$ and $D \subseteq \mathrm{CSt}(Q_s, \mathcal{E})$. But $P_s \subseteq \mathrm{St}(P_s, \mathcal{E})$, $\mathrm{CSt}(Q_s, \mathcal{E}) \subseteq Q_s$, while $P_s \not\subseteq Q_s$ as $(S, \mathcal{S})$ is plain, so $s \in C \setminus D$. From $s \notin D$ we obtain $D \notin \mathcal{P}_s$, and hence $P_{\mathcal{P}_s} \not\subseteq \widetilde{D}$. Thus $\widetilde{C} \subseteq \mathrm{St}(P_{\mathcal{P}_s}, \widetilde{\mathcal{D}})$, and since $E \subseteq C$ we obtain $\widetilde{E} \subseteq \mathrm{St}(P_{\mathcal{P}_s}, \widetilde{\mathcal{D}})$. Likewise, $s \in C$ leads to $\mathrm{CSt}(Q_{\mathcal{P}_s}, \widetilde{\mathcal{D}}) \subseteq \widetilde{F}$, and we have verified that $\widetilde{\mathcal{E}} \prec \widetilde{\mathcal{D}}^{\Delta}$.

Finally, let $\mathcal{F} \in \widetilde{S}$ and choose $(C, D) \in \mathcal{D} \cap (\mathcal{F} \times (\mathcal{S} \setminus \mathcal{F}))$, $A \mathcal{C} B$ with $\mathrm{St}(C, \mathcal{D}) \subseteq A$ and $B \subseteq \mathrm{CSt}(D, \mathcal{D})$. For $C' \mathcal{D} D'$, $P_{\mathcal{F}} \not\subseteq \widetilde{D'} \Longrightarrow D' \in \mathcal{S} \setminus \mathcal{F} \Longrightarrow C \not\subseteq D' \Longrightarrow C' \subseteq \mathrm{St}(C, \mathcal{D}) \subseteq A$, and so $\mathrm{St}(P_{\mathcal{F}}, \widetilde{\mathcal{D}}) \subseteq \widetilde{A}$. Likewise $\widetilde{B} \subseteq \mathrm{CSt}(Q_{\mathcal{F}}, \widetilde{\mathcal{D}})$, and we have established $\widetilde{\mathcal{D}}^{\Delta} \prec \widetilde{\mathcal{C}}$. □

Proposition 2.5. *Let υ be a dicovering uniformity on $(S, \mathcal{S})$ and $\beta = \{\widetilde{\mathcal{C}} \mid \mathcal{C} \in \upsilon\}$. Then β is a base for a dicovering uniformity $\widetilde{\upsilon}$ on $(\widetilde{S}, \widetilde{\mathcal{S}})$.*

Proof. We verify (1), (3) and (4) of [10, Lemma 3.5] for β.

(1) Take $\mathcal{C} \in \upsilon$. Then we may choose $\mathcal{D} \in \upsilon$ with $\mathcal{D} \prec_{(\star)} \mathcal{C}$, and then $\mathcal{E}$ anchored with $\mathcal{E} \prec_{(\star)} \mathcal{D}$. By [10, Lemma 2.2(2)] we have $\mathcal{E} \prec_{(\Delta)} \mathcal{D}$, so by Lemma 2.4(3) we have $\widetilde{\mathcal{E}} \prec \widetilde{\mathcal{D}}^{\Delta} \prec \widetilde{\mathcal{C}}$. Since $\widetilde{\mathcal{D}}^{\Delta}$ is an anchored dicover by [10, Corollary 2.8(2)] we see that $\widetilde{\upsilon}$ has a base of anchored dicovers of $(\widetilde{S}, \widetilde{\mathcal{S}})$.

(3) For $\mathcal{C}, \mathcal{D} \in \upsilon$ we have $\mathcal{C} \wedge \mathcal{D} \in \upsilon$, and it is trivial to verify that $\widetilde{\mathcal{C} \wedge \mathcal{D}} = \widetilde{\mathcal{C}} \wedge \widetilde{\mathcal{D}}$, so $\widetilde{\mathcal{C}} \wedge \widetilde{\mathcal{D}} \in \beta$.

(4) For $\mathcal{C} \in \upsilon$ there exists $\mathcal{D} \in \upsilon$ with $\mathcal{D} \prec_{(\star)} \mathcal{C}$, and $\widetilde{\mathcal{D}} \prec_{(\star)} \widetilde{\mathcal{C}}$ by Lemma 2.4(2). □

Theorem 2.6. *With the dicovering uniformity $\widetilde{\upsilon}$ defined as above*:

(1) $(\widetilde{S}, \widetilde{\mathcal{S}}, \widetilde{\upsilon})$ *is dicomplete.*
(2) $(e, E) : (S, \mathcal{S}, \upsilon) \to (\widetilde{S}, \widetilde{\mathcal{S}}, \widetilde{\upsilon})$ *is uniformly bicontinuous.*
(3) $\widetilde{\upsilon}|_{\epsilon(S)} = \epsilon(\upsilon)$.
(4) $\epsilon(S)$ *is dense in $\widetilde{S}$ under the uniform ditopology of $\widetilde{\upsilon}$.*

Proof. (1) Let $\Phi \times \Gamma$ be a regular $\widetilde{\upsilon}$-Cauchy difilter on $(\widetilde{S}, \widetilde{\mathcal{S}})$, $\mathcal{F} = \{A \in \mathcal{S} \mid \widetilde{A} \in \Phi\}$ and $\mathcal{G} = \{B \in \mathcal{S} \mid \widetilde{B} \in \Gamma\}$. In view of Lemma 2.2 it is clear that $\mathcal{F} \times \mathcal{G}$ is a regular difilter on $(S, \mathcal{S})$. To show $\mathcal{F} \times \mathcal{G}$ is υ-Cauchy take $\mathcal{C} \in \upsilon$. Then $\widetilde{\mathcal{C}} \in \widetilde{\upsilon}$, so there exists $A \mathcal{C} B$ with $(\widetilde{A}, \widetilde{B}) \in \widetilde{\mathcal{C}} \cap (\Phi \times \Gamma)$. Hence $(A, B) \in \mathcal{C} \cap (\mathcal{F} \times \mathcal{G}) \neq \emptyset$, which verifies that $\mathcal{F} \times \mathcal{G}$ is Cauchy. By [12, Proposition 2.17] there exists a maximal regular difilter $\mathcal{H} \times \mathcal{K}$ on $(S, \mathcal{S})$ with $\mathcal{F} \times \mathcal{G} \subseteq \mathcal{H} \times \mathcal{K}$, and clearly $\mathcal{H} \times \mathcal{K}$ is Cauchy also. Hence $\mathcal{K} = \mathcal{S} \setminus \mathcal{H}$ and $\mathcal{H} \in \widetilde{S}$. We verify that $\Phi \to \mathcal{H}$. Take $G \in \tau_{\widetilde{\upsilon}}$ with $P_{\mathcal{H}} \subseteq G$. By [10, Definition 4.5] we have $\mathcal{C} \in \upsilon$ with $\mathrm{St}(\widetilde{\mathcal{C}}, P_{\mathcal{H}}) \subseteq G$, and there exists $A \mathcal{C} B$ with $(A, B) \in \mathcal{F} \times \mathcal{G}$ since $\mathcal{F} \times \mathcal{G}$ is υ-Cauchy. Clearly

$$B \in \mathcal{G} \subseteq \mathcal{K} \implies B \notin \mathcal{H} \implies \mathcal{H} \notin \widetilde{B} \implies P_{\mathcal{H}} \not\subseteq \widetilde{B} \implies \widetilde{A} \subseteq \mathrm{St}(\widetilde{\mathcal{C}}, P_{\mathcal{H}}) \subseteq G.$$

On the other hand $A \in \mathcal{F}$ gives $\widetilde{A} \in \Phi$, so $G \in \Phi$. Hence $\eta^*(\mathcal{H}) = \eta(\mathcal{H}) \subseteq \Phi$, so $\Phi \to \mathcal{H}$. Dually $\mu^*(\mathcal{H}) = \mu(\mathcal{H}) \subseteq \Gamma$, so $\Gamma \to \mathcal{H}$ and since $P_{\mathcal{H}} \not\subseteq Q_{\mathcal{H}}$, we deduce that $\Phi \times \Gamma$ is diconvergent. Hence $\widetilde{\upsilon}$ is dicomplete.

(2) Clear because for $\mathcal{C} \in \upsilon$ we have $\mathcal{C} = \epsilon^{-1}\widetilde{\mathcal{C}} = (e, E)^{-1}\widetilde{\mathcal{C}}$.

(3) Straightforward by Lemma 2.3(4).

(4) We must show that for $H \in \tau_{\widetilde{\upsilon}}$, $K \in \kappa_{\widetilde{\upsilon}}$ with $H \not\subseteq K$ there exists $s \in S$ with $H \not\subseteq Q_{\epsilon(s)}$ and $P_{\epsilon(s)} \not\subseteq K$. Now there exists $\mathcal{F} \in \widetilde{S}$ with $H \not\subseteq Q_{\mathcal{F}}$, $P_{\mathcal{F}} \not\subseteq K$, and so we have $\mathcal{C} \in \upsilon$ with $\mathrm{St}(\widetilde{\mathcal{C}}, P_{\mathcal{F}}) \subseteq H$ and $K \subseteq \mathrm{CSt}(\widetilde{\mathcal{C}}, Q_{\mathcal{F}})$. Since $\mathcal{F}$ is Cauchy we have $A \mathcal{C} B$ with $A \in \mathcal{F}$, $B \notin \mathcal{F}$. Hence $A \not\subseteq B$, whence we have $s \in S$ with $A \not\subseteq Q_s$ and $P_s \not\subseteq B$. This leads easily to $\epsilon(s) = \mathcal{P}_s \in \widetilde{A} \subseteq \mathrm{St}(\widetilde{\mathcal{C}}, P_{\mathcal{F}}) \subseteq H$, and likewise $\epsilon(s) = \mathcal{P}_s \notin K$. Hence $H \not\subseteq Q_{\epsilon(s)}$, $P_{\epsilon(s)} \not\subseteq K$ as $(\widetilde{S}, \widetilde{\mathcal{S}})$ is plain. □

Definition 2.7. $(\widetilde{S}, \widetilde{\mathcal{S}}, \widetilde{\upsilon})$ is called the *prime dicompletion* of the di-uniform plain texture space $(S, \mathcal{S}, \upsilon)$.

The following two lemmas will be useful in what follows.

Lemma 2.8. *Let $(S, \mathcal{S})$, $(T, \mathcal{T})$ be plain, $\varphi : S \to T$ a point function satisfying*

(a) $P_s \subseteq P_u \Longrightarrow P_{\varphi(s)} \subseteq P_{\varphi(u)}$.

If υ is a dicovering uniformity on $(S, \mathcal{S})$ and ν a dicovering uniformity on $(T, \mathcal{T})$, then the following are equivalent:

(1) *The difunction (f_φ, F_φ) is υ–ν uniformly bicontinuous.*
(2) $\mathcal{D} \in \nu \Longrightarrow \varphi^{-1}\mathcal{D} = \{(\varphi^{-1}[C], \varphi^{-1}[D]) \mid C \,\mathcal{D}\, D\} \in \upsilon$.
(3) $\mathcal{D} \in \nu \Longrightarrow (f_\varphi, F_\varphi)^{-1}(\mathcal{D}) \in \upsilon$.

Proof. (2) $\Longleftrightarrow$ (3) Clear since $\varphi^{-1}[B] = f_\varphi^{\leftarrow} B = F^{\leftarrow} B$ for all $B \in \mathcal{T}$.

(1) $\Longrightarrow$ (2) Suppose that (f_φ, F_φ) is uniformly bicontinuous and take $\mathcal{D} \in \nu$. Now we have $\mathcal{E} \in \nu$ with $\mathcal{E}$ anchored so that $\mathcal{E} \prec_{(\star)} \mathcal{D}$, whence $\mathcal{E} \prec_{(\Delta)} \mathcal{D}$ by [10, Lemma 2.2(2)]. We show that $(\varphi^{-1}\mathcal{E})^\Delta \prec \varphi^{-1}\mathcal{D}$. To this end take $s \in S$ and $C \,\mathcal{D}\, D$ with $\mathrm{St}(\mathcal{E}, P_{\varphi(s)}) \subseteq C$ and $D \subseteq \mathrm{CSt}(\mathcal{E}, Q_{\varphi(s)})$. Now for $E \,\mathcal{E}\, F$ with $P_s \not\subseteq \varphi^{-1}[F]$ we have $P_{\varphi(s)} \not\subseteq F$ by the condition on φ, whence $E \subseteq \mathrm{St}(\mathcal{E}, P_{\varphi(s)}) \subseteq C$ and so $\varphi^{-1}[E] \subseteq \varphi^{-1}[C]$. This gives $\mathrm{St}(\varphi^{-1}\mathcal{E}, P_s) \subseteq \varphi^{-1}[C]$, and dually $\varphi^{-1}[D] \subseteq \mathrm{CSt}(\varphi^{-1}\mathcal{E}, Q_s)$, so $(\varphi^{-1}\mathcal{E})^\Delta \prec \varphi^{-1}\mathcal{D}$ as stated. Since (f_φ, F_φ) is uniformly bicontinuous we have $(\varphi^{-1}\mathcal{E})^\Delta = ((f_\varphi, F_\varphi)^{-1}\mathcal{E})^\Delta \in \upsilon$ by [10, Definition 5.21], whence $\varphi^{-1}\mathcal{D} \in \upsilon$.

(3) $\Longrightarrow$ (1) Take $\mathcal{D} \in \nu$. Then $(f_\varphi, F_\varphi)^{-1}\mathcal{D} \in \upsilon$ by hypothesis, so we have $\mathcal{C} \in \upsilon$ with $\mathcal{C}$ anchored and $\mathcal{C} \prec (f_\varphi, F_\varphi)^{-1}\mathcal{D}$. By [10, Definition 2.1(2)] we have $\mathcal{C} \prec \mathcal{C}^\Delta$, whence $\mathcal{C} \prec (f_\varphi, F_\varphi)^{-1}\mathcal{D}^\Delta$. Hence $(f_\varphi, F_\varphi)^{-1}\mathcal{D}^\Delta \in \upsilon$ and so (f_φ, F_φ) is uniformly bicontinuous. □

Lemma 2.9. *Let $(S, \mathcal{S})$ be plain. Then in the definition of dicovering uniformity the condition that υ has a base of anchored dicovers may be replaced by the condition that it has a base of excluding dicovers $\mathcal{C}$ satisfying $\mathcal{P} \prec \mathcal{C}$.*

Proof. We recall [12, Definition 3.9(1)] that $\mathcal{C}$ is excluding if $A \,\mathcal{C}\, B \Longrightarrow A \not\subseteq B$. It is easy to see that if $\mathcal{C}$ is excluding and $\mathcal{P} \prec \mathcal{C}$ then $\mathcal{P} \prec \mathcal{C} \prec \mathcal{C}^\Delta$, whence $\mathcal{C}$ is anchored. Conversely, given $\mathcal{C} \in \upsilon$ we may choose $\mathcal{D}, \mathcal{E} \in \upsilon$, $\mathcal{E}$ anchored, with $\mathcal{E} \prec_{(\star)} \mathcal{D} \prec_{(\star)} \mathcal{C}$. Now $\mathcal{E} \prec \mathcal{D}^\Delta \prec \mathcal{C}$ and $P_s \subseteq \mathrm{St}(\mathcal{D}, P_s)$, $\mathrm{CSt}(\mathcal{D}, Q_s) \subseteq Q_s$ and $P_s \not\subseteq Q_s$ shows that $\mathcal{D}^\Delta$ is excluding and satisfies $\mathcal{P} \prec \mathcal{D}^\Delta$. □

We will refer to dicovers satisfying the two conditions of Lemma 2.9 as being *strongly anchored*. As is clear from the proof of Lemma 2.9, a strongly anchored dicover is anchored.

Now let $(S, \mathcal{S}, \upsilon)$, $(T, \mathcal{T}, \nu)$ be dicovering uniform spaces and $\varphi : S \to T$ a point function satisfying condition (a), and the equivalent conditions of Lemma 2.8. We wish to extend φ to a function from $\widetilde{S}$ to $\widetilde{T}$. For $\mathcal{F} \in \widetilde{S}$ consider the following definition:

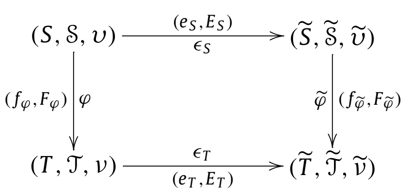
$$\widetilde{\varphi}(\mathcal{F}) = \{B \in \mathcal{T} \mid \varphi^{-1}[B] \in \mathcal{F}\}.$$

It is easy to verify that $\widetilde{\varphi}(\mathcal{F})$ is a prime $\mathcal{T}$-filter, and we omit the details. In order to show $\widetilde{\varphi}(\mathcal{F}) \in \widetilde{T}$, it remains to show it is ν-Cauchy. Take $\mathcal{D} \in \nu$. Then by hypothesis $\varphi^{-1}\mathcal{D} \in \upsilon$ and so $\varphi^{-1}\mathcal{D} \cap (\mathcal{F} \times (\mathcal{S} \setminus \mathcal{F})) \neq \emptyset$ since $\mathcal{F}$ is υ-Cauchy. Hence we have $B_1 \,\mathcal{D}\, B_2$ with $\varphi^{-1}[B_1] \in \mathcal{F}$ and $\varphi^{-1}[B_2] \notin \mathcal{F}$, whence $(B_1, B_2) \in \mathcal{D} \cap (\widetilde{\varphi}(\mathcal{F}) \times (\mathcal{T} \setminus \widetilde{\varphi}(\mathcal{F}))) \neq \emptyset$, which verifies that $\widetilde{\varphi}(\mathcal{F})$ is ν-Cauchy.

In this way we obtain the point function $\widetilde{\varphi} : \widetilde{S} \to \widetilde{T}$.

Proposition 2.10. *With the notation as above*:

(1) *The diagram below is commutative, that is $\epsilon_T \circ \varphi = \widetilde{\varphi} \circ \epsilon_S$.*
(2) *The point function $\widetilde{\varphi} : \widetilde{S} \to \widetilde{T}$ satisfies condition (a), and for the corresponding difunction $(f_{\widetilde{\varphi}}, F_{\widetilde{\varphi}}) : (\widetilde{S}, \widetilde{\mathcal{S}}) \to (\widetilde{T}, \widetilde{\mathcal{T}})$ we have $(e_T, E_T) \circ (f_\varphi, F_\varphi) = (f_{\widetilde{\varphi}}, F_{\widetilde{\varphi}}) \circ (e_S, E_S)$.*
(3) *$\widetilde{\varphi}$ and $(f_{\widetilde{\varphi}}, F_{\widetilde{\varphi}})$ are $\widetilde{\upsilon}$–$\widetilde{\nu}$ uniformly bicontinuous.*



Proof. (1) Immediate from the definitions.

(2) For $\mathcal{F}, \mathcal{G} \in \widetilde{S}$ let $P_\mathcal{F} \subseteq P_\mathcal{G}$. Then $\mathcal{F} \in P_\mathcal{G}$, so by Lemma 2.3(2) we have $\mathcal{G} \subseteq \mathcal{F}$. From the definition of $\widetilde{\varphi}$ we deduce $\widetilde{\varphi}(\mathcal{G}) \subseteq \widetilde{\varphi}(\mathcal{F})$, whence $\widetilde{\varphi}(\mathcal{F}) \in P_{\widetilde{\varphi}(\mathcal{G})}$ and so $P_{\widetilde{\varphi}(\mathcal{F})} \subseteq P_{\widetilde{\varphi}(\mathcal{G})}$. This verifies (a), and it is known that the composition of the difunctions corresponds to that of the respective point functions [6, Theorem 3.10].

(3) We need only verify the condition of Lemma 2.8(2). If $B \in \mathcal{T}$ then

$$\mathcal{F} \in \widetilde{\varphi}^{-1}[\widetilde{B}] \iff \widetilde{\varphi}(\mathcal{F}) \in \widetilde{B} \iff B \in \widetilde{\varphi}(\mathcal{F}) \iff \varphi^{-1}[B] \in \mathcal{F} \iff \mathcal{F} \in \widetilde{\varphi^{-1}[B]},$$

so $\widetilde{\varphi}^{-1}[\widetilde{B}] = \widetilde{\varphi^{-1}[B]}$. Hence for $\mathcal{D} \in \nu$, that is $\widetilde{\mathcal{D}} \in \widetilde{\nu}$, we obtain $\widetilde{\varphi}^{-1}(\widetilde{\mathcal{D}}) = \widetilde{\varphi^{-1}(\mathcal{D})} \in \widetilde{\upsilon}$ since $\varphi^{-1}(\mathcal{D}) \in \upsilon$ by hypothesis. □

It is easy to see that if φ is the identity on S then $\widetilde{\varphi}$ is the identity on $\widetilde{S}$, while for $\varphi : S \to T$, $\psi : T \to U$ satisfying (a) we have $\widetilde{\psi} \circ \widetilde{\varphi} = \widetilde{\psi \circ \varphi}$. This verifies that the operation of forming the prime dicompletion is functorial. Specifically, if we denote by **PDiUni** the category whose objects are plain di-uniform texture spaces and whose morphisms are uniformly bicontinuous point functions between the base sets satisfying (a), or the corresponding difunctions, as the case may be, and by $\overline{\textbf{PDiUni}}$ the subcategory of dicomplete spaces, then we have shown:

Theorem 2.11. *$\sim$: **PDiUni** $\to$ $\overline{\textbf{PDiUni}}$ defined by $\sim((S, \mathcal{S}, \upsilon)) = (\widetilde{S}, \widetilde{\mathcal{S}}, \widetilde{\upsilon})$, $\sim(\varphi) = \widetilde{\varphi}$, or $\sim(f_\varphi, F_\varphi) = (f_{\widetilde{\varphi}}, F_{\widetilde{\varphi}})$, is a functor.*

Now we consider the case where the di-uniformity υ is totally bounded [12, Definition 3.5].

Theorem 2.12. *Let the dicovering uniformity υ on $(S, \mathcal{S})$ be totally bounded. Then the dicovering uniformity $\widetilde{\upsilon}$ on $(\widetilde{S}, \widetilde{\mathcal{S}})$ is also totally bounded, and so the uniform ditopology $(\widetilde{\tau}, \widetilde{\kappa})$ is dicompact.*

Proof. Since $(S, \mathcal{S})$ is plain, by Lemma 2.9 we see that υ has a base of excluding dicovers. Hence, by [12, Proposition 3.11], it has a base of finite, cofinite dicovers. If $\mathcal{C} \in \upsilon$ is finite and cofinite then so is $\widetilde{\mathcal{C}}$, so $\widetilde{\upsilon}$ has a base of finite, cofinite dicovers. A dicover which is refined by a finite, cofinite dicover certainly has a sub-dicover which is finite and cofinite, so $\widetilde{\upsilon}$ is totally bounded. It is also dicomplete by Theorem 2.6, so the uniform ditopology is dicompact by [12, Theorem 3.8]. □

Corollary 2.13. *The functor $\sim$: **PDiUni** $\to$ $\overline{\textbf{PDiUni}}$ of Theorem 2.11 specializes to a functor from the category **PCBiReg** of plain completely biregular ditopological texture spaces and bicontinuous morphisms to the category **PDiComp** of plain dicompact ditopological texture spaces and bicontinuous morphisms.*

3. Separated di-uniformities

In this section we consider the dicompletion of separated di-uniformities on a plain texture. The definition of a separated di-uniformity is given in terms of direlational uniformities in [10], but a necessary and sufficient condition for a direlational uniformity to be separated is that the uniform ditopology be T_0 [10, Theorem 4.16], and this condition is equally applicable to dicovering uniformities.

As we will see in Example 4.8, the prime dicompletion of a separated di-uniformity need not be separated. However we may obtain a T_0 quotient of the uniform ditopology on the prime dicompletion $(\widetilde{S}, \widetilde{\mathcal{S}}, \widetilde{\upsilon})$ of $(S, \mathcal{S}, \upsilon)$, as in [2, Theorem 4.2]. We denote this T_0 quotient by $(\overline{S}, \overline{\mathcal{S}}, \overline{\tau}, \overline{\kappa})$, and seek to define a compatible dicovering uniformity on this space. The quotient $(\overline{S}, \overline{\mathcal{S}})$ is taken modulo the equivalence direlation (r, R) given by

$$r = \bigcup \{\overline{P}_{(\mathcal{F},\mathcal{G})} \mid \mathcal{G} \in X \text{ or } \mathcal{F} \notin X,\ \forall X \in \tau_{\widetilde{\upsilon}} \cup \kappa_{\widetilde{\upsilon}}\},$$

$$R = \bigcap \{\overline{Q}_{(\mathcal{F},\mathcal{G})} \mid \mathcal{F} \in X \text{ or } \mathcal{G} \notin X,\ \forall X \in \tau_{\widetilde{\upsilon}} \cup \kappa_{\widetilde{\upsilon}}\},$$

where we have used the fact that $(\widetilde{S}, \widetilde{\mathcal{S}})$ is plain to simplify these expressions. The elements $\overline{\mathcal{F}} = \varphi(\mathcal{F})$, $\mathcal{F} \in \widetilde{S}$, of $\overline{S}$ are now the equivalences classes for the equivalence point relation ρ on $\widetilde{S}$ given by $\mathcal{F} \rho \mathcal{G} \iff P_{\mathcal{F}} \subseteq r^{\to} P_{\mathcal{G}}$ and $P_{\mathcal{G}} \subseteq r^{\to} P_{\mathcal{F}}$, and

$$\overline{\mathcal{S}} = \{Y \subseteq \overline{S} \mid \varphi^{-1}[Y] \in \mathcal{R}\} = \{\varphi[X] \mid X \in \mathcal{R}\},$$

where $\mathcal{R} = \{X \in \widetilde{\mathcal{S}} \mid r^{\to} X = X\}$. The sets of $\mathcal{R}$ are known to be saturated with respect to ρ, and in the present case $\mathcal{R}$ consists of arbitrary intersections of arbitrary unions of sets in $\tau_{\widetilde{\upsilon}} \cup \kappa_{\widetilde{\upsilon}}$.

According to [2, Lemma 2.5] we have $P_{\overline{\mathcal{F}}} = \varphi[r^{\to} P_{\widetilde{\mathcal{F}}}]$, and bearing in mind that $(\widetilde{S}, \widetilde{\mathcal{S}})$ is plain, $Q_{\overline{\mathcal{F}}} = \varphi[R^{\to} Q_{\widetilde{\mathcal{F}}}]$. A simple calculation now shows that $P_{\overline{\mathcal{F}}} \not\subseteq Q_{\overline{\mathcal{F}}}$ for all $\mathcal{F} \in \widetilde{S}$, whence the texture $(\overline{S}, \overline{\mathcal{S}})$ is also plain.

Proposition 3.1. *Let υ be separated and denote by $\overline{\upsilon}$ the set of dicovers $\mathcal{C}$ of $(\overline{S}, \overline{\mathcal{S}})$ satisfying $\varphi^{-1}\mathcal{C} \in \widetilde{\upsilon}$. Then $\overline{\upsilon}$ is a dicovering uniformity compatible with the T_0 quotient ditopology $(\overline{\tau}, \overline{\kappa})$, and is therefore separated.*

Proof. Condition (2) of [10, Lemma 3.5] is immediate, and (3) follows easily from the fact that φ is onto. For (1) and (4) take $\mathcal{C} \in \overline{\upsilon}$. Then $\varphi^{-1}\mathcal{C} \in \widetilde{\upsilon}$ so we may take $\mathcal{E} \in \widetilde{\upsilon}$ with $\mathcal{E} \prec_{(\star)} \varphi^{-1}\mathcal{C}$. By [10, Proposition 4.8] we may take $\mathcal{E}$ to be open and coclosed, while by Lemma 2.9 we may assume $\mathcal{E}$ is excluding and that $\widetilde{\mathcal{P}} = \{(P_{\mathcal{P}_s}, Q_{\mathcal{P}_s}) \mid s \in S\} \prec \mathcal{E}$. Let $\mathcal{D} = \{(\varphi[E], \varphi[F]) \mid E\, \mathcal{E}\, F\}$. Since for $E\, \mathcal{E}\, F$ we have $E, F \in \tau_{\widetilde{\upsilon}} \cup \kappa_{\widetilde{\upsilon}} \subseteq \mathcal{R}$, these sets are saturated, whence it is easy to check that $\mathcal{D}$ is a dual cover of $(\overline{S}, \overline{\mathcal{S}})$ for which $\mathcal{E} = \varphi^{-1}\mathcal{D}$. Thus $\mathcal{D} \in \overline{\upsilon}$, and as we easily have $\mathcal{D} \prec_{(\star)} \mathcal{C}$ this establishes (4). On the other hand $E \not\subseteq F \implies \varphi[E] \not\subseteq \varphi[F]$ by saturation, so $\mathcal{D}$ is excluding. Finally, for $\mathcal{F} \in \widetilde{S}$ we have $E\, \mathcal{E}\, F$ with $P_{\mathcal{F}} \subseteq E$, $F \subseteq Q_{\mathcal{F}}$. Now $r^{\to} P_{\mathcal{F}} \subseteq r^{\to} E = E$ since $E \in \mathcal{R}$, so $P_{\overline{\mathcal{F}}} = \varphi[r^{\to} P_{\mathcal{F}}] \subseteq \varphi[E]$. Dually, $\varphi[F] \subseteq Q_{\overline{\mathcal{F}}}$ and we see that $\overline{\mathcal{P}} = \{(P_{\overline{\mathcal{F}}}, Q_{\overline{\mathcal{F}}}) \mid \mathcal{F} \in \widetilde{S}\} \prec \mathcal{D}$. In particular $\mathcal{D}$ is anchored (see Lemma 2.9), and by [10, Lemma 2.2(1)] we have $\mathcal{D} \prec \mathcal{C}$, so (1) is satisfied too.

Now take $P_{\overline{\mathcal{F}}} \subseteq G \in \overline{\tau}$. Then by [2, Definition 3.1] we have $P_{\mathcal{F}} \subseteq \varphi^{-1}[G] \in \tau_{\widetilde{\upsilon}}$ and so we have $\mathcal{E} \in \widetilde{\upsilon}$ with $\mathrm{St}(\mathcal{E}, P_{\mathcal{F}}) \subseteq \varphi^{-1}[G]$. Without loss of generality we may take $\mathcal{E}$ to be open and coclosed, and defining $\mathcal{D}$ as above gives us $\mathcal{D} \in \overline{\upsilon}$ satisfying $\mathrm{St}(\mathcal{D}, P_{\overline{\mathcal{F}}}) \subseteq G$. Hence $G \in \tau_{\overline{\upsilon}}$.

Conversely, take $G \in \tau_{\overline{\upsilon}}$ and $P_{\mathcal{F}} \subseteq \varphi^{-1}[G]$. Then $P_{\overline{\mathcal{F}}} \subseteq G$ and we have $\mathcal{C} \in \overline{\upsilon}$ satisfying $\mathrm{St}(\mathcal{C}, P_{\overline{\mathcal{F}}}) \subseteq G$. It follows easily that $\mathrm{St}(\varphi^{-1}\mathcal{C}, P_{\mathcal{F}}) \subseteq \varphi^{-1}[G]$, so $\varphi^{-1}[G] \in \tau_{\widetilde{\upsilon}}$ and hence $G \in \overline{\tau}$. This establishes $\tau_{\overline{\upsilon}} = \overline{\tau}$, and dually it may be shown that $\kappa_{\overline{\upsilon}} = \overline{\kappa}$. In particular, $\overline{\upsilon}$ is separated by [10, Theorem 4.16]. □

Let us set $\overline{\epsilon} = \varphi \circ \epsilon$, so that $\overline{\epsilon} : S \to \overline{S}$. The point function $\overline{\epsilon}$ satisfies the condition (a). Indeed if $P_s \subseteq P_{s'}$, then $P_{\epsilon(s)} \subseteq P_{\epsilon(s')}$ since ϵ satisfies (a), so $P_{\overline{\epsilon}(s)} = P_{\overline{\epsilon(s)}} = \varphi[r^{\to} P_{\epsilon(s)}] \subseteq \varphi[r^{\to} P_{\epsilon(s')}] = P_{\overline{\epsilon(s')}} = P_{\overline{\epsilon}(s')}$, as required. Hence we may define the difunction $(\overline{e}, \overline{E}) = (f_{\overline{\epsilon}}, F_{\overline{\epsilon}}) : (S, \mathcal{S}) \to (\overline{S}, \overline{\mathcal{S}})$.

Proposition 3.2. *Let υ be separated. Then with $\overline{\epsilon}$ and $(\overline{e}, \overline{E})$ defined as above*:

(1) *$(\overline{e}, \overline{E})$ is injective.*
(2) *The mapping $\overline{\epsilon} : S \to \overline{S}$ is a textural isomorphism between $(S, \mathcal{S})$ and $(\overline{S}, \overline{\mathcal{S}})$.*

Proof. We establish (1), leaving the proof of (2) to the interested reader.

Take $s, s' \in S$, $\mathcal{F} \in \widetilde{S}$, with $\overline{e} \nsubseteq \overline{Q}_{(s,\varphi(\mathcal{F}))}$, $\overline{P}_{(s',\varphi(\mathcal{F}))} \nsubseteq \overline{E}$. Hence $\varphi(\mathcal{F}) \in P_{\overline{\epsilon}(s)}$ and $\varphi(\mathcal{F}) \notin Q_{\overline{\epsilon}(s')}$. Using the formulae for these sets now gives $\mathcal{G}, \mathcal{H} \in \widetilde{S}$ with $r \nsubseteq \overline{Q}_{(\mathcal{G},\mathcal{F})}$, $P_{\epsilon(s)} \nsubseteq Q_{\mathcal{G}}$ and $\overline{P}_{(\mathcal{H},\mathcal{F})} \nsubseteq R$, $P_{\mathcal{H}} \nsubseteq Q_{\epsilon(s')}$. On the one hand this gives

$$\forall L \in \tau_{\widetilde{\upsilon}} \cup \kappa_{\widetilde{\upsilon}}, \quad (\mathcal{F} \in L \text{ or } \mathcal{G} \notin L) \quad \text{and} \quad (\mathcal{H} \in L \text{ or } \mathcal{F} \notin L),$$

so $P_{\mathcal{H}} \subseteq L$ or $L \subseteq Q_{\mathcal{G}}$ for all $L \in \tau_{\widetilde{\upsilon}} \cup \kappa_{\widetilde{\upsilon}}$, while on the other we have $P_{\epsilon(s')} \subseteq P_{\mathcal{H}}$ and $Q_{\mathcal{G}} \subseteq Q_{\epsilon(s)}$. Hence, $\epsilon(s') \in L$ or $\epsilon(s) \notin L$ for all $L \in \tau_{\widetilde{\upsilon}} \cup \kappa_{\widetilde{\upsilon}}$.

Now suppose that $P_{s'} \nsubseteq P_s$. Then since the uniform ditopology of υ is T_0 we have $Z \in \tau_{\upsilon} \cup \kappa_{\upsilon}$ with $s \in Z$ and $s' \notin Z$ [8]. By Theorem 2.6(3) the restriction to $\epsilon(S)$ of the uniform ditopology of $\widetilde{\upsilon}$ is the image under ϵ of the uniform ditopology of υ. Hence there exists $L \in \tau_{\widetilde{\upsilon}} \cup \kappa_{\widetilde{\upsilon}}$ with $\epsilon(Z) = \epsilon(S) \cap L$, whence we obtain the contradiction $\epsilon(s) \in L$ and $\epsilon(s') \notin L$. This verifies $P_{s'} \subseteq P_s$, so $(\overline{e}, \overline{E})$ is injective. □

Theorem 3.3. *Let υ be separated. Then*:

(1) *$(\overline{S}, \overline{\mathcal{S}}, \overline{\upsilon})$ is dicomplete.*
(2) *$(\overline{e}, \overline{E}) : (S, \mathcal{S}, \upsilon) \to (\overline{S}, \overline{\mathcal{S}}, \overline{\upsilon})$ is uniformly bicontinuous.*
(3) *$\overline{\upsilon}|_{\overline{\epsilon}(S)} = \overline{\epsilon}(\upsilon)$.*
(4) *$\overline{\epsilon}(S)$ is dense in $\overline{S}$ for the uniform ditopology of $\overline{\upsilon}$.*

Proof. We verify (1), leaving the proofs of the other results to the interested reader.

Let $\Phi \times \Gamma$ be an $\overline{\upsilon}$-Cauchy regular difilter on $(\overline{S}, \overline{\mathcal{S}})$. Then $\{\varphi^{-1}[X] \mid X \in \Phi\}$ is a base for an $\widetilde{\mathcal{S}}$-filter which we will denote by $\varphi^{-1}\Phi$, and likewise we obtain the $\widetilde{\mathcal{S}}$-cofilter $\varphi^{-1}\Gamma$. Clearly the difilter $\varphi^{-1}\Phi \times \varphi^{-1}\Gamma$ is regular, we show it is $\widetilde{\upsilon}$-Cauchy. Take $\mathcal{C} \in \widetilde{\upsilon}$ and an open, coclosed dicover $\mathcal{E}$ in $\widetilde{\upsilon}$ satisfying $\mathcal{E} \prec \mathcal{C}$. If we define $\mathcal{D} \in \overline{\upsilon}$ as above we obtain $E\,\mathcal{E}\,F$ with $(\varphi[E], \varphi[F]) \in \mathcal{D} \cap (\Phi \times \Gamma)$, whence $(E, F) \in \mathcal{E} \cap (\varphi^{-1}\Phi \times \varphi^{-1}\Gamma) \neq \emptyset$. This shows $\mathcal{C} \cap (\varphi^{-1}\Phi \times \varphi^{-1}\Gamma) \neq \emptyset$, so $\varphi^{-1}\Phi \times \varphi^{-1}\Gamma$ is $\widetilde{\upsilon}$-Cauchy as required.

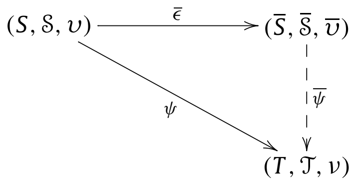
Since $\widetilde{\upsilon}$ is dicomplete we have $\mathcal{F} \in \widetilde{S}$ with $\varphi^{-1}\Phi \to \mathcal{F}$ and $\varphi^{-1}\Gamma \to \mathcal{F}$. A straightforward argument now leads to $\Phi \to \overline{\mathcal{F}}$, $\Gamma \to \overline{\mathcal{F}}$, whence $\overline{\upsilon}$ is dicomplete since $(\overline{S}, \overline{\mathcal{S}})$ is plain. □

Definition 3.4. We call $(\overline{S}, \overline{\mathcal{S}}, \overline{\upsilon})$ the *prime separated dicompletion* of the separated di-uniform plain texture space $(S, \mathcal{S}, \upsilon)$.

We denote by $\mathbf{PDiUni}_0$ the subcategory of **PDiUni** whose objects are separated di-uniform spaces, and likewise $\overline{\mathbf{PDiUni}}_0$ is the subcategory of $\overline{\mathbf{PDiUni}}$ obtained by restricting the objects to be separated complete di-uniform spaces.

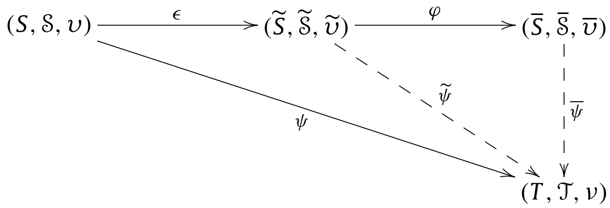
Theorem 3.5. *The category $\overline{\mathbf{PDiUni}}_0$ is a full reflective subcategory of $\mathbf{PDiUni}_0$.*

Proof. It is clear that $\overline{\mathbf{PDiUni}}_0$ is a full subcategory of $\mathbf{PDiUni}_0$. Take $(S, \mathcal{S}, \upsilon) \in \mathrm{Ob}\,\mathbf{PDiUni}_0$. We will show that $\overline{\epsilon} : (S, \mathcal{S}, \upsilon) \to (\overline{S}, \overline{\mathcal{S}}, \overline{\upsilon})$ is a reflection [1, Definition 4.16]. Take $(T, \mathcal{T}, \nu) \in \mathrm{Ob}\,\overline{\mathbf{PDiUni}}_0$ and $\psi \in \mathbf{PDiUni}_0((S, \mathcal{S}, \upsilon), (T, \mathcal{T}, \nu))$. We must show the existence of a unique $\overline{\mathbf{PDiUni}}_0$ morphism $\overline{\psi}$ so that the following diagram is commutative.



We begin by defining a mapping $\widetilde{\psi} : \widetilde{S} \to T$. Let $\psi_{\mathcal{F}} = \{B \in \mathcal{T} \mid \psi^{-1}[B] \in \mathcal{F}\}$. It is trivial to verify that $\psi_{\mathcal{F}}$ is a prime $\mathcal{T}$-filter, so has a limit in T. Moreover, since the uniform ditopology is T_0 and $(T, \mathcal{T})$ is plain, it may be verified that the limit is unique. We denote it by $\widetilde{\psi}(\mathcal{F})$, thereby defining a point function $\widetilde{\psi} : \widetilde{S} \to T$. Take $G \in \tau_\nu$, $K \in \kappa_\nu$ with $G \subseteq K$. We claim that $\widetilde{\psi^{-1}[G]} \subseteq \widetilde{\psi}^{-1}[K]$. Take $\mathcal{F} \in \widetilde{\psi^{-1}[G]}$. Then $\psi^{-1}[G] \in \mathcal{F}$, so $G \in \psi_{\mathcal{F}}$. Since $\psi_{\mathcal{F}} \to \widetilde{\psi}(\mathcal{F})$ this is a cluster point so $\widetilde{\psi}(\mathcal{F}) \in K$, that is $\mathcal{F} \in \widetilde{\psi}^{-1}[K]$, as required. It is easy to deduce that $\widetilde{\psi}$ satisfies condition (a). Also, to establish that this mapping is uniformly bicontinuous it is sufficient to show that for a closed, co-open dicover $\mathcal{C} \in \nu$ we have $\widetilde{\psi}^{-1}\mathcal{C} \in \widetilde{\upsilon}$. Take $\mathcal{D} \in \nu$ open, coclosed with $\mathcal{D} \prec \mathcal{C}$. By the above we have $\widetilde{\psi^{-1}\mathcal{D}} \prec \widetilde{\psi}^{-1}\mathcal{C}$, which establishes the uniform bicontinuity of $\widetilde{\psi}$.

In particular, $(T, \mathcal{T}, \tau_\nu, \kappa_\nu)$ is T_0, and $\widetilde{\psi} : \widetilde{S} \to T$ is bicontinuous, so by the proof of [2, Theorem 4.3] the T_0 reflection φ leads to a mapping $\overline{\psi} : \overline{S} \to T$, satisfying condition (a) and for which $\overline{\psi}^{-1}[B] = \varphi[\widetilde{\psi}^{-1}[B]]$.



If now $\mathcal{C}$ is an open, coclosed element of ν, $\widetilde{\psi}^{-1}\mathcal{C}$ is an open, coclosed element of $\widetilde{\upsilon}$, and so $\overline{\psi}^{-1}\mathcal{C} = \varphi[\widetilde{\psi}^{-1}\mathcal{C}] \in \overline{\upsilon}$ since open and closed subsets of $\widetilde{S}$ are saturated. This verifies that $\overline{\psi} \in \mathrm{Mor}\,\overline{\mathbf{PDiUni}}_0$, and the commutativity of the above diagrams and the uniqueness of this morphism are clear, so we have established that $\overline{\epsilon}$ is a reflection. □

Finally we note the following:

Theorem 3.6. *If $(S, \mathcal{S}, \upsilon)$ is plain, separated and totally bounded then $(\overline{S}, \overline{\mathcal{S}}, \tau_{\overline{\upsilon}}, \kappa_{\overline{\upsilon}})$ is a bi-T_2 dicompactification of $(S, \mathcal{S}, \tau_\upsilon, \kappa_\upsilon)$.*

Proof. Straightforward. □

4. Complementation

Throughout this section we consider a complemented simple texture, $(S, \mathcal{S}, \sigma)$, and suppose that the dicovering uniformity υ is complemented. Our aim is to find a condition under which the complementation σ may be extended to the prime dicompletion $(\widetilde{S}, \widetilde{\mathcal{S}})$ in such a way that $\widetilde{\upsilon}$ is also a complemented di-uniformity. First we note the following:

Lemma 4.1. *For $\mathcal{F} \in \widetilde{S}$ let $\mathcal{F}' = \mathcal{S} \setminus \sigma[\mathcal{F}]$. Then $\mathcal{F}' \in \widetilde{S}$.*

Proof. Since $\mathcal{F}$ is a prime $\mathcal{S}$-filter, it is straightforward to verify that $\mathcal{F}'$ is also a prime $\mathcal{S}$-filter, and we omit the details. It remains to show that $\mathcal{F}' \times (\mathcal{S} \setminus \mathcal{F}') = (\mathcal{S} \setminus \sigma[\mathcal{F}]) \times \sigma[\mathcal{F}]$ is a υ-Cauchy difilter. Take $\mathcal{C} \in \upsilon$. Since $\upsilon' = \upsilon$, by [11, Definition 2.16] there exists $\mathcal{D} \in \upsilon$ with $(\mathcal{D}')^\Delta \prec \mathcal{C}$, where $\mathcal{D}' = \{(\sigma(D), \sigma(C)) \mid C\,\mathcal{D}\,D\}$. Since $\mathcal{F} \times (\mathcal{S} \setminus \mathcal{F})$ is υ-Cauchy we have $C\,\mathcal{D}\,D$ with $C \in \mathcal{F}$ and $D \in \mathcal{S} \setminus \mathcal{F}$. In particular $C \not\subseteq D$, whence $\sigma(D) \not\subseteq \sigma(C)$ and there exists $s \in S$ with $\sigma(D) \not\subseteq Q_s$ and $P_s \not\subseteq \sigma(C)$. From $(\mathcal{D}')^\Delta \prec \mathcal{C}$ we have $A\,\mathcal{C}\,B$ satisfying $\mathrm{St}(\mathcal{D}', P_s) \subseteq A$, $B \subseteq \mathrm{CSt}(\mathcal{D}', Q_s)$, while $\sigma(D)\,\mathcal{D}'\,\sigma(C)$ gives $\sigma(D) \subseteq \mathrm{St}(\mathcal{D}', P_s)$ and $B \subseteq \mathrm{CSt}(\mathcal{D}', Q_s)$. Hence $D \notin \mathcal{F} \Longrightarrow \sigma(D) \in \mathcal{S} \setminus \sigma[\mathcal{F}] \Longrightarrow A \in \mathcal{S} \setminus \sigma[\mathcal{F}]$, and likewise $B \in \sigma[\mathcal{F}]$. This establishes that $\mathcal{F}'$ is Cauchy. □

It is clear that $(\mathcal{F}')' = \mathcal{F}$, whence the mapping $\mathcal{F} : \widetilde{S} \to \widetilde{S}$ is an involution.

Proposition 4.2. *For $X = \bigcup_{j \in J} P_{\mathcal{F}_j} \in \widetilde{\mathcal{S}}$ let $\widetilde{\sigma}(X) = \bigcap_{j \in J} \bigcup_{A \in \mathcal{F}_j} \widetilde{\sigma(A)}$. Then:*

(1) *The mapping $\widetilde{\sigma} : \widetilde{\mathcal{S}} \to \widetilde{\mathcal{S}}$ is well defined.*
(2) *$\widetilde{\sigma}$ is a complementation on $(\widetilde{S}, \widetilde{\mathcal{S}})$.*

Proof. (1) Let $\bigcup_{j\in J} P_{\mathcal{F}_j} = \bigcup_{k\in K} P_{\mathcal{G}_k}$. If $\bigcap_{j\in J}\bigcup_{A\in\mathcal{F}_j}\widetilde{\sigma(A)} \not\subseteq \bigcap_{k\in K}\bigcup_{A\in\mathcal{G}_k}\widetilde{\sigma(A)}$ then we have $\mathcal{H}\in\bigcap_{j\in J}\bigcup_{A\in\mathcal{F}_j}\widetilde{\sigma(A)}$ for which there exists $k\in K$ satisfying $A\in\mathcal{G}_k \Longrightarrow \sigma(A)\notin\mathcal{H}$. Now $\mathcal{G}_k\in P_{\mathcal{G}_k}\subseteq\bigcup_{k\in K}P_{\mathcal{G}_k} = \bigcup_{j\in J}P_{\mathcal{F}_j}$, so there exists $j\in J$ with $\mathcal{G}_k\in P_{\mathcal{F}_j}$, whence $\mathcal{F}_j\subseteq\mathcal{G}_k$ for this j and k. Now $\mathcal{H}\in\bigcup_{A\in\mathcal{F}_j}\widetilde{\sigma(A)}$ so we have $A\in\mathcal{F}_j$ with $\sigma(A)\in\mathcal{H}$. But now $A\in\mathcal{G}_k$, which gives the contradiction $\sigma(A)\notin\mathcal{H}$ by the above implication. By symmetry this is sufficient to show the equality of the two expressions for $\widetilde{\sigma}(X)$, so $\widetilde{\sigma}$ is well defined.

(2) First take $X, Y\in\widetilde{\mathcal{S}}$ with $X\subseteq Y$. Using the natural representations $X=\bigcup_{\mathcal{F}\in X}P_{\mathcal{F}}$, $Y=\bigcup_{\mathcal{F}\in Y}P_{\mathcal{F}}$ leads immediately to $\widetilde{\sigma}(Y)\subseteq\widetilde{\sigma}(X)$.

It remains to show $\widetilde{\sigma}(\widetilde{\sigma}(X)) = X$ for all $X\in\widetilde{\mathcal{S}}$. Using the natural representations for $\widetilde{\sigma}(X)$ and X gives

$$\widetilde{\sigma}\big(\widetilde{\sigma}(X)\big) = \bigcap_{\mathcal{G}\in\widetilde{\sigma}(X)}\ \bigcup_{A\in\mathcal{G}}\widetilde{\sigma(A)},\quad\text{where}$$

$$\widetilde{\sigma}(X) = \bigcap_{\mathcal{F}\in X}\ \bigcup_{B\in\mathcal{F}}\widetilde{\sigma(B)}.$$

Take $\mathcal{F}\in X$ and suppose that $\mathcal{F}\notin\widetilde{\sigma}(\widetilde{\sigma}(X))$. Then we have $\mathcal{G}\in\widetilde{\sigma}(X)$ satisfying $\mathcal{F}\notin\bigcup_{A\in\mathcal{G}}\widetilde{\sigma(A)}$. On the other hand $\mathcal{G}\in\bigcup_{B\in\mathcal{F}}\widetilde{\sigma(B)}$ gives $B\in\mathcal{F}$ with $\sigma(B)\in\mathcal{G}$, and then setting $A=\sigma(B)$ in the union above gives the contradiction $B=\sigma(A)\notin\mathcal{F}$. Hence, $X\subseteq\widetilde{\sigma}(\widetilde{\sigma}(X))$.

To prove the reverse inclusion suppose we have $\mathcal{H}\in\widetilde{\sigma}(\widetilde{\sigma}(X))$ with $\mathcal{H}\notin X$. Then for any $\mathcal{F}\in X$ we have $\mathcal{H}\notin P_{\mathcal{F}}$, so $\mathcal{F}\not\subseteq\mathcal{H}$ and we have $B\in\mathcal{F}$ with $B\notin\mathcal{H}$. Now $\sigma(B)\in\mathcal{S}\setminus\sigma[\mathcal{H}]=\mathcal{H}'$, whence $\mathcal{H}'\in\bigcup_{B\in\mathcal{F}}\widetilde{\sigma(B)}$. This gives $\mathcal{H}'\in\widetilde{\sigma}(X)$ and we obtain $\mathcal{H}\in\bigcup_{A\in\mathcal{H}'}\widetilde{\sigma(A)}$. Hence for some $A\in\mathcal{H}'$ we have $\sigma(A)\in\mathcal{H}$, which gives the contradiction $A\in\sigma[\mathcal{H}]=\mathcal{S}\setminus\mathcal{H}'$. This completes the proof of $\widetilde{\sigma}(\widetilde{\sigma}(X))=X$. □

Proposition 4.3. *The complementation $\widetilde{\sigma}$ on $(\widetilde{S},\widetilde{\mathcal{S}})$ has the properties*:

(1) $\widetilde{\sigma}(P_{\mathcal{F}}) = Q_{\mathcal{F}'}$ *for all* $\mathcal{F}\in\widetilde{S}$.
(2) $\widetilde{\sigma}(\widetilde{A}) = \widetilde{\sigma(A)}$ *for all* $A\in\mathcal{S}$.
(3) $\widetilde{\sigma}(\widetilde{A})\cap\epsilon(S)=\epsilon(\sigma(A))$ *for all* $A\in\mathcal{S}$.

Proof. (1) We note that $\widetilde{\sigma}(P_{\mathcal{F}})=\bigcup_{B\in\mathcal{F}}\widetilde{\sigma(B)}$ and $Q_{\mathcal{F}'}=\bigcup_{\mathcal{F}'\notin P_{\mathcal{G}}}P_{\mathcal{G}}$. Hence if $\widetilde{\sigma}(P_{\mathcal{F}})\not\subseteq Q_{\mathcal{F}'}$ we have $\mathcal{H}\in\widetilde{S}$ for which there exists $B\in\mathcal{F}$ with $\sigma(B)\in\mathcal{H}$, and for which $\mathcal{F}'\notin P_{\mathcal{G}}\Longrightarrow\mathcal{H}\notin P_{\mathcal{G}}$. Taking $\mathcal{G}=\mathcal{H}$ in this implication gives $\mathcal{F}'\in P_{\mathcal{H}}$, whence $\mathcal{H}\subseteq\mathcal{F}'$ by Lemma 2.3(2). Now $\sigma(B)\in\mathcal{F}'$, and we obtain the contradiction $B\notin\mathcal{F}$.

Conversely, suppose $Q_{\mathcal{F}'}\not\subseteq\widetilde{\sigma}(P_{\mathcal{F}})$. Then we have $\mathcal{H}\in Q_{\mathcal{F}'}$ with $\mathcal{H}\notin\bigcup_{B\in\mathcal{F}}\widetilde{\sigma(B)}$. Now we have $\mathcal{G}\in\widetilde{S}$ with $\mathcal{H}\in P_{\mathcal{G}}$ and $\mathcal{F}'\notin P_{\mathcal{G}}$, whence $\mathcal{G}\not\subseteq\mathcal{F}'$, $\mathcal{G}\subseteq\mathcal{H}$, and so $\mathcal{H}\not\subseteq\mathcal{F}'$. Hence we have $A\in\mathcal{H}$ with $A\notin\mathcal{F}'$. If we set $B=\sigma(A)$ we obtain $B\in\mathcal{F}$, whence $\mathcal{H}\notin\widetilde{\sigma(B)}$, that is $A=\sigma(B)\notin\mathcal{H}$, which is a contradiction.

(2) From $\widetilde{A}=\bigcup_{\mathcal{F}\in\widetilde{A}}P_{\mathcal{F}}$ we obtain $\widetilde{\sigma}(\widetilde{A})=\bigcap_{\mathcal{F}\in\widetilde{A}}\bigcup_{B\in\mathcal{F}}\widetilde{\sigma(B)}$. Hence it is clear that $\widetilde{\sigma(A)}\subseteq\widetilde{\sigma}(\widetilde{A})$. Suppose we have $\mathcal{H}\in\widetilde{\sigma}(\widetilde{A})$ with $\mathcal{H}\notin\widetilde{\sigma(A)}$. Then $\sigma(A)\notin\mathcal{H}$ so $A\in\mathcal{H}'$ and we deduce that $\mathcal{H}\in\bigcup_{B\in\mathcal{H}'}\widetilde{\sigma(B)}$. This gives $B\in\mathcal{H}'$ with $\sigma(B)\in\mathcal{H}$, and hence the contradiction $B\in\sigma[\mathcal{H}]=\mathcal{S}\setminus\mathcal{H}'$.

(3) Immediate from (2). □

This last result shows that when restricted to $(S,\mathcal{S})$, the complementation $\widetilde{\sigma}$ coincides with σ.

Definition 4.4. We will call a complementation σ on $(S,\mathcal{S})$ *grounded* if there is an involution $s\mapsto s'$ on S so that $\sigma(P_s)=Q_{s'}$ for all $s\in S$.

Proposition 4.3(1) now says that the complementation $\widetilde{\sigma}$ on $(\widetilde{S},\widetilde{\mathcal{S}})$ is grounded. Many common textures, such as $(X,\mathcal{P}(X),\pi_X)$, $(\mathbb{I},\mathcal{I},\iota)$ and $(\mathbb{R},\mathcal{R},\rho)$ have grounded complementations, but on the other hand the complementation of [7, Example 2.14] is easily seen to be not grounded.

Proposition 4.5. *Let $(S,\mathcal{S},\sigma)$ be a plain texture with grounded complementation σ. Then a dicovering uniformity υ on $(S,\mathcal{S},\sigma)$ is complemented if and only if it has a base of dicovers of the form $\mathcal{C}'$, $\mathcal{C}\in\upsilon$.*

Proof. The given condition is clearly sufficient, even if σ is not grounded, so we prove necessity.

Take $\mathcal{C}\in\upsilon$. Then, since $\upsilon=\upsilon'$, we have $\mathcal{D}\in\upsilon$ with $(\mathcal{D}')^{\Delta}\prec\mathcal{C}$, and there is no loss of generality in assuming that $\mathcal{D}$ is strongly anchored. For $s\in S$ we have $A\,\mathcal{C}\,B$ with $\mathrm{St}(\mathcal{D}',P_s)\subseteq A$, $B\subseteq\mathrm{CSt}(\mathcal{D}',Q_s)$, so $\sigma(\mathrm{CSt}(\mathcal{D}',Q_s))\subseteq\sigma(B)$ and $\sigma(A)\subseteq\sigma(\mathrm{St}(\mathcal{D}',P_s))$. Now

$$\begin{aligned}\sigma\big(\mathrm{St}\big(\mathcal{D}', P_s\big)\big) &= \sigma\Big(\bigcup\big\{\sigma(D) \bigm| \exists C \text{ with } C\,\mathcal{D}\,D,\ P_s \not\subseteq \sigma(C)\big\}\Big)\\ &= \bigcap\big\{D \bigm| \exists C \text{ with } C\,\mathcal{D}\,D,\ P_s \not\subseteq \sigma(C)\big\}\\ &= \bigcap\{D \mid \exists C \text{ with } C\,\mathcal{D}\,D,\ C \not\subseteq Q_{s'}\}\\ &= \mathrm{CSt}(\mathcal{D}, Q_{s'}),\end{aligned}$$

so $\sigma(A) \subseteq \mathrm{CSt}(\mathcal{D}, Q_{s'})$, and likewise $\mathrm{St}(\mathcal{D}, P_{s'}) \subseteq \sigma(B)$. Since the mapping $s \to s'$, being an involution is onto, we deduce that $\mathcal{D}^{\Delta} \prec \mathcal{C}'$. Also, $\mathcal{D}$ is anchored so we have $\mathcal{D} \prec \mathcal{D}^{\Delta}$, hence $\mathcal{D} \prec \mathcal{C}'$ and since $\mathcal{D} \in \upsilon$ we obtain $\mathcal{C}' \in \upsilon$.

Since $\mathcal{C}$ above was arbitrary we also have $\mathcal{D}' \in \upsilon$, so to complete the proof it will suffice to show that $\mathcal{D}' \prec \mathcal{C}$. Take $C\,\mathcal{D}\,D$. By hypothesis $\mathcal{D}$ is excluding, so $C \not\subseteq D$ and hence $\sigma(D) \not\subseteq \sigma(C)$. We now have $s \in S$ with $\sigma(D) \not\subseteq Q_s$ and $P_s \not\subseteq \sigma(C)$, so $\sigma(D) \subseteq \mathrm{St}(\mathcal{D}', P_s)$, $\mathrm{CSt}(\mathcal{D}', Q_s) \subseteq \sigma(C)$. Since $\sigma(D)\,\mathcal{D}'\,\sigma(C)$ and $(\mathcal{D}')^{\Delta} \prec \mathcal{C}$ we see that $\mathcal{D}' \prec \mathcal{C}$, as required. □

Theorem 4.6. *Let $(S, \mathcal{S})$ be a plain texture and σ a grounded complementation on $(S, \mathcal{S})$. If υ is a complemented dicovering uniformity on $(S, \mathcal{S}, \sigma)$, and $\widetilde{\sigma}$ is defined on $(\widetilde{S}, \widetilde{\mathcal{S}})$ as in Proposition 4.2, then $\widetilde{\upsilon}$ is a complemented di-uniformity on $(\widetilde{S}, \widetilde{\mathcal{S}}, \widetilde{\sigma})$.*

Proof. Take $\mathcal{C} \in \upsilon$. Then by Proposition 4.5 we have $\mathcal{D}, \mathcal{E} \in \upsilon$ with $\mathcal{E} \prec \mathcal{D}' \prec \mathcal{C}$. By Lemma 2.2(1) we obtain $\widetilde{\mathcal{E}} \prec \widetilde{\mathcal{D}'} \prec \widetilde{\mathcal{C}}$, whence $\widetilde{\mathcal{E}} \prec (\widetilde{\mathcal{D}})' \prec \widetilde{\mathcal{C}}$ since $(\widetilde{\mathcal{D}})' = \widetilde{\mathcal{D}'}$ by Proposition 4.3(2). Hence $\widetilde{\upsilon}$ is complemented by Proposition 4.5. □

The authors do not know if this theorem holds without the restriction that σ be grounded.

Theorem 4.7. *Let $(S, \mathcal{S}, \sigma, \upsilon)$ be a plain complemented separated di-uniform space with σ grounded, and $(\widetilde{S}, \widetilde{\mathcal{S}}, \widetilde{\sigma}, \widetilde{\upsilon})$ its complemented prime dicompletion. If $(\overline{S}, \overline{\mathcal{S}}, \overline{\upsilon})$ is the separated quotient of $(\widetilde{S}, \widetilde{\mathcal{S}}, \widetilde{\upsilon})$, then the complementation $\widetilde{\sigma}$ may be extended to a complementation $\overline{\sigma}$ on $(\widetilde{S}, \widetilde{\mathcal{S}})$ in such a way that $\overline{\upsilon}$ is complemented.*

Proof. The elements of $\overline{\mathcal{S}}$ may be uniquely written in the form $\varphi[X]$, $X \in \mathcal{R}$. Because the uniform ditopology of $\overline{\upsilon}$ is complemented, $\widetilde{\sigma}$ maps $\tau_{\widetilde{\upsilon}} \cup \kappa_{\widetilde{\upsilon}}$ to itself. Hence, since $\mathcal{R}$ may be generated by taking unions of intersections of elements of $\tau_{\widetilde{\upsilon}} \cup \kappa_{\widetilde{\upsilon}}$, it follows that $\mathcal{R}$ is mapped into itself by $\widetilde{\sigma}$. Hence we may define $\overline{\sigma}$ by setting $\overline{\sigma}(\varphi[X]) = \varphi[\widetilde{\sigma}(X)]$, and it is straightforward to verify that $\overline{\sigma}$ is a complementation on $(\overline{S}, \overline{\mathcal{S}})$.

Let $\mathcal{A} \in \overline{\upsilon}$. Then $\varphi^{-1}\mathcal{A} \in \widetilde{\upsilon}$ so, since $\widetilde{\upsilon}$ is complemented, by Proposition 4.5 there exists $\mathcal{D}, \mathcal{E} \in \widetilde{\upsilon}$ with $\mathcal{E} \prec \mathcal{D}' \prec \varphi^{-1}\mathcal{A}$. Also, since the uniform ditopology of $\widetilde{\upsilon}$ is complemented, $\mathcal{D}'$ is open, coclosed if and only if the same is true of $\mathcal{D}$, so there is no loss of generality in assuming that $\mathcal{D}$ and $\mathcal{E}$ are open, coclosed. Since sets in $\tau_{\widetilde{\upsilon}}$ and $\kappa_{\widetilde{\upsilon}}$ belong to $\mathcal{R}$, and are therefore saturated, we see that $\mathcal{B} = \{(\varphi[C], \varphi[D]) \mid C\,\mathcal{D}\,D\}$, $\mathcal{C} = \{(\varphi[E], \varphi[F]) \mid E\,\mathcal{E}\,F\}$ are dicovers of $(\overline{S}, \overline{\mathcal{S}}, \overline{\sigma})$ belonging to $\overline{\upsilon}$ since $\varphi^{-1}\mathcal{B} = \mathcal{D}$ and $\varphi^{-1}\mathcal{C} = \mathcal{E}$. Also $\varphi^{-1}\mathcal{B}' = D'$, so $\mathcal{C} \prec \mathcal{B}' \prec \mathcal{A}$ and therefore $\overline{\upsilon}$ has a base of dicovers of the form $\mathcal{B}'$, $\mathcal{B} \in \overline{\upsilon}$. Hence $\overline{\upsilon}$ is complemented by Proposition 4.5. □

We end with an example that illustrates the constructions described above.

Example 4.8. Consider the real texture $(\mathbb{R}, \mathcal{R}, \rho)$ defined in the introduction. This is a plain complemented texture, and the complementation ρ is certainly grounded. We will define a totally bounded [12] complemented dicovering uniformity on $(\mathbb{R}, \mathcal{R}, \rho)$, compatible with the usual complemented ditopology $\tau_{\mathbb{R}} = \{(-\infty, r) \mid r \in \mathbb{R}\}$, $\kappa_{\mathbb{R}} = \{(-\infty, r] \mid r \in \mathbb{R}\}$. Since this ditopology is clearly not dicompact, such a di-uniformity will not be complete and we may construct its prime dicompletion.

To this end, for $N \in \mathbb{N}$, $N > 0$, define

$$\mathcal{D}_N = \big\{(P_{r_{n+1}}, Q_{r_{n-1}}) \bigm| -N \cdot 2^N < n < N \cdot 2^N\big\} \cup \big\{(\mathbb{R}, Q_N)\big\} \cup \big\{(P_{-N}, \emptyset)\big\},$$

where $r_n = n \cdot 2^{-N}$. It is straightforward, if somewhat tedious, to verify that:

(i) $\mathcal{D}_N$ is a dicover of $(\mathbb{R}, \mathcal{R})$.
(ii) $\mathcal{D}_{N+1} \prec_{(\star)} \mathcal{D}_N$.
(iii) $\mathcal{D}_N \prec \mathcal{D}_{N'}$ for $N' < N$.

It follows that the family $\mathcal{D}_N$, $N > 0$, is a base for a dicovering uniformity υ on $(\mathbb{R}, \mathcal{R})$. Clearly υ is complemented and compatible with the usual ditopology $(\tau_{\mathbb{R}}, \kappa_{\mathbb{R}})$. Since this ditopology is T_0, υ is separated. It is also totally bounded since the dicovers $\mathcal{D}_N$ are finite.

We begin by describing the prime dicompletion of $(\mathbb{R}, \mathcal{R}, \rho, \upsilon)$. The prime filters are seen to be as follows:

(a) $\mathcal{P}_r = \{F \in \mathcal{R} \mid P_r \subseteq F\}$, $r \in \mathbb{R}$,
(b) $\mathcal{P}_r^o = \{F \in \mathcal{R} \mid Q_r \subseteq F\}$, $r \in \mathbb{R}$,
(c) $\mathcal{L} = \mathcal{R} \setminus \{\emptyset\}$,
(d) $\mathcal{U} = \{\mathbb{R}\}$.

It may be verified that all these filters are υ-Cauchy, so $\widetilde{\mathbb{R}} = \{\mathcal{P}_r, \mathcal{P}_r^o \mid r \in \mathbb{R}\} \cup \{\mathcal{L}, \mathcal{U}\}$. It may also be noted that $\mathcal{P}_r' = \mathcal{P}_{-r}$, $(\mathcal{P}_r^o)' = \mathcal{P}_{-r}^o$, $\mathcal{L}' = \mathcal{U}$, and $\mathcal{U}' = \mathcal{L}$.

We must find the open and closed sets for the uniform ditopology of $\widetilde{\upsilon}$. For $r \in \mathbb{R}$ let $G_r = \{\mathcal{P}_s, \mathcal{P}_s^o \mid s < r\} \cup \{\mathcal{L}\}$. Clearly, $G_r = \bigcup\{\widetilde{Q}_{r-\frac{1}{n}} \mid n = 1, 2, \ldots\} \in \widetilde{\mathcal{R}}$. If we take $\mathcal{P}_s \in G_r$ then $s < r$ so we may choose $N > 0$ with $s + 2^{-N} < r$ and $s \in [-N, N]$. It follows that $\mathrm{St}(\widetilde{\mathcal{D}}_N, P_{\mathcal{P}_s}) \subseteq G_r$, with the same result for $\mathcal{P}_s^o$ in place of $\mathcal{P}_s$, so $G_r \in \tau_{\widetilde{\upsilon}}$. On the other hand, if $\mathcal{P}_s \in G \in \tau_{\widetilde{\upsilon}}$ we have $N > 0$ with $\mathrm{St}(\widetilde{\mathcal{D}}_N, P_{\mathcal{P}_s}) \subseteq G$, whence $\mathcal{P}_s \in G_{r+2^{-N}} \subseteq G$. Since that same result is again true with $\mathcal{P}_s^o$ in place of $\mathcal{P}_s$ we see that the sets G_r, $r \in \mathbb{R}$, form a base of $\tau_{\widetilde{\upsilon}}$. Finally this family, together with $\widetilde{\mathbb{R}}$, is closed under arbitrary unions, so $\tau_{\widetilde{\upsilon}} = \{G_r \mid r \in \mathbb{R}\} \cup \{\emptyset, \widetilde{\mathbb{R}}\}$.

If we let $K_r = \{\mathcal{P}_s, \mathcal{P}_s^o \mid s \leqslant r\} \cup \{\mathcal{L}\}$, then by showing that $K_r = \widetilde{\sigma}(G_{-r})$, or by using a direct argument dual to the above, we obtain $\kappa_{\widetilde{\upsilon}} = \{K_r \mid r \in \mathbb{R}\} \cup \{\emptyset, \widetilde{\mathbb{R}}\}$. It is now clear that there is no open or closed set separating the points $\mathcal{P}_s$ and $\mathcal{P}_s^o$, whence the ditopology $(\tau_{\widetilde{\upsilon}}, \kappa_{\widetilde{\upsilon}})$ is not T_0. It follows that $\widetilde{\upsilon}$ is not separated, so this verifies that the prime dicompletion of a separated di-uniformity need not be separated.

Finally let us identify the T_0 quotient. It is clear that the equivalence classes are $\{\mathcal{P}_r, \mathcal{P}_r^o\}$, $r \in \mathbb{R}$, $\{\mathcal{L}\}$ and $\{\mathcal{U}\}$. Clearly $\overline{\mathcal{P}_s} = \overline{\mathcal{P}_s^o}$ may be identified with $r \in \mathbb{R}$, while we may identify $\overline{\mathcal{L}}$, $\overline{\mathcal{U}}$ with points $-\infty$, ∞, respectively, outside $\mathbb{R}$. The uniform ditopology becomes $\tau = \{[-\infty, r) \mid r \in \mathbb{R}\} \cup \{\emptyset, \mathbb{R} \cup \{-\infty, \infty\}\}$, $\kappa = \{[-\infty, r] \mid r \in \mathbb{R}\} \cup \{\emptyset, \mathbb{R} \cup \{-\infty, \infty\}\}$, which is dicompact by Theorem 3.6, and T_0 and therefore bi-T_2 [8]. It is clearly the textural analogue of the standard two-point Hausdorff compactification of $\mathbb{R}$, and is isomorphic in **dfPDitop** and **fPDitop** to the unit interval ditopological texture space $(\mathbb{I}, \mathcal{I}, \iota, \tau_{\mathbb{I}}, \kappa_{\mathbb{I}})$.

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