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Real dicompactifications of ditopological texture spaces

Filiz Yıldız, Lawrence M. Brown *

Hacettepe University, Faculty of Science, Mathematics Department, Beytepe, Ankara, Turkey

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ABSTRACT

Real dicompactifications and dicompactifications of a ditopological texture space are defined and studied.

Section 2 considers nearly plain extensions of a ditopological texture space $(S, \mathcal{S}, \tau, \kappa)$. Spaces that possess a nearly plain extension are shown to have a property, called here almost plainness, that is weaker than that of near plainness, but which shares with near plainness the existence of an associated plain space $(S_p, \mathcal{S}_p, \tau_p, \kappa_p)$. Some properties of the class of almost plain ditopological texture spaces are established, a notion of canonical nearly plain extension of an almost plain ditopological texture space, projective and injective pre-orderings and the concept of isomorphism on such canonical nearly plain extensions are defined.

In Section 3 the notion of nearly plain extension is specialized to that of real dicompactification and dicompactification, and the spaces that have such extensions are characterized. Working in terms of a specific representation of the canonical real dicompactifications and dicompactifications of a completely biregular bi-T_2 almost plain ditopological space, the interrelation between sub-T-lattices of the T-lattice of ω-preserving bicontinuous real mappings on the associated plain space and the real dicompactifications and dicompactifications are investigated. In particular generalizations of the Hewitt realcompactification and Stone–Čech compactification are obtained, and shown to be reflectors for the appropriate categories.

* Corresponding author.
E-mail addresses: yfiliz@hacettepe.edu.tr (F. Yıldız), brown@hacettepe.edu.tr (L.M. Brown).

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1. Introduction

An adequate introduction to the theory of ditopological texture spaces, and the motivation for its study may be obtained from [4,5,7–9].

If S is a set, a *texturing* $\mathcal{S}$ of S is a subset of $\mathcal{P}(S)$ which is a point-separating, complete, completely distributive lattice containing S and $\emptyset$, and for which meet coincides with intersection and finite joins with union. The pair $(S, \mathcal{S})$ is then called a *texture*.

We regard a texture as a framework in which to do mathematics.

For a texture $(S, \mathcal{S})$, most properties are conveniently defined in terms of the *p-sets* and *q-sets*

$$P_s = \bigcap\{A \in \mathcal{S} \mid s \in A\}, \qquad Q_s = \bigvee\{A \in \mathcal{S} \mid s \notin A\}.$$

However, as noted in [3,14] we may associate with $(S, \mathcal{S})$ the C-space [15] (core-space) $(S, \mathcal{S}^c)$, and then the frequently occurring relationship $P_{s'} \not\subseteq Q_s$, $s, s' \in S$, is equivalent to $s\, \omega_S\, s'$, where ω_S is the *interior relation* for $(S, \mathcal{S}^c)$. In particular if $(S, \mathcal{S})$, $(T, \mathcal{T})$ are textures and $\varphi : S \to T$ a point function, then φ is called *ω-preserving* if $s_1\, \omega_S\, s_2 \Rightarrow \varphi(s_1)\, \omega_T\, \varphi(s_2)$.

Since a texturing $\mathcal{S}$ need not be closed under the operation of taking the set complement, the notion of topology is replaced by that of *dichotomous topology* or *ditopology*, namely a pair (τ, κ) of subsets of $\mathcal{S}$, where the set of *open sets* τ satisfies

(1) $S, \emptyset \in \tau$,
(2) $G_1, G_2 \in \tau \Rightarrow G_1 \cap G_2 \in \tau$ and
(3) $G_i \in \tau$, $i \in I \Rightarrow \bigvee_i G_i \in \tau$,

and the set of *closed sets* κ satisfies

(1) $S, \emptyset \in \kappa$,
(2) $K_1, K_2 \in \kappa \Rightarrow K_1 \cup K_2 \in \kappa$ and
(3) $K_i \in \kappa$, $i \in I \Rightarrow \bigcap K_i \in \kappa$.

Hence a ditopology is essentially a "topology" for which there is no *a priori* relation between the open and closed sets.

It is shown in the general references given above that ditopological texture spaces provide a unified setting for the study of topology, bitopology [18] and topology on Hutton algebras.

Background material specific to the topic of this paper may be found in [20,21], and the reader is also referred to [10,11]. Due to lack of space, none of this material is repeated here. The classical theory of realcompact topological spaces, with an emphasis on the relation with the ideal structure of the ring of continuous real-valued functions, is given in the renowned book of Gillman and Jerison [17]. The second author's PhD thesis [2] gives a generalization of some of this material to the bitopological case, ring ideals being replaced by dual ideals in the T-lattice of pairwise continuous real-valued functions. This theory of dual ideals in a T-lattice developed in [2] also plays an important role in [20,21], and in the present paper. It is shown in [21] that the bitopological theory given in [2] is categorically equivalent to the ditopological theory restricted to the subclass of plain ditopological texture spaces. The reader is also referred to [13] for a more categorically based treatment of bitopological real compactness.

The reader is referred to [16] for terms from lattice theory not defined here, and our overall reference for category theory is [1].

Section 2 considers nearly plain extensions of a ditopological texture space $(S, \mathcal{S}, \tau, \kappa)$. The class of spaces that possess a nearly plain extension is identified in terms of a notion of almost plain texture. This notion is weaker than that of nearly plain texture defined in [20, p. 174], but shares with near plainness the existence of an associated plain space $(S_p, \mathcal{S}_p, \tau_p, \kappa_p)$. Some properties of the class of almost plain ditopological texture spaces are established, a notion of canonical nearly plain extension of an almost plain ditopological texture space, projective and injective pre-orderings and the concept of isomorphism on such canonical nearly plain extensions are defined.

In Section 3 the notion of nearly plain extension is specialized to that of real dicompactification and dicompactification, and the spaces that have such extensions are characterized. It is shown that, up to isomorphism, the canonical real dicompactifications (dicompactifications) of a completely biregular bi-T_2 almost plain ditopological space $(S, \mathcal{S}, \tau, \kappa)$ have the form $(H_{\langle B\rangle}, \mathcal{H}_{\langle B\rangle}, \tau_{\langle B\rangle}, \kappa_{\langle B\rangle})$ for B a bigenerating subset of $\mathrm{BA}(S_p)$ (respectively, $\mathrm{BA}^*(S_p)$). See [21] for the necessary definitions. The remainder of this section is devoted to an investigation of the interrelation between the sub-T-lattices $\langle B\rangle$ and the (real) dicompactifications. In particular generalizations of the Hewitt realcompactification and Stone–Čech compactification are established, and shown to be reflectors for the appropriate categories.

2. Nearly plain extensions

The fact that dicompact, or more generally real dicompact, bi-T_2 spaces are nearly plain, focuses our attention on nearly plain ditopological texture spaces. We begin, therefore, by considering nearly plain extensions of ditopological texture spaces. First we make definite what we will mean by a "dense subspace" in this context.

In [6] the notion of *elementary subtexture* is defined. If $(U, \mathcal{U})$ is an elementary subtexture of $(S, \mathcal{S})$ then, in particular, $U \subseteq S$ and $\mathcal{U} = \mathcal{S}_U = \{A \cap U \mid A \in \mathcal{S}\}$ is a texturing of U. We make the following definition:

Definition 2.1. $(U, \mathcal{U})$ is an *induced subtexture of* $(S, \mathcal{S})$ if $U \subseteq S$ and $\mathcal{U} = \mathcal{S}_U$ is a texturing of U.

Hence, an elementary subtexture of $(S, \mathcal{S})$ is an induced subtexture, but not conversely since elementary subtextures satisfy an additional condition that need not hold for induced subtextures.

In case $(S, \mathcal{S}, \tau, \kappa)$ is a ditopological texture space and $(U, \mathcal{U})$ an induced subtexture of $(S, \mathcal{S})$, then clearly $\tau_U = \{G \cap U \mid G \in \tau\}$, $\kappa_U = \{K \cap U \mid K \in \kappa\}$ defines a ditopology (τ_U, κ_U) on $(U, \mathcal{U})$ which we will refer to as the *induced ditopology* on $(U, \mathcal{U})$.

Definition 2.2. Let $(V, \mathcal{V}, \nu, \mu)$ be a nearly plain ditopological texture space. By a *dense subspace* of $(V, \mathcal{V}, \nu, \mu)$ we mean an induced subtexture $(U, \mathcal{V}_U)$ with the induced ditopology (ν_U, μ_U) that satisfies the conditions:

(1) $\varphi_p^V(U) \subseteq U_p$, and
(2) U is dense in V under the joint topology of (ν, μ).

Note that the joint topology on $(V, \mathcal{V}, \nu, \mu)$ is defined as in [20, p. 186], but without the restriction to V_p.

Lemma 2.3. *If $(U, \mathcal{V}_U, \nu_U, \mu_U)$ is a dense subspace of the nearly plain ditopological texture space $(V, \mathcal{V}, \nu, \mu)$ then $(U, \mathcal{V}_U)$ is nearly plain and $U_p = V_p \cap U$.*

Proof. For $u \in U \subseteq V$ we have $Q_u^U = Q_u^V \cap U = Q^V_{\varphi_p^V(u)} \cap U = Q^U_{\varphi_p^V(u)}$, and $\varphi_p^V(u) \in U_p$ by Definition 2.2(1). Hence $(U, \mathcal{V}_U)$ is nearly plain and $\varphi_p^U(u) = \varphi_p^V(u)$ for all $u \in U$. The equality $U_p = V_p \cap U$ is now easily shown. $\square$

Definition 2.4. The nearly plain ditopological texture space $(V, \mathcal{V}, \nu, \mu)$ will be called a *nearly plain extension* of $(S, \mathcal{S}, \tau, \kappa)$ if $(S, \mathcal{S}, \tau, \kappa)$ is dihomeomorphic [3, Definition 4.3] to a dense subspace of $(V, \mathcal{V}, \nu, \mu)$.

Since by [20, Proposition 2.7] a nearly plain texture $(V, \mathcal{V})$ is isomorphic in **dfTex** to the plain texture $(V_p, \mathcal{V}_p)$ we see from Lemma 2.3 that if $(S, \mathcal{S}, \tau, \kappa)$ has a nearly plain extension then $(S, \mathcal{S})$ is, in particular, isomorphic to a plain texture. In view of this the following result will help us characterize those spaces with a nearly plain extension.

Lemma 2.5. *The following are equivalent for a texture $(S, \mathcal{S})$.*

(1) *There exists a plain texture $(Z, \mathcal{Z})$ and a surjective difunction $(f, F) : (Z, \mathcal{Z}) \to (S, \mathcal{S})$.*
(2) *Given $s_1, s_2 \in S$ with $s_1\, \omega_S\, s_2$ there exists $u \in S_p$ with $s_1\, \omega_S\, u$ and $u\, \omega_S\, s_2$.*
(3) *$\mathcal{S}_p = \{A \cap S_p \mid A \in \mathcal{S}\}$ is a plain texturing of S_p, the identity $\epsilon : (S_p, \mathcal{S}_p) \to (S, \mathcal{S})$ is ω-preserving and $(f_\epsilon, F_\epsilon) : (S_p, \mathcal{S}_p) \to (S, \mathcal{S})$ is bijective.*

Proof. (1) ⇒ (2). Since $(Z, \mathcal{Z})$ is plain, by [7, Proposition 3.7] there exists an ω-preserving point function $\psi : Z \to S$ with $f = f_\psi$, $F = F_\psi$. For $z \in Z$ we have $z\, \omega_Z\, z$, whence $\psi(z)\, \omega_S\, \psi(z)$ and so $\psi(Z) \subseteq S_p$, the set of plain points of S. Now take $P_{s_2} \not\subseteq Q_{s_1}$. Since (f_ψ, F_ψ) is surjective we have $z \in Z$ satisfying $f_\psi \not\subseteq \overline{Q}_{(z,s_1)}$ and $\overline{P}_{(z,s_2)} \not\subseteq F_\psi$. But

$$f_\psi = \bigvee \{\overline{P}_{(u,v)} \mid P_{\psi(u)} \not\subseteq Q_v\}$$

and we deduce that $P_{\psi(z)} \not\subseteq Q_{s_1}$. Likewise, $P_{s_2} \not\subseteq Q_{\psi(z)}$ and we have shown (2).

(2) ⇒ (3). The stated properties of $\mathcal{S}_p$ follow as in the proof of [20, Proposition 3.6], and we again obtain $P_s^p = P_s \cap S_p$, $Q_s^p = Q_s \cap S_p$ for $s \in S_p$ so ϵ is ω-preserving. It remains to show that (f_ϵ, F_ϵ) is bijective. Since $(S_p, \mathcal{S}_p)$ is plain we clearly have

$$f_\epsilon = \bigvee \{\overline{P}_{(u,s)} \mid P_u \not\subseteq Q_s\}, \qquad F_\epsilon = \bigcap \{\overline{Q}_{(u,s)} \mid P_s \not\subseteq Q_u\}. \tag{2.1}$$

Take $s, s' \in S$ with $P_s \not\subseteq Q_{s'}$. By (2) we have $u \in S_p$ with $P_s \not\subseteq Q_u$ and $P_u \not\subseteq Q_{s'}$. Since $P_u \not\subseteq Q_u$ we have by (2.1) that $\overline{P}_{(u,u)} \subseteq f_\epsilon$, $F_\epsilon \subseteq \overline{Q}_{(u,u)}$, whence $f_\epsilon \not\subseteq \overline{Q}_{(u,s')}$, $\overline{P}_{(u,s)} \not\subseteq F_\epsilon$. This shows that (f_ϵ, F_ϵ) is surjective.

Now take $u, u' \in S_p$, $s \in S$ with $f_\epsilon \not\subseteq \overline{Q}_{(u,s)}$ and $\overline{P}_{(u',s)} \not\subseteq F_\epsilon$. Using (2.1) we easily obtain $P_u \not\subseteq Q_{u'}$, whence (f_ϵ, F_ϵ) is injective.

(3) ⇒ (1). Immediate on taking $(Z, \mathcal{Z}) = (S_p, \mathcal{S}_p)$ and $(f, F) = (f_\epsilon, F_\epsilon)$. $\square$

Definition 2.6. A texture $(S, \mathcal{S})$ satisfying the equivalent conditions of Lemma 2.5 will be called *almost plain*. For an almost plain texture $(S, \mathcal{S})$, the plain texture $(S_p, \mathcal{S}_p)$ will be referred to as the *associated plain texture*, the inclusion $\epsilon : S_p \to S$ and the **dfTex** isomorphism $(f_\epsilon, F_\epsilon) : (S_p, \mathcal{S}_p) \to (S, \mathcal{S})$ as the *canonical inclusion* and *canonical isomorphism*, respectively.

Clearly the conditions in Lemma 2.5 are also equivalent to [20, Lemma 2.1(1)]. Hence by [20, Lemma 2.4], every nearly-plain texture is almost plain. The following example shows that the converse is false.

Example 2.7. Denote by $(B_s, \mathcal{B}_s)$, $0 < s < 1$, the principal subtexture of $(M_\mathcal{J}, \mathcal{M}_\mathcal{J})$ on $B_s \in \mathcal{M}_\mathcal{J}$ [8, Example 2.4]. Then $B_s = \{(r,0) \mid 0 \leqslant r < s\} \cup \{(r,1) \mid 0 < r \leqslant s\}$, and it is easy to verify that the p-sets and q-sets in $(B_s, \mathcal{B}_s)$ of the points of B_s, being the intersection with B_s of the p-sets and q-sets of these points in $(M_\mathcal{J}, \mathcal{M}_\mathcal{J})$, are the same as these p-sets and q-sets. Now take $(r_1,k_1), (r_2,k_2) \in B_s$ with $P_{(r_1,k_1)} \not\subseteq Q_{(r_2,k_2)}$. Since by [20, Examples 2.3(2)] the texture $(M_\mathcal{J}, \mathcal{M}_\mathcal{J})$ is nearly plain and hence almost plain there exists a plain point $(r,0)$ satisfying $P_{(r_1,k_1)} \not\subseteq Q_{(r,0)}$ and $P_{(r,0)} \not\subseteq Q_{(r_2,k_2)}$. To deduce that $(B_s, \mathcal{B}_s)$ is almost plain it remains only to verify that $(r,0) \in B_s$. In case $k_1 = 0$, $P_{(r_1,k_1)} \not\subseteq Q_{(r,0)} \Rightarrow A_{r_1} \not\subseteq B_r \Rightarrow r \leqslant r_1 < s$, while $k_1 = 1$ gives $B_{r_1} \not\subseteq Q_r$ and so $r < r_1 \leqslant s$. Hence $r < s$, and $(r,0) \in B_s$ as required.

On the other hand consider the point $(s,1) \in B_s$. It is clear that there is no point in $(B_s)_p = \{(r,0) \mid 0 \leqslant r < s\}$ whose q-set is equal to $Q_{(s,1)}$, so $(B_s, \mathcal{B}_s)$ is not nearly plain.

This example also shows that a principal subtexture of a nearly plain texture need not be nearly plain.

The following lemma further clarifies the relation between almost plain and nearly plain textures.

Lemma 2.8. *Let $(S, \mathcal{S})$ be almost plain. The following are equivalent*:

(1) *There exists an ω-preserving point function $\psi : S \to S_p$ which is the identity on S_p.*
(2) *$(S, \mathcal{S})$ is nearly plain.*
(3) *The inverse isomorphism $(f_\epsilon, F_\epsilon)^\leftarrow : (S, \mathcal{S}) \to (S_p, \mathcal{S}_p)$ is representable.*

Proof. $(1) \Rightarrow (2)$. To prove that $(S, \mathcal{S})$ is nearly plain it will be sufficient to prove $Q_s = Q_{\psi(s)}$ for $s \in S$.

Suppose first that $Q_s \not\subseteq Q_{\psi(s)}$. Then $P_s \not\subseteq P_{\psi(s)}$ and we have $u \in S_p$ with $P_s \not\subseteq Q_u$ and $P_u \not\subseteq P_{\psi(s)}$ by textural density. On the other hand, ψ is ω-preserving and $\psi(u) = u$, so $P_{\psi(s)} \not\subseteq Q_{\psi(u)} = Q_u$, and we have the contradiction $P_u \subseteq P_{\psi(s)}$. Secondly, suppose that $Q_{\psi(s)} \not\subseteq Q_s$. Then we have $u \in S_p$ by textural density so that $Q_{\psi(s)} \not\subseteq Q_u$ and $P_u \not\subseteq Q_s$. Again using the fact that ψ is ω-preserving and $\psi(u) = u$ we obtain $P_u = P_{\psi(u)} \not\subseteq Q_{\psi(s)}$, which gives the contradiction $Q_{\psi(s)} \subseteq Q_u$. This completes the proof that $(S, \mathcal{S})$ is nearly plain, and indeed shows that $\psi = \varphi_p$.

$(2) \Rightarrow (3)$. When $(S, \mathcal{S})$ is nearly plain we have the ω-preserving point function $\varphi_p : S \to S_p$ for which $\varphi_p|_{S_p} = \epsilon$ is the identity on S_p. By [20, Proposition 2.7] we obtain $(f_\epsilon, F_\epsilon)^\leftarrow = (f_p, F_p)$, and (f_p, F_p) is representable by φ_p.

$(3) \Rightarrow (1)$. Let $(F_\epsilon^\leftarrow, f_\epsilon^\leftarrow) = (f_\psi, F_\psi)$, where $\psi : S \to S_p$ is ω-preserving. We must show that $\psi(s) = s$ for all $s \in S_p$.

Take $s \in S_p \subseteq S$. Since $P_s \not\subseteq Q_s$ and $P_{\psi(s)} \not\subseteq Q_{\psi(s)}$ as $s, \psi(s) \in S_p$ we clearly have $f_\psi \not\subseteq \overline{Q}_{(s,\psi(s))}$, whence $F_\epsilon^\leftarrow = f_\psi \not\subseteq \overline{Q}_{(s,\psi(s))}$. This now gives $\overline{P}_{(\psi(s),s)} \not\subseteq F_\epsilon = \bigcap\{\overline{Q}_{(v,s)} \mid P_s \not\subseteq Q_v\}$, so for some $s' \in S_p$, $\overline{P}_{(\psi(s),s)} \not\subseteq \overline{Q}_{(\psi(s),s')}$ and $P_{s'} \not\subseteq Q_{\psi(s)}$. We deduce $P_s \not\subseteq Q_{\psi(s)}$, while $P_{\psi(s)} \not\subseteq Q_s$ follows by a dual argument using $f_\epsilon^\leftarrow = F_\psi$, and we have established $P_s = P_{\psi(s)}$, so $s = \psi(s)$ as required. $\square$

Corollary 2.9. *A difunction between two almost plain textures is not necessarily representable.*

Proof. By Lemma 2.8 the inverse isomorphism $(f_\epsilon, F_\epsilon)^\leftarrow : (B_s, \mathcal{B}_s) \to ((B_s)_p, (\mathcal{B}_s)_p)$, where $(B_s, \mathcal{B}_s)$, $0 < s < 1$, is the texture of Example 2.7, cannot be representable since $(B_s, \mathcal{B}_s)$ is not nearly plain. $\square$

This should be contrasted with the contrary result for nearly plain textures [20, Theorem 2.10].

In view of Lemma 2.8(3), properties of nearly plain spaces given in [20,21] that do not depend on the point function φ_p will also be valid for almost plain spaces. This applies, in particular, to [21, Proposition 2.17 and Corollary 2.18] that we will use in the sequel.

We shall denote by **dfApTex** the full, isomorphism closed subcategory of **dfTex** whose objects are almost plain textures. The embedding **dfPTex** $\to$ **dfNpTex** is known to be an equivalence by [20, Theorem 2.8], and it is an easy consequence of Lemma 2.5(3) and the fact that a nearly plain texture is almost plain that the embedding **dfNpTex** $\to$ **dfApTex** is also an equivalence. Hence, the categories **dfPTex**, **dfNpTex** and **dfApTex** are equivalent to one another.

If $(S_j, \mathcal{S}_j)$, $j \in J$, are almost plain textures and $s = (s_j)$, $s' = (s'_j) \in S = \prod_{j \in J} S_j$, then since $P_s \not\subseteq Q_{s'} \Leftrightarrow P_{s_j} \not\subseteq Q_{s'_j}$, $\forall j \in J$ by [7, Corollary 1.4(2)] we have $S_p = \prod_{j \in J} (S_j)_p$ and it is easy to deduce that the product texture $(S, \mathcal{S})$ is almost plain. It follows from [8, Theorem 3.10] that the textural products of almost plain textures, together with the projection difunctions, are products in **dfApTex**. In a similar way, disjoint sums of almost plain textures, together with the inclusion difunctions, are easily shown to be coproducts in **dfApTex**.

We now consider the full subcategory **dfApDitop** of **dfDitop** whose objects are ditopological almost plain texture spaces. We regard this as a concrete category over **dfApTex**. Since **dfDitop** is topological [1,12] over **dfTex** [8, Theorem 3.6] we see at once that **dfApDitop** is topological over **dfApTex**. In particular, **dfApDitop** has products and coproducts that are obtained from those of **dfApTex** by giving them the product ditopology and disjoint sum ditopology, respectively.

Lemma 2.10. *Let $(S, \mathcal{S}, \tau, \kappa)$ be an almost plain ditopological texture space, $(S_p, \mathcal{S}_p)$ the associated plain texture. Denoting by $\tau_p = \{G \cap S_p \mid G \in \tau\}$, $\kappa_p = \{K \cap S_p \mid K \in \kappa\}$ the induced ditopology, the canonical isomorphism $(f_\epsilon, F_\epsilon) : (S_p, \mathcal{S}_p, \tau_p, \kappa_p) \to (S, \mathcal{S}, \tau, \kappa)$ is a dihomeomorphism.*

Proof. Since $(S_p, \mathcal{S}_p)$ is plain the canonical inclusion $\epsilon : S_p \to S$ satisfies the hypotheses of [7, Lemma 3.9], so for $A \in \mathcal{S}$ we have $f_\epsilon^{\leftarrow} A = \epsilon^{-1}[A] = A \cap S_p$. By the definition of (τ_p, κ_p) we now have $f_\epsilon^{\leftarrow} G \in \tau_p \Leftrightarrow G \in \tau$ and $f_\epsilon^{\leftarrow} K \in \kappa_p \Leftrightarrow K \in \kappa$, whence (f_ϵ, F_ϵ) is a dihomeomorphism by [7, Proposition 2.12(v)]. □

As an immediate consequence of this lemma we see that **dfApDitop** is isomorphism closed in **dfDitop**. Also, the inclusion **dfNpDitop** → **dfApDitop** is an equivalence, whence by [20, Theorem 4.3] the categories **dfPDitop**, **dfNpDitop** and **dfApDitop** are equivalent to one another.

Lemma 2.10 also tells us that if $(S, \mathcal{S}, \tau, \kappa)$ is an almost plain ditopological texture space, then $(S_p, \mathcal{S}_p, \tau_p, \kappa_p)$ is plain, and hence a nearly plain extension of $(S, \mathcal{S}, \tau, \kappa)$. Combining this with Lemma 2.5 gives:

Theorem 2.11. *A ditopological texture space has a nearly plain extension if and only if it is almost plain.*

Clearly a nearly plain space is a nearly plain extension of itself.

If $(V, \mathcal{V}, \nu, \mu)$ is a nearly plain extension of $(S, \mathcal{S}, \tau, \kappa)$ we have the following commutative diagram.

$$\begin{tikzcd}
(S, \mathcal{S}, \tau, \kappa) \arrow[r, "{(f,F)}"] & (U, \mathcal{V}_U, \nu_U, \mu_U) \arrow[r] \arrow[d, "{(f_p,F_p)}"] & (V, \mathcal{V}, \nu, \mu) \\
(S_p, \mathcal{S}_p, \tau_p, \kappa_p) \arrow[u, "{(f_\epsilon,F_\epsilon)}"] \arrow[r, "{(f_\psi,F_\psi)}"] & (U_p, (\mathcal{V}_U)_p, (\nu_U)_p, (\mu_U)_p) \arrow[r] & (V_p, \mathcal{V}_p, \nu_p, \kappa_p) \arrow[u]
\end{tikzcd}$$

Here (f, F) is the postulated dihomeomorphism, (f_ϵ, F_ϵ), (f_p, F_p) the dihomeomorphisms introduced above and in [21], respectively, and the unnamed difunctions are inclusions. Finally, $(f_p, F_p) \circ (f, F) \circ (f_\epsilon, F_\epsilon)$ is a dihomeomorphism between plain textures and hence represented by some unique **fDitop**-isomorphism ψ, that is $(f_\psi, F_\psi) = (f_p, F_p) \circ (f, F) \circ (f_\epsilon, F_\epsilon)$. Since an **fDitop**-isomorphism is a textural isomorphism in the sense of [4] it may be used to rename the points of U_p with the points of S_p, the sets of $(\mathcal{V}_U)_p$ with the sets of $\mathcal{S}_p$, and finally the ditopology $((\nu_U)_p, (\mu_U)_p)$ with the ditopology (τ_p, κ_p). Hence we may regard $(S_p, \mathcal{S}_p, \tau_p, \kappa_p)$ as a subspace of $(V_p, \mathcal{V}_p, \nu_p, \mu_p)$. It is straightforward to see that $(U_p, (\mathcal{V}_U)_p, (\nu_U)_p, (\mu_U)_p)$ is a dense subspace of $(V, \mathcal{V}, \nu, \mu)$ in the sense of Definition 2.2, whence we may regard $(S_p, \mathcal{S}_p, \tau_p, \kappa_p)$ itself as a dense subspace of $(V, \mathcal{V}, \nu, \mu)$. When we do this we will refer to $(V, \mathcal{V}, \nu, \mu)$ as a *canonical nearly plain extension of* $(S, \mathcal{S}, \tau, \kappa)$.

Let $(f, F) : (V_1, \mathcal{V}_1, \nu_1, \mu_1) \to (V_2, \mathcal{V}_2, \nu_2, \mu_2)$ be a difunction between the canonical nearly plain extensions $(V_1, \mathcal{V}_1, \nu_1, \mu_1)$, $(V_2, \mathcal{V}_2, \nu_2, \mu_2)$ of $(S, \mathcal{S}, \tau, \kappa)$. We know by [20, Theorem 2.10] that there exists an ω-preserving point function $\varphi : V_1 \to V_2$ with $(f, F) = (f_\varphi, F_\varphi)$. In general φ need not be unique, but it clearly maps $(V_1)_p$ to $(V_2)_p$ since φ is ω-preserving, and its restriction to $(V_1)_p$ is unique. Indeed, suppose that ψ is a second such representative of (f, F), and that $\varphi(u) \neq \psi(u)$ for some $u \in (V_1)_p$. We may assume without loss of generality that $P_{\varphi(u)} \not\subseteq P_{\psi(u)}$, and hence obtain $v \in (V_1)_p$ with $P_{\varphi(u)} \not\subseteq Q_v$ and $P_v \not\subseteq P_{\psi(u)}$. By the formula for f_φ we easily obtain $\overline{P}_{(u,v)} \subseteq f_\varphi$, whence $f_\psi = f_\varphi \not\subseteq \overline{Q}_{(u,v)}$ as $P_v \not\subseteq Q_v$, and now the formula for f_ψ leads to the contradiction $P_v \subseteq P_{\psi(u)}$. In case this unique restriction to $(V_1)_p$ leaves the subset S_p pointwise fixed, that is $\varphi(s) = s$ for all $s \in S_p$, we will say that (f, F) *leaves S_p pointwise fixed.*

We may now define various pre-orderings on the canonical nearly plain extensions of $(S, \mathcal{S}, \tau, \kappa)$. These are natural generalizations of the corresponding pre-orderings in the classical case [19].

Definition 2.12. Let $(V, \mathcal{V}, \nu, \mu)$, $(W, \mathcal{W}, \varpi, \eta)$ be canonical nearly plain extensions of $(S, \mathcal{S}, \tau, \kappa)$.

(1) $(W, \mathcal{W}, \varpi, \eta)$ is said to be *projectively larger* than $(V, \mathcal{V}, \nu, \mu)$ if there is a bicontinuous surjection $(f, F) : (W, \mathcal{W}, \varpi, \eta) \to (V, \mathcal{V}, \nu, \mu)$ that leaves S_p pointwise fixed.
In case (f, F) is a dihomeomorphism, $(W, \mathcal{W}, \varpi, \eta)$ is said to be *isomorphic* to $(V, \mathcal{V}, \nu, \mu)$.

(2) $(W, \mathcal{W}, \varpi, \eta)$ is said to be *injectively larger* than $(V, \mathcal{V}, \nu, \mu)$ if there is an induced subspace $(U, \mathcal{U}, \upsilon, \varrho)$ of $(W, \mathcal{W}, \varpi, \eta)$ and a dihomeomorphism $(f, F) : (V, \mathcal{V}, \nu, \mu) \to (U, \mathcal{U}, \upsilon, \varrho)$ that leaves S_p pointwise fixed.

3. Real dicompactifications

Throughout this paper all real dicompact spaces $(V, \mathcal{V}, \nu, \mu)$ will be assumed to be bi-T_2, hence in particular nearly plain, and will refer to a nearly plain extension $(V, \mathcal{V}, \nu, \mu)$ of $(S, \mathcal{S}, \tau, \kappa)$ that is real dicompact (dicompact) as a *real dicompactification* (*dicompactification*) of $(S, \mathcal{S}, \tau, \kappa)$. By Theorem 2.11, if $(S, \mathcal{S}, \tau, \kappa)$ has a real dicompactification it must be almost plain. It must also be completely biregular and bi-T_2, for these properties are possessed by the real dicompactification and are preserved by induced subspaces and dihomeomorphisms.

Theorem 3.1. *The ditopological texture space $(S, \mathcal{S}, \tau, \kappa)$ has a real dicompactification if and only if it is an almost plain completely biregular bi-T_2 space.*

Proof. Necessity has been established above, so we prove sufficiency. Let $(S, \mathcal{S}, \tau, \kappa)$ be an almost plain, completely biregular bi-T_2 space, and consider the dihomeomorphic plain space $(S_p, \mathcal{S}_p, \tau_p, \kappa_p)$ which is also completely biregular and bi-T_2. If $B \subseteq BA(S_p)$ is bigenerating [21, Definition 2.2], we claim that $(H_{\langle B\rangle}, \mathcal{H}_{\langle B\rangle}, \tau_{\langle B\rangle}, \kappa_{\langle B\rangle})$ is a real dicompactification of $(S_p, \mathcal{S}_p, \tau_p, \kappa_p)$, and hence of $(S, \mathcal{S}, \tau, \kappa)$. To see this we note that $(H_{\langle B\rangle}, \mathcal{H}_{\langle B\rangle}, \tau_{\langle B\rangle}, \kappa_{\langle B\rangle})$ is a jointly closed subspace of a product of copies of $(\mathbb{R}, \mathcal{R}, \tau_{\mathbb{R}}, \kappa_{\mathbb{R}})$ by [21, Proposition 2.17(2)]. It is certainly a completely biregular bi-T_2 $*$-space, whence by [21, Theorem 2.19] it is real dicompact. Further, recalling from [21] the mapping $\xi_{\langle B\rangle} : S_p \to \xi_{\langle B\rangle}(S_p)$, we see from [21, Proposition 2.17] that $\xi_{\langle B\rangle}(S_p)$ is dense in $H_{\langle B\rangle}$ for the joint topology of $(\tau_{\langle B\rangle}, \kappa_{\langle B\rangle})$, and that $\xi_{\langle B\rangle} : S_p \to \xi_{\langle B\rangle}(S_p)$ is an **fDitop** isomorphism which therefore gives rise to a dihomeomorphism. Finally, Definition 2.4 is automatically satisfied for plain textures, and our claim is justified. $\square$

It is straightforward to verify that precisely the same conditions are necessary and sufficient to ensure that $(S, \mathcal{S}, \tau, \kappa)$ has a dicompactification.

Clearly a real dicompact (dicompact) space is a real dicompactification (dicompactification) of itself. Also, if $(S, \mathcal{S}, \tau, \kappa)$ is almost plain then $(S, \mathcal{S}, \tau, \kappa)$ and $(S_p, \mathcal{S}_p, \tau_p, \kappa_p)$ have the same real dicompactifications and the same dicompactifications.

The notions of *canonical real dicompactification*, *projectively larger, injectively larger* and *isomorphic* canonical real dicompactifications of $(S, \mathcal{S}, \tau, \kappa)$ are specializations of the corresponding notions for canonical nearly plain extensions. Note that if $B \subseteq BA(S_p)$ is bigenerating we may regard $(H_{\langle B\rangle}, \mathcal{H}_{\langle B\rangle}, \tau_{\langle B\rangle}, \kappa_{\langle B\rangle})$ as a canonical real dicompactification of $(S, \mathcal{S}, \tau, \kappa)$ via the **fDitop**-isomorphism $\xi_{\langle B\rangle} : S_p \to \xi_{\langle B\rangle}(S_p)$. We now show that up to isomorphism all canonical real dicompactifications of $(S, \mathcal{S}, \tau, \kappa)$ have this form.

Theorem 3.2. *Let $(S, \mathcal{S}, \tau, \kappa)$ be an almost plain completely biregular bi-T_2 ditopological texture space. Then every canonical real dicompactification is isomorphic to $(H_{\langle B\rangle}, \mathcal{H}_{\langle B\rangle}, \tau_{\langle B\rangle}, \kappa_{\langle B\rangle})$ for some bigenerating set $B \subseteq BA(S_p)$.*

Proof. Let $(V, \mathcal{V}, \nu, \mu)$ be a canonical real dicompactification of $(S, \mathcal{S}, \tau, \kappa)$. Since we are working up to an isomorphism there is no loss of generality in assuming that $(V, \mathcal{V}, \nu, \mu)$ is plain, since otherwise it may be replaced by $(V_p, \mathcal{V}_p, \nu_p, \mu_p)$. By hypothesis $S_p \subseteq V$, so we may define $B = \{\varphi|_{S_p} \mid \varphi \in BA(V)\}$. It is trivial to verify that B is a bigenerating sub-T-lattice of $BA(S_p)$, and we omit the details. It remains to show that $(H_B, \mathcal{H}_B, \tau_B, \kappa_B)$ and $(V, \mathcal{V}, \nu, \mu)$ are isomorphic as canonical real dicompactifications of $(S, \mathcal{S}, \tau, \kappa)$. We first note that the mapping $\alpha : BA(V) \to B$ defined by $\alpha(\varphi) = \varphi|_{S_p}$, $\varphi \in BA(V)$, is a T-lattice isomorphism. Indeed, it is clearly surjective and preserves the T-lattice operations, so it remains to show that it is injective. Suppose on the contrary that for some $\varphi, \psi \in BA(V)$ we have $\varphi|_{S_p} = \psi|_{S_p}$ but that $\varphi \neq \psi$. Now we have $v \in V$ with $\varphi(v) \neq \psi(v)$, and we may assume without loss of generality that $\varphi(v) < \psi(v)$ and take $r \in \mathbb{R}$ with $\varphi(v) < r < \psi(v)$. We now have $\varphi^{-1}(-\infty, r) \in \nu$, $\psi^{-1}(-\infty, r] \in \mu$ so $G = (\varphi^{-1}(-\infty, r)) \cap (V \setminus \psi^{-1}(-\infty, r])$ is an open set for the joint topology of (ν, μ) on V which is non-empty as it contains v. By the density of S_p in V under the joint topology there exists $s \in S_p \cap G$, which leads to the contradiction $\varphi(s) < \psi(s)$.

It is now straightforward to verify that the T-lattice isomorphism α sets up a **fDitop** isomorphism $\beta : (H_{BA(V)}, \mathcal{H}_{BA(V)}, \tau_{BA(V)}, \kappa_{BA(V)}) \to (H_B, \mathcal{H}_B, \tau_B, \kappa_B)$ defined by $\beta(h)(\alpha(\varphi)) = h(\varphi)$, $\varphi \in BA(V)$, and we again omit the details. Combining this with the mapping $\xi_{BA(V)} : V \to H_{BA(V)}$ which is surjective as $(V, \mathcal{V}, \nu, \mu)$ is real dicompact we obtain the **fDitop** isomorphism $\beta \circ \xi_{BA(V)} : (V, \mathcal{V}, \nu, \mu) \to (H_B, \mathcal{H}_B, \tau_B, \kappa_B)$. Finally, for $s \in S_p$ we clearly have $(\beta \circ \xi_{BA(V)})(s) = \xi_B(s)$, whence $\beta \circ \xi_{BA(V)}$ leaves S_p pointwise fixed and is therefore an isomorphism between the canonical real dicompactifications $(V, \mathcal{V}, \nu, \mu)$ and $(H_B, \mathcal{H}_B, \tau_B, \kappa_B)$, as required. $\square$

The real dicompactification $(H_{BA(S_p)}, \mathcal{H}_{BA(S_p)}, \tau_{BA(S_p)}, \kappa_{BA(S_p)})$ is special in that S_p is $\mathrm{BA}(S_p)$-embedded in $H_{BA(S_p)}$, that is, every element of $\mathrm{BA}(S_p)$ can be extended to an element of $\mathrm{BA}(H_{\mathrm{BA}(S_p)})$. Indeed, $\xi_{\mathrm{BA}(S_p)}$ is an isomorphism between $(S_p, \mathcal{S}_p, \tau_p, \kappa_p)$ and its image in $(H_{\mathrm{BA}(S_p)}, \mathcal{H}_{\mathrm{BA}(S_p)}, \tau_{\mathrm{BA}(S_p)}, \kappa_{\mathrm{BA}(S_p)})$ by [21, Proposition 2.17(1)], whence $\mu \in \mathrm{BA}(S_p) \mapsto \mu \circ \xi^{-1}_{\mathrm{BA}(S_p)} \in \mathrm{BA}(\xi_{\mathrm{BA}(S_p)}(S_p))$ is a T-lattice isomorphism. Hence $\mathrm{BA}(\xi_{\mathrm{BA}(S_p)}(S_p)) = \{\hat{\mu} \mid \mu \in \mathrm{BA}(S_p)\}$, $\hat{\mu}(\hat{s}) = \mu(s)$, and setting $\hat{\mu}(h) = h(\mu)$ for $h \in H_{\mathrm{BA}(S_p)}$ gives us the required extension $\hat{\mu}$ of μ to an element of $\mathrm{BA}(H_{\mathrm{BA}(S_p)})$, since $\hat{\mu}(\hat{s}) = \hat{s}(\mu) = \mu(s)$, for $s \in S_p$.

In view of the analogous property of the Hewitt realcompactification of a topological space [17], we shall refer to $(H_{BA(S_p)}, \mathcal{H}_{BA(S_p)}, \tau_{BA(S_p)}, \kappa_{BA(S_p)})$ as the *Hewitt real dicompactification* of $(S, \mathcal{S}, \tau, \kappa)$.

We will denote by $\mathbf{dfRdiComp_2}$ the category of real dicompact bi-T_2 spaces and bicontinuous difunctions, and by $\mathbf{dfApCbiR_2}$ that of almost plain completely biregular bi-T_2 spaces and bicontinuous difunctions.

Proposition 3.3. $\mathbf{dfRdiComp_2}$ *is a reflective subcategory of* $\mathbf{dfApCbiR_2}$.

Proof. Take $(S, \mathcal{S}, \tau, \kappa) \in \mathrm{Ob}\,\mathbf{dfApCbiR_2}$ and consider the Hewitt real dicompactification $(H_{BA(S_p)}, \mathcal{H}_{BA(S_p)}, \tau_{BA(S_p)}, \kappa_{BA(S_p)})$ and the morphism

$$(f_{\xi_{\mathrm{BA}(S_p)}}, F_{\xi_{\mathrm{BA}(S_p)}}) \circ (f_{\epsilon^S}, F_{\epsilon^S})^{\leftarrow} : (S, \mathcal{S}, \tau, \kappa) \to (H_{BA(S_p)}, \mathcal{H}_{BA(S_p)}, \tau_{BA(S_p)}, \kappa_{BA(S_p)}).$$

To show that this is a $\mathbf{dfRdiComp_2}$-reflection take $(V, \mathcal{V}, \tau_V, \kappa_V) \in \mathrm{Ob}\,\mathbf{dfRdiComp_2}$, a morphism $(f, F) : (S, \mathcal{S}, \tau, \kappa) \to (V, \mathcal{V}, \tau_V, \kappa_V)$, and consider the following diagram:

$$\begin{tikzcd}
S \arrow[r, bend left, "(f_{\epsilon^S}, F_{\epsilon^S})^{\leftarrow}"] \arrow[dr, "(f,F)"'] & S_p \arrow[l, bend left, "(f_{\epsilon^S}, F_{\epsilon^S})"] \arrow[rr, "(f_{\xi_{\mathrm{BA}(S_p)}}, F_{\xi_{\mathrm{BA}(S_p)}})"] & & H_{\mathrm{BA}(S_p)} \arrow[d, dashed, "(f_\theta, F_\theta)"] \\
& V \arrow[r, bend left, "(f_{\epsilon^V}, F_{\epsilon^V})^{\leftarrow}"] & V_p \arrow[l, bend left, "(f_{\epsilon^V}, F_{\epsilon^V})"] \arrow[r, bend right, "(f_{\xi_{\mathrm{BA}(V_p)}}, F_{\xi_{\mathrm{BA}(V_p)}})"'] & H_{\mathrm{BA}(V_p)} \arrow[l, bend right, "(f_{\xi^{-1}_{\mathrm{BA}(V_p)}}, F_{\xi^{-1}_{\mathrm{BA}(V_p)}})"']
\end{tikzcd}$$

Here we note that $\xi_{\mathrm{BA}(V_p)}$ maps onto $H_{\mathrm{BA}(V_p)}$ since $(V, \mathcal{V}, \tau_V, \kappa_V)$ is real dicompact, so in addition to $(f_{\epsilon^S}, F_{\epsilon^S})$ and $(f_{\epsilon^V}, F_{\epsilon^V})$, $(f_{\xi_{\mathrm{BA}(V_p)}}, F_{\xi_{\mathrm{BA}(V_p)}})$ is also a dihomeomorphism.

To prove the existence of a morphism from $(H_{\mathrm{BA}(S_p)}, \mathcal{H}_{\mathrm{BA}(S_p)}, \tau_{\mathrm{BA}(S_p)}, \kappa_{\mathrm{BA}(S_p)})$ to $(V, \mathcal{V}, \tau_V, \kappa_V)$ whose composition with $(f_{\xi_{\mathrm{BA}(S_p)}}, F_{\xi_{\mathrm{BA}(S_p)}}) \circ (f_{\epsilon^S}, F_{\epsilon^S})^{\leftarrow}$ is (f, F) it will be sufficient to prove the existence of an ω-preserving bicontinuous point function $\theta : H_{\mathrm{BA}(S_p)} \to H_{\mathrm{BA}(V_p)}$ for which the difunction (f_θ, F_θ) makes the diagram commutative.

Since S_p, V_p are plain, $(f_{\epsilon^V}, F_{\epsilon^V})^{\leftarrow} \circ (f, F) \circ (f_{\epsilon^S}, F_{\epsilon^S}) = (f_\delta, F_\delta)$ for some (unique) ω-preserving bicontinuous point function $\delta : S_p \to V_p$. For $\mu \in \mathrm{BA}(V_p)$ it is straightforward to verify that $\mu \circ \delta \in \mathrm{BA}(S_p)$. Hence we may define $\theta : H_{\mathrm{BA}(S_p)} \to H_{\mathrm{BA}(V_p)}$ by $\theta(h)(\mu) = h(\mu \circ \delta)$ for $h \in H_{\mathrm{BA}(S_p)}$ and $\mu \in \mathrm{BA}(V_p)$. The mapping θ is easily seen to be ω-preserving and bicontinuous, so it remains only to verify that the above diagram is commutative. It is clearly sufficient to show that $\delta = \xi^{-1}_{\mathrm{BA}(V_p)} \circ \theta \circ \xi_{\mathrm{BA}(S_p)}$. However, for $s \in S_p$ we have $\xi_{\mathrm{BA}(S_p)}(s) = \hat{s}$, $\theta(\hat{s}) = \widehat{\delta(s)}$ since for $\mu \in \mathrm{BA}(V_p)$ we have $\theta(\hat{s})(\mu) = \hat{s}(\mu \circ \delta) = \mu(\delta(s)) = \widehat{\delta(s)}(\mu)$, and finally $\xi^{-1}_{\mathrm{BA}(V_p)}(\widehat{\delta(s)}) = \delta(s)$. This verifies the existence of a suitable morphism from $(H_{\mathrm{BA}(S_p)}, \mathcal{H}_{\mathrm{BA}(S_p)}, \tau_{\mathrm{BA}(S_p)}, \kappa_{\mathrm{BA}(S_p)})$ to $(V, \mathcal{V}, \tau_V, \kappa_V)$.

To prove uniqueness, assume there is a second such morphism (g, G). We must show that

$$(g, G) = (f_{\epsilon^V}, F_{\epsilon^V}) \circ (f_{\xi^{-1}_{\mathrm{BA}(V_p)}}, F_{\xi^{-1}_{\mathrm{BA}(V_p)}}) \circ (f_\theta, F_\theta). \tag{3.1}$$

We note that $(g, G) = (f_\psi, F_\psi)$ for some ω-preserving bicontinuous point function $\psi : H_{\mathrm{BA}(S_p)} \to V$ since the domain is plain. Also, since the space $(V, \mathcal{V}, \tau_V, \kappa_V)$ is real dicompact it is nearly plain by [21, Proposition 2.9], whence $(f_{\epsilon^V}, F_{\epsilon^V})^{\leftarrow} = (f_{\varphi_p^V}, F_{\varphi_p^V})$ by Lemma 2.8. Consider the point functions $\varphi_p^V \circ \psi$ and $\varphi_p^V \circ \epsilon^V \circ \xi^{-1}_{\mathrm{BA}(V_p)} \circ \theta$ which map from $H_{\mathrm{BA}(S_p)}$ to V_p. To show these are equal, it will be sufficient to show their restrictions to $\xi_{\mathrm{BA}(S_p)}(S_p)$ are equal. Indeed, $\xi_{\mathrm{BA}(S_p)}(S_p)$ is dense in $H_{\mathrm{BA}(S_p)}$ under the joint topology, the given mappings are bicontinuous and $(V, \mathcal{V}, \tau_V, \kappa_V)$ is bi-T_2 so the conclusion follows much as in the classical case. However, commutativity of the diagram involving (g, G) leads to $\varphi_p^V \circ \psi \circ \xi_{\mathrm{BA}(S_p)} = \delta$ and a straightforward calculation shows that for $s \in S_p$,

$$\varphi_p^V \circ \psi(\hat{s}) = \delta(s) = \varphi_p^V \circ \epsilon^V \circ \xi^{-1}_{\mathrm{BA}(V_p)} \circ \theta(\hat{s}),$$

whence $\varphi_p^V \circ \psi = \varphi_p^V \circ \epsilon^V \circ \xi^{-1}_{\mathrm{BA}(V_p)} \circ \theta$. Passing to the corresponding difunctions gives

$$\begin{aligned}
(f_{\epsilon^V}, F_{\epsilon^V})^{\leftarrow} \circ (g, G) &= (f_{\epsilon^V}, F_{\epsilon^V})^{\leftarrow} \circ (f_{\epsilon^V}, F_{\epsilon^V}) \circ (f_{\xi^{-1}_{\mathrm{BA}(V_p)}}, F_{\xi^{-1}_{\mathrm{BA}(V_p)}}) \circ (f_\theta, F_\theta) \\
&= (f_{\xi^{-1}_{\mathrm{BA}(V_p)}}, F_{\xi^{-1}_{\mathrm{BA}(V_p)}}) \circ (f_\theta, F_\theta),
\end{aligned}$$

whence (3.1) is obtained by taking the composition of each side on the left with $(f_{\epsilon^V}, F_{\epsilon^V})$.

It will be appropriate to refer to the reflector given by this theorem as the *Hewitt-reflector*. □

Now let B, B' be bigenerating sub-T-lattices of $\mathrm{BA}(S_p)$ with $B \subseteq B'$ and B' a finite ρ_b-refinement of B, meaning effectively that every real bi-ideal in B has a unique extension to a real bi-ideal in B (see [2] or [21, Introduction]). For $h' \in H_{B'}$, define the mapping $\theta : H_{B'} \to H_B$ by $\theta(h') = h'|B$. Then:

Lemma 3.4. *The mapping θ is an ω-preserving bicontinuous bijection that preserves S_p in the sense that $\theta(\xi_{B'}(s)) = \xi_B(s)$, $s \in S_p$.*

Proof. Immediate from the definitions. □

Corollary 3.5. *$(H_{B'}, \mathcal{H}_{B'}, \tau_{B'}, \kappa_{B'})$ is projectively larger than $(H_B, \mathcal{H}_B, \tau_B, \kappa_B)$.*

Proof. Clearly, (f_θ, F_θ) has the properties required by Definition 2.12. □

In general we cannot assume θ^{-1} is ω-preserving. Indeed,

Lemma 3.6. *θ^{-1} is ω-preserving if and only if*

(i) $h'|B \leqslant k'|B \Rightarrow h' \leqslant k', \forall h', k' \in H_{B'}$.

Proof. Clear from the definitions. □

Proposition 3.7. *Suppose that $B \subseteq \mathrm{BA}^*(S)$ and that condition* (i) *holds. Then*

(ii) *(f_θ, F_θ) is a dihomeomorphism,*

whence $(H_{B'}, \mathcal{H}_{B'}, \tau_{B'}, \kappa_{B'})$ is isomorphic to $(H_B, \mathcal{H}_B, \tau_B, \kappa_B)$.

Proof. Under the given conditions (f_θ, F_θ) is a bijective bicontinuous difunction. Since $B \subseteq \mathrm{BA}^*(S_p)$ and $\mathrm{BA}^*(S_p)$ is finitely ρ_b-complete we have $B' \subseteq \mathrm{BA}^*(S_p)$. Hence $(H_{B'}, \mathcal{H}_{B'}, \tau_{B'}, \kappa_{B'})$ is dicompact, and since $(H_B, \mathcal{H}_B, \tau_B, \kappa_B)$ is bi-T_2 we see (f_θ, F_θ) is a dihomeomorphism by [10, Corollary 4.6]. □

Even if we assume (i), we cannot obtain (ii) without the additional assumption $B \subseteq \mathrm{BA}^*(S_p)$. To study the general case we will find it useful to consider the mapping $\nu : \mathrm{BA}(H_B) \to \mathrm{BA}(S_p)$ given by $\nu(\varphi) = \varphi \circ \xi_B$, $\varphi \in \mathrm{BA}(H_B)$. This is the analogue of the mapping with the same name defined in [2], and will be seen to have similar properties in this new setting.

Lemma 3.8.

(a) *ν is an injective T-lattice homomorphism.*
(b) *$\nu(\mathrm{BA}(H_B))$ is a finite ρ_b-refinement of B in $\mathrm{BA}(S_p)$.*

Proof. (a) For $\varphi \in \mathrm{BA}(H_B)$, $\varphi \circ \xi_B$ is certainly ω-preserving and bicontinuous, hence an element of $\mathrm{BA}(S_p)$. Thus, ν is well defined. It is trivial to verify that ν is a T-lattice homomorphism, and we omit the details. Suppose that for $\varphi, \psi \in \mathrm{BA}(H_B)$ we have $\nu(\varphi) = \nu(\psi)$ but $\varphi \neq \psi$. Then for some $h \in H_B$ we have $\varphi(h) \neq \psi(h)$, and without loss of generality we may assume $\varphi(h) < r < \psi(h)$ for some $r \in \mathbb{R}$. Now $h \in \varphi^{-1}[Q_r] \cap (H_B \setminus \psi^{-1}[P_r])$ is a non-empty open set for the joint topology of (τ_B, κ_B) on H_B, so since $\xi_B(S_p)$ is a dense subset for this topology by [21, Proposition 2.17(1)] there exists $s \in S_p$ with $\xi_B(s) \in \varphi^{-1}[Q_r] \cap (H_B \setminus \psi^{-1}[P_r])$. On the other hand $\varphi(\xi_B(s)) = \psi(\xi_B(s))$, which gives an immediate contradiction.

(b) Firstly $B \subseteq \nu(\mathrm{BA}(H_B))$, since for $b \in B$ the projection π_b belongs to $\mathrm{BA}(H_B)$ and clearly $b = \nu(\pi_b)$. Now let (L, M) be a real bi-ideal in B. Then $(L, M) = (L^h, M^h)$ where $h \in H_B$ is the B-resolution of (L, M). Corresponding to $h \in H_B$ we have $\hat{h} \in H_{\mathrm{BA}(H_B)}$ defined by $\hat{h}(\varphi) = \varphi(h)$ for all $\varphi \in \mathrm{BA}(H_B)$, and hence the real bi-ideal $(L^{\hat{h}}, M^{\hat{h}})$ in $\mathrm{BA}(H_B)$. By (i) it is easy to see that $\nu : \mathrm{BA}(H_B) \to \nu(\mathrm{BA}(H_B))$ is a T-lattice isomorphism and so $(\nu(L^{\hat{h}}), \nu(M^{\hat{h}}))$ is a real bi-ideal in $\nu(\mathrm{BA}(H_A))$. However it is easy to verify that

$$\nu\big(L^{\hat{h}}\big) \cap B = L^h \quad \text{and} \quad \nu\big(M^{\hat{h}}\big) \cap B = M^h$$

so $(\nu(L^{\hat{h}}), \nu(M^{\hat{h}}))$ is a real extension of (L, M) to $\nu(\mathrm{BA}(H_B))$. On the other hand if (L', M') is any real extension of (L, M) to $\nu(\mathrm{BA}(H_B))$ then $(\nu^{-1}[L'], \nu^{-1}[M'])$ is a real bi-ideal in $\mathrm{BA}(H_B)$. Since $H_{\mathrm{BA}(H_B)}$ is a real dicompact plain space and hence a $*$-space, $(\nu^{-1}[L'], \nu^{-1}[M'])$ is difixed by some $g \in H_B$. Thus $(\nu^{-1}[L'], \nu^{-1}[M']) = (L(g), M(g)) = (L^{\hat{g}}, M^{\hat{g}})$, and we deduce that $(L', M') = (\nu(L^{\hat{g}}), \nu(M^{\hat{g}}))$. On the other hand,

$$\big(L^h, M^h\big) = (L' \cap B, M' \cap B) = \big(\nu\big(L^{\hat{g}}\big) \cap B, \nu\big(M^{\hat{g}}\big) \cap B\big) = \big(L^g, M^g\big)$$

implies $h = g$ and hence $\hat{h} = \hat{g}$. Thus (L', M') is unique, and $\nu(\mathrm{BA}(H_B))$ is a finite ρ_b-refinement of B as required. □

Corollary 3.9. *If $(S, \mathcal{S}, \tau, \kappa)$ is a dicompact bi-T_2 space then $\mathrm{BA}(S) = \mathrm{BA}^*(S)$, the set of bounded elements of $\mathrm{BA}(S)$.*

Proof. A simple topological proof of this result has been given in [20, Theorem 4.2]. Here we give a structural proof based on properties of T-lattices. Firstly we note that it will suffice to show $\mathrm{BA}(S_p) = \mathrm{BA}^*(S_p)$. By the Lemma 3.8 applied to $B = \mathrm{BA}^*(S_p)$ we see that $\nu(\mathrm{BA}(H_{\mathrm{BA}^*(S_p)}))$ is a finite ρ_b-refinement of $\mathrm{BA}^*(S_p)$. However, $\mathrm{BA}^*(S_p)$ is finitely ρ_b-complete by [2, Propositions 3.1.9 and 3.3.1] so $\nu(\mathrm{BA}(H_{\mathrm{BA}^*(S_p)})) = \mathrm{BA}^*(S_p)$. It follows easily that $\mathrm{BA}(H_{\mathrm{BA}^*(S_p)}) = \mathrm{BA}^*(H_{\mathrm{BA}^*(S_p)})$. However, since $(S, \mathcal{S}, \tau, \kappa)$ is $\mathrm{BA}^*(S_p)$-real dicompact, $H_{\mathrm{BA}^*(S_p)}$ is isomorphic to S_p, so $\mathrm{BA}(S_p) = \mathrm{BA}^*(S_p)$ also. □

Note 3.10. The above results imply that the dicompactifications of $(S, \mathcal{S}, \tau, \kappa)$, that is the nearly plain extensions that are dicompact, are, up to isomorphism, precisely the spaces $(H_B, \mathcal{H}_B, \tau_B, \kappa_B)$, where B is a bigenerating sub-T-lattice of $\mathrm{BA}^*(S_p)$.

Clearly, if $(S, \mathcal{S}, \tau, \kappa)$ is dicompact then the dicompactifications of $(S, \mathcal{S}, \tau, \kappa)$ are **dfDitop**-isomorphic to $(S, \mathcal{S}, \tau, \kappa)$, and hence to each other.

Proposition 3.11. *Let B be a bigenerating sub-T-lattice of* $\mathrm{BA}(S_p)$ *and B' a finite ρ_b-refinement of B in* $\mathrm{BA}(S_p)$. *Then the following are equivalent*:

(iii) $B' \subseteq \nu(\mathrm{BA}(H_B))$.
(iv) $(L_{B'}, M_{B'})$ *is nearly prime for every real bi-ideal* (L, M) *in B.*

Here we recall from [2] *that a bi-ideal* (L, M) *is called nearly prime if* (L^+, M^+) *is prime, that is* L^+, M^+ *are prime as a lattice ideal, dual ideal respectively.*

Proof. (iii) $\Rightarrow$ (iv). If $B' \subseteq \nu(\mathrm{BA}(S_p))$ then $B \subseteq B' \subseteq \nu(\mathrm{BA}(S_p))$. Let $(L, M) = (L^h, M^h)$, $h \in H_B$, be a real bi-ideal in B and define $\hat{h} \in H_{\mathrm{BA}(H_B)}$ by $\hat{h}(\varphi) = \varphi(h)$, $\varphi \in \mathrm{BA}(H_B)$, as in the proof of Lemma 3.8(b). By the definition of the ditopology on H_B it is clear that $\pi_B = \{\pi_b \mid b \in B\}$ is a bigenerating subset of $\mathrm{BA}(H_B)$, and it is not difficult to check that π_B is actually a sub-T-lattice of $\mathrm{BA}(H_B)$. Since $(H_B, \mathcal{H}_B)$ is plain we have $h\, \omega_{H_B}\, h$ and so by [21, Proposition 2.4] we have

$$\big(L(h), M(h)\big) \preccurlyeq \big(\big[L^{\hat{h}|_{\pi_B}}_{\mathrm{BA}(H_B)}\big]^+, \big[M^{\hat{h}|_{\pi_B}}_{\mathrm{BA}(H_B)}\big]^+\big). \tag{3.2}$$

Since a plain space is a $*$-space and π_B is a T-lattice we have

$$L^{\hat{h}|_{\pi_B}}_{\mathrm{BA}(H_B)} = \big(L^{\hat{h}|_{\pi_B}}\big)_{\mathrm{BA}(H_B)} = \big\{\varphi \in \mathrm{BA}(H_B) \bigm| \exists \pi_b \in L^{\hat{h}|_{\pi_B}},\ r > 0 \text{ with } \varphi \wedge r \leqslant \pi_b\big\},$$

while $\hat{h}|_{\pi_B} \in H_{\pi_B}$, so

$$L^{\hat{h}|_{\pi_B}} = \big\{\pi_b \bigm| b \in B,\ \hat{h}(\pi_b) \leqslant 0\big\} = \big\{\pi_b \bigm| b \in B,\ h(b) \leqslant 0\big\}. \tag{3.3}$$

Hence if $\varphi \in L^{\hat{h}|_{\pi_B}}_{\mathrm{BA}(H_B)}$ we have $b \in B$ with $h(b) \leqslant 0$ and $r > 0$ with $\varphi \wedge r \leqslant \pi_b$, whence $\varphi(h) \wedge r \leqslant \pi_b(h) = h(b) \leqslant 0$, so $\varphi(h) \leqslant 0$ and $\varphi \in L(h)$. Hence $L^{\hat{h}|_{\pi_B}}_{\mathrm{BA}(H_B)} \subseteq L(h)$, and a dual proof which we omit establishes $M^{\hat{h}|_{\pi_B}}_{\mathrm{BA}(H_B)} \subseteq M(h)$, so

$$\big(L^{\hat{h}|_{\pi_B}}_{\mathrm{BA}(H_B)}, M^{\hat{h}|_{\pi_B}}_{\mathrm{BA}(H_B)}\big) \preccurlyeq \big(L(h), M(h)\big). \tag{3.4}$$

From (3.2) and (3.4) we deduce $([L^{\hat{h}|_{\pi_B}}_{\mathrm{BA}(H_B)}]^+, [M^{\hat{h}|_{\pi_B}}_{\mathrm{BA}(H_B)}]^+) = (L(h), M(h))$ since $(L(h), M(h))$ is real and hence maximal ρ_b-regular, so $(L^{\hat{h}|_{\pi_B}}_{\mathrm{BA}(H_B)}, M^{\hat{h}|_{\pi_B}}_{\mathrm{BA}(H_B)})$ is nearly total by [2, Theorem 3.1.2], and hence nearly prime.

Take $g \in B' \subseteq \nu(\mathrm{BA}(H_B))$. Then $g = \nu(\varphi)$ for some $\varphi \in \mathrm{BA}(H_B)$ and $T_\epsilon(\varphi) \in L^{\hat{h}|_{\pi_B}}_{\mathrm{BA}(H_B)}$, $\forall \epsilon > 0$ or $T_{-\epsilon}(\varphi) \in M^{\hat{h}|_{\pi_B}}_{\mathrm{BA}(H_B)}$, $\forall \epsilon > 0$. We assume the first case, so given $\epsilon > 0$ we have as above $b \in B$ and $r > 0$ with $T_\epsilon(\varphi) \wedge r \leqslant \pi_b$, and $h(b) \leqslant 0$ by (3.3). Then $T_\epsilon(g) \wedge r \leqslant b$, and $b \in L^h = L$ so $T_\epsilon(g) \in L_{B'}$ for all $\epsilon > 0$. We deduce that $(L_{B'}, M_{B'})$ is nearly total, and hence nearly prime by [2, Proposition 3.1.3]. The second case leads to the same conclusion, so (ii) is established.

(iv) $\Rightarrow$ (iii). Since B' is a finite ρ_b-refinement of B it is easy to see that the mapping $h' \mapsto h'|_B$, $h' \in H_{B'}$ is a bijection from $H_{B'}$ to H_B. Take $b' \in B'$. Then we may define $\varphi : H_B \to \mathbb{R}$ by $\varphi(h'|_B) = h'(b')$, $h' \in H_{B'}$. We prove $\varphi \in \mathrm{BA}(H_B)$, from which we have $b' = \nu(\varphi) \in \nu(\mathrm{BA}(H_B))$ by the definition of φ.

Suppose that for $r \in \mathbb{R}$ and $h = h'|_B \in H_B$, $h' \in H_{B'}$, we have $\varphi^{\leftarrow} Q_r \nsubseteq Q_h$. Then $\varphi(h) < r$ and we may set $\epsilon = r - \varphi(h) > 0$. Since (L^h, M^h) is real it has a unique maximal ρ_b-regular extension to B' and so $((L^h)_{B'}, (M^h)_{B'})$ is ρ_b-outer prime. It is also clearly finite, while by hypothesis it is nearly prime, so by [2, Proposition 3.1.7, Corollary 3], $([(L^h)_{B'}]^+, [(L^h)_{B'}]^+)$ is real. It follows that $([(L^h)_{B'}]^+, [(L^h)_{B'}]^+) = (L^{h'}, M^{h'})$, whence $T_{h'(b') + \frac{\epsilon}{2}} \in (L^h)_{B'}$. Hence for some $0 < \delta < \frac{\epsilon}{2}$ and $\psi \in L^h$ we have

$$T_{h'(b') + \frac{\epsilon}{2}}(b') \wedge \delta \leqslant \psi.$$

We deduce at once that

$$\pi_\psi^{\leftarrow} Q_{\pi_\psi(h) + \delta} \subseteq \varphi^{\leftarrow} Q_r \quad \text{and} \quad \pi_\psi^{\leftarrow} Q_{\pi_\psi(h) + \delta} \nsubseteq Q_h,$$

whence φ is continuous as $\pi_\psi^{\leftarrow} Q_{\pi_\psi(h)+\delta} \in \tau_B$. A dual proof shows that φ is co-continuous, and the proof is complete. □

We recall from [2] the following definition:

Definition 3.12. ([2, Definition 3.3.2]) If A is a T-lattice and B, B' sub-T-lattices of A then B' is called a *finite* ρ_b*-prime-refinement of* B if it is a finite ρ_b-refinement and $(L_{B'}, M_{B'})$ is nearly prime in B' for every real bi-ideal (L, M) in B.

The relation of being a finite ρ_b-prime-refinement is easily seen to be transitive, whence we have the following corollary to Lemma 3.8.

Corollary 3.13. *If* $B \subseteq \mathrm{BA}(S_p)$ *is a bigenerating sub-*T*-lattice then* $\nu(\mathrm{BA}(H_B))$ *is the finite* ρ_b*-prime-completion of* B.

We also have

Corollary 3.14. *Properties* (iii) *and* (iv) *of Proposition* 3.11 *are each equivalent to property* (ii) *of Proposition* 3.7.

Proof. If θ is a **fPDitop**-isomorphism then $\mathrm{BA}(H_B)$ and $\mathrm{BA}(H_{B'})$ are isomorphic T-lattices under the correspondence $\varphi \leftrightarrow \varphi'$ defined by $\varphi(h) = \varphi'(h')$ such that $h'|_B = h$. It follows at once that $B' \subseteq \nu'(\mathrm{BA}(H_{B'})) = \nu(\mathrm{BA}(H_B))$, using an obvious notation.

To establish the converse it will clearly suffice to show $(H_B, \mathcal{H}_B, \tau_B, \kappa_B)$ is **fPDitop**-isomorphic to $(H_{B''}, \mathcal{H}_{B''}, \tau_{B''}, \kappa_{B''})$ for the case $B'' = \nu(\mathrm{BA}(H_B))$. Now by Lemma 3.8(b) and Proposition 3.11, applied to B'' in place of B we know that $\nu''(\mathrm{BA}(H_{B''}))$ is a finite ρ_b-prime-refinement of B''. However, $\nu(\mathrm{BA}(H_B))$ is finitely ρ_b-prime-complete so $\nu(\mathrm{BA}(H_B)) = \nu''(\mathrm{BA}(H_{B''}))$, and $\mathrm{BA}(H_B)$ and $\mathrm{BA}(H_{B''})$ are isomorphic T-lattices under the correspondence $\varphi \leftrightarrow \varphi''$ defined by $\varphi(h) = \varphi''(h'')$. It follows that $h \leftrightarrow h''$ is a **fPDitop**-isomorphism, as required. □

Note 3.15. The above results show that the canonical real dicompactifications of $(S, \mathcal{S}, \tau, \kappa)$ are in one-to-one correspondence with the sub-T-lattices $\nu(\mathrm{BA}(H_B))$ of $\mathrm{BA}(S_p)$ for B a bigenerating sub-T-lattice of $\mathrm{BA}(S_p)$. Moreover, these are characterized internally amongst the bigenerating sub-T-lattices of $\mathrm{BA}(S_p)$ by the requirement that they be finitely ρ_b-prime-complete.

It will be observed that if $(S, \mathcal{S}, \tau, \kappa)$ is real dicompact, then the bigenerating sub-T-lattices of $\mathrm{BA}(S_p)$ may themselves be characterized in terms of the internal T-lattice structure of $\mathrm{BA}(S_p)$, one such characterization being obtained explicitly by applying [21, Proposition 2.4] to the **fPDitop**-isomorphic space $(H_{\mathrm{BA}(S_p)}, \mathcal{H}_{\mathrm{BA}(S_p)}, \tau_{\mathrm{BA}(S_p)}, \kappa_{\mathrm{BA}(S_p)})$. In this case all the canonical real dicompactifications of $(S, \mathcal{S}, \tau, \kappa)$, including the space itself, of course, can be obtained up to isomorphism from the T-lattice structure of $\mathrm{BA}(S_p)$.

In case B is a bigenerating sub-T-lattice of $\mathrm{BA}^*(S_p)$, we see that the properties (i), (ii), (iii) and (iv) are mutually equivalent.

By the above analysis the family of all canonical real dicompactifications of $(S, \mathcal{S}, \tau, \kappa)$ is seen to be in one-to-one correspondence with the set $\mathcal{B}$ of bigenerating finitely ρ_b-prime-complete sub-T-lattices of $\mathrm{BA}(S_p)$. We shall also set $\mathcal{B}^* = \{A \in \mathcal{B} \mid A \subseteq \mathrm{BA}^*(S_p)\}$, and for a finitely ρ_b-complete $C \in \mathcal{B}$ we let $\mathcal{B}_C = \{B \in \mathcal{B} \mid C \text{ is a finite } \rho_b\text{-completion of } B\}$. The sets $\mathcal{B}_C$ form a partition of $\mathcal{B}$, and $\mathcal{B}$ is an upper semi-lattice. Likewise, $\mathcal{B}^*$ and each $\mathcal{B}_C$ are upper sub-semi-lattices of $\mathcal{B}$. Thus, the canonical dicompactifications of $(S, \mathcal{S}, \tau, \kappa)$ are in one-to-one correspondence with the elements of $\mathcal{B}^*$.

Lemma 3.16. *For* $B, B' \in \mathcal{B}$ *we have* $B \subseteq B'$ *if and only if there exists an* ω*-preserving bicontinuous point function* $\varphi : (H_{B'}, \mathcal{H}_{B'}, \tau_{B'}, \kappa_{B'}) \to (H_B, \mathcal{H}_B, \tau_B, \kappa_B)$ *that preserves the points of* S_p.

Proof. If $B \subseteq B'$ the required function is clearly $\varphi(h') = h'|B$.

Conversely, let φ have the stated properties and for $\mu \in B$ define $\mu' : H_{B'} \to \mathbb{R}$ by

$$\mu'(h') = \pi_\mu\big(\varphi(h')\big), \quad h' \in H_{B'}.$$

Since μ' is the composition of ω-preserving bicontinuous mappings it is itself ω-preserving and bicontinuous, that is $\mu' \in \mathrm{BA}(H_{B'})$. On the other hand, since φ preserves S_p we may immediately verify $\mu = \nu'(\mu') \in \nu'(\mathrm{BA}(H_{B'})) = B'$. Hence $B \subseteq B'$, as required. □

The mapping $\varphi : H_{B'} \to H_B$ defined as above when $B \subseteq B'$ need not be onto, so in particular $(H_{B'}, \mathcal{H}_{B'}, \tau_{B'}, \kappa_{B'})$ may not be projectively larger than $(H_B, \mathcal{H}_B, \tau_B, \kappa_B)$. Three special cases where we can be more specific are covered in the following corollaries.

Corollary 3.17. *For* $B, B' \in \mathcal{B}^*$ *the mapping* φ *is surjective. Hence the ordering in* $\mathcal{B}^*$ *reflects the projective ordering of the corresponding dicompactifications.*

Proof. For $h \in H_B$, $(L^h_{B'}, M^h_{B'})$ has a (not necessarily unique) maximal ρ_b-regular refinement in B'. Since $B' \subseteq \mathrm{BA}^*(S_p)$, such a refinement has the form $(L^{h'}, M^{h'})$ for some $h' \in H_{B'}$, and clearly $h'|B = h$. □

Under the projective ordering the largest dicompactification is $(H_{\mathrm{BA}^*(S_p)}, \mathcal{H}_{\mathrm{BA}^*(S_p)}, \tau_{\mathrm{BA}^*(S_p)}, \kappa_{\mathrm{BA}^*(S_p)})$, which is also the dicompactification in which S_p is $\mathrm{BA}^*(S_p)$-embedded. It is natural to refer to this as the *Stone–Čech dicompactification of* $(S, \mathcal{S}, \tau, \kappa)$.

As in [20] we will denote by $\mathbf{dfDicomp_2}$ the category of dicompact bi-T_2 spaces and bicontinuous difunctions.

Proposition 3.18. $\mathbf{dfDicomp_2}$ *is a reflective subcategory of* $\mathbf{dfApCbiR_2}$.

Proof. The proof is essentially similar to that of Proposition 3.3, but uses the Stone–Čech dicompactification instead of the Hewitt real dicompactification to produce the *Stone–Čech-reflector*. □

Corollary 3.19. *If* $B, B' \in \mathcal{B}_C$ *then* φ *is bijective.*

In the classical case $\mathcal{B}_C$ contains only one element, as may be verified by applying (i) to the space $(X, \mathcal{P}(X), \mathcal{T}, \mathcal{T}^c)$. In the general case, however, it seems likely that this set could contain more than one element.

Corollary 3.20. *Let* $B \in \mathcal{B}$ *and* $B' = B \cap \mathrm{BA}^*(S_p)$. *Then* $B' \in \mathcal{B}^*$, *and the dicompactification* $(H_{B'}, \mathcal{H}_{B'}, \tau_{B'}, \kappa_{B'})$ *is injectively larger than the real dicompactification* $(H_B, \mathcal{H}_B, \tau_B, \kappa_B)$. *In particular, the Hewitt real dicompactification is the injectively smallest among the real dicompactifications that are injectively smaller than the Stone–Čech dicompactification of* $(S, \mathcal{S}, \tau, \kappa)$.

Proof. Left to the interested reader. □

Corollary 3.21. *Under the hypotheses of Corollary* 3.20, $(H_{B'}, \mathcal{H}_{B'}, \tau_{B'}, \kappa_{B'})$ *is the Stone–Čech dicompactification of* $(H_B, \mathcal{H}_B, \tau_B, \kappa_B)$.

In particular, just as in the classical case $(H_{\mathrm{BA}^*(S_p)}, \mathcal{H}_{\mathrm{BA}^*(S_p)}, \tau_{\mathrm{BA}^*(S_p)}, \kappa_{\mathrm{BA}^*(S_p)})$ is the Stone–Čech dicompactification of the Hewitt real dicompactification $(H_{\mathrm{BA}(S_p)}, \mathcal{H}_{\mathrm{BA}(S_p)}, \tau_{\mathrm{BA}(S_p)}, \kappa_{\mathrm{BA}(S_p)})$ of $(S, \mathcal{S}, \tau, \kappa)$.

A detailed analysis of the relation between the theory of (real) dicompactifications as given here and the bitopological and topological case, and also its implications for the theory of topologies on Hutton algebras is planned for a future paper.

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