Real dicompact textures
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A notion of real compactness for completely biregular bi-$T_2$ ditopological texture spaces is defined and studied under the name real dicompactness. In particular it is shown that real dicompact spaces are nearly plain $s^*$-spaces, and an important characterization is presented. Finally the connection of this work with topological and bitopological real compactness is discussed in a categorical setting.

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1. Introduction

The interplay between various topological properties of a space $X$ and certain types of ideal in the ring $C(X)$ of continuous real-valued functions on $X$ is well known [15], and is intimately involved in the definition and study of real compactness. When seeking to establish a corresponding theory for bitopological spaces, the second author considered in [2, Chapter 3] the set $P(X)$ of pairwise continuous functions from a bitopological space $(X, u, v)$ in the sense of Kelly [17] to the real bitopological space $(\mathbb{R}, s, t)$, where $s = \{(-\infty, r) | r \in \mathbb{R}\} \cup \{\mathbb{R}, \emptyset\}$ and $t = \{(r, \infty) | r \in \mathbb{R}\} \cup \{\mathbb{R}, \emptyset\}$. Unlike $C(X)$ this is not a ring, but an additive lattice, and this property of $P(X)$ was further abstracted to the notion of a $T$-lattice, that is a distributive lattice $A$ with distinguished element 0 and a suitable family of translations $T_r : A \rightarrow A, r \in \mathbb{R}$. In the theory developed in [2] the role of the ring ideals is played by the bi-ideals, pairs $(L, M)$ consisting of a lattice ideal $L$ and a dual lattice ideal $M$ satisfying $0 \in L \cap M$. Various notions of regularity for bi-ideals were introduced, including a notion of real bi-ideal, and these were used to define bireal compactness. A characterization of bireal compactness was presented which shows that this notion coincides with the bitopological real compactness considered by Brümmer and Salbany in [10], and several other properties of this class of spaces were investigated.

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The aim of this paper is to carry over the concepts and some of the results given in [2] to the much more general framework of ditopological texture spaces. As in the authors’ recent paper [23] on dicompact spaces we work within the same general class of completely biregular bi-$T_2$ spaces, and this enables us to use many of the notions and results given in that paper. Indeed, the layout of [23] was designed explicitly to provide the foundation necessary for the study of real compactness in a ditopological setting, as well as presenting important results on the more specific class of dicompact spaces. In particular the study in [23] highlights the importance for dicompact bi-$T_2$ spaces of the notion of a nearly plain texture $(S, \bar{S}, \tau, \kappa)$ and its associated plain space $(S_p, S_p', \tau_p, \kappa_p)$, and also of $*$-spaces, the $T$-lattices $BA(S)$ and $BA(S_p)$, and the notion of a bi-ideal being difixed. All of these are of equal importance in the present study.

The layout of this paper is as follows. The remainder of this introduction is given over to some background material. No attempt is made at completeness, our aim being to give just enough material to enable a casual reader to gain a general idea of the contents of the paper, although an exception is made for the required material on bi-ideals from [2], since this is not currently available as a paper. Section 2 gives the definition of $B$-real dicompactness for a bigenerating subset $B$ of $BA(S)$ and various fundamental results are given, including a characterization in terms of powers of the real texture $(\mathbb{R}, \mathbb{R}, \tau_\mathbb{R}, \kappa_\mathbb{R})$. Finally, Section 3 gives various results of a categorical nature. In particular, the relation between the ditopological theory presented here and the bitopological study in [2] is investigated in some detail.

An earlier version of the main results given in Section 2, and the Hewitt Isomorphism Theorem from Section 3, occur in the PhD thesis of the first author [22], written with the partial support of Grant Number 06 T03 604005 awarded by Hacettepe University.

**Ditopological texture spaces**

There is now a considerable literature on the theory of ditopological texture spaces, and an adequate introduction to this theory and the motivation for its study may be obtained from [4–8].

Briefly, if $S$ is a set, a **texturing** $\mathcal{S}$ of $S$ is a subset of $P(S)$ which is a point-separating, complete, completely distributive lattice containing $S$ and $\emptyset$, and for which meet coincides with intersection and finite joins with union. The pair $(S, \mathcal{S})$ is then called a **texture**. We regard a texture $(S, \mathcal{S})$ as a framework in which to do mathematics.

For a texture $(S, \mathcal{S})$, most properties are conveniently defined in terms of the $p$-sets $P_s = \bigcap\{A \in \mathcal{S} \mid s \in A\}$ and the $q$-sets, $Q_s = \sqrt{\{A \in \mathcal{S} \mid s \notin A\}}$. However, as noted in [3] we may associate with $(S, \mathcal{S})$ the C-space (core-space) $[11–13,16,18]$ $(S, \mathcal{S}^c)$, and then the frequently occurring relationship $Q_s \subseteq Q_{s'}$, $s, s' \in S$, is equivalent to $s \mathcal{S} s'$, where $s \mathcal{S}$ is the interior relation for $(S, \mathcal{S})$. In this paper we will use whichever notation seems to be the more convenient in each particular instance.

In general a texturing $\mathcal{S}$ need not be closed under the operation of taking the set complement, so in the context of a texture $(S, \mathcal{S})$ the notion of topology is replaced by that of dichotomous topology. A **dichotomous topology**, or ditopology for short, on a texture $(S, \mathcal{S})$ is a pair $(\tau, \kappa)$ of subsets of $\mathcal{S}$, where the set of open sets $\tau$ satisfies

1. $S, \emptyset \in \tau$,
2. $G_1, G_2 \in \tau \Rightarrow G_1 \cap G_2 \in \tau$, and
3. $G_i \in \tau$, $i \in I \Rightarrow \bigvee_i G_i \in \tau$,

and the set of closed sets $\kappa$ satisfies

1. $S, \emptyset \in \kappa$,
2. $K_1, K_2 \in \kappa \Rightarrow K_1 \cup K_2 \in \kappa$, and
3. $K_i \in \kappa$, $i \in I \Rightarrow \bigcap_i K_i \in \kappa$.

Hence a ditopology is essentially a “topology” for which there is no a priori relation between the open and closed sets. As will be clear from the references given above, ditopological texture spaces provide a unified setting for the study of topology, bitopology and fuzzy topology. We will not be concerned with the links with fuzzy topology in this paper, but the relation with bitopology will be considered in some detail in the final section. See also [20] in this context.

We recall the product of textures and of ditopological texture spaces. Let $(S_j, \mathcal{S}_j)$, $j \in J$, be textures and $S = \prod_{j \in J} S_j$. If $A_k \in \mathcal{S}_k$ for some $k \in J$ we write

$$E(k, A_k) = \prod_{j \in J} Y_j \quad \text{where} \quad Y_j = \begin{cases} A_j, & \text{if } j = k, \\ S_j, & \text{otherwise.} \end{cases}$$

Then the **product texturing** $\mathcal{S} = \bigotimes_{j \in J} \mathcal{S}_j$ of $S$ consists of arbitrary intersections of elements of the set

$$\mathcal{E} = \bigcap_{j \in J} \{E(j, A_j) \mid A_j \in \mathcal{S}_j \text{ for } j \in J\}.$$

Let $(S_j, \mathcal{S}_j)$, $j \in J$ be textures and $(S, \mathcal{S})$ their product. Then for $s = (s_j) \in S$, $s \mathcal{S} s'$
In case \((\tau_j, \kappa_j)\) is a ditopology on \((S_j, S_j^c)\), \(j \in J\), the \textit{product ditopology} on the product texture \((S, S)\) has subbase \(\{E(j, G) \mid G \in \tau_j, j \in J\}\), cosubbase \(\gamma' = \{E(j, K) \mid K \in \kappa_j, j \in J\}\).

We recall from [8, Theorem 4.17] that a ditopological space \((S, S, \tau, \kappa)\) is bi-\(T_2\) if given \(s, s' \in S\) with \(Q_s \not\subseteq Q_{s'}\) there exists \(H \in \tau, K \in \kappa\) with \(H \subset K, P_s \not\subseteq K\) and \(H \not\subseteq Q_{s'}\). This is the form of the Hausdorff property considered in [8], and it arises naturally in various contexts such as that of separated di-uniformities and of dimetries [19].

Various special classes of textures have been considered. Here we will be concerned primarily with plain textures and nearly plain textures. The texture \((S, S)\) is \textit{plain} if \(S\) is closed under arbitrary unions, equivalently if the corresponding C-space is an Alexandroff-discrete [9] or \(A\)-space [12], or if the interior relation \(\omega_S\) is reflexive. The more general class of nearly plain textures was introduced in [23]. The texture \((S, S)\) is \textit{nearly plain} if given \(s \in S\) there exists a point \(w \in S\) satisfying \(Q_s = Q_w\) and \(\omega_S(w) = w\) gives a mapping from \(S\) to the set \(S_p\) of plain points. The texture \(S\) on \(S\) induces a plain texture \(S_p\) on \(S_p\), and if \((\tau, \kappa)\) is a ditopology on \((S, S)\) we obtain the induced ditopology \((\tau_p, \kappa_p)\) on \((S_p, S_p)\). The plain ditopological space \((S_p, S_p, \tau_p, \kappa_p)\) will play an important role in this paper as it does in [23], and the reader is referred to that paper for a detailed discussion of the relation between the spaces \((S, S, \tau, \kappa)\) and \((S_p, S_p, \tau_p, \kappa_p)\). We recall from [23] the \textit{joint topology} \(\mathcal{J}_{\tau\kappa}\) on \(S_p\), which is defined in terms of its family \(J_{\tau\kappa}^p\) of closed sets by the condition

\[
W \in J_{\tau\kappa}^p \iff (s \in S_p, G \in \eta^*(s), K \in \mu^*(s) \Rightarrow G \cap W \not\subseteq K) \Rightarrow s \in W,
\]

where

\[
\eta^*(s) = \{A \in S \mid \exists G_k \in \tau \text{ with } G_k \not\subseteq Q_s, 1 \leq k \leq n \text{ and } G_1 \cap \cdots \cap G_n \subseteq A\},
\]

and

\[
\mu^*(s) = \{A \in S \mid \exists G_k \in \tau \text{ with } G_k \not\subseteq F_k, 1 \leq k \leq n \text{ and } A \subseteq F_1 \cup \cdots \cup F_n\}.
\]

Nearly plain textures share with plain textures the property that a difunction \([6, \text{Definition 2.22}]\) between them may be represented by an \(\omega\)-preserving point function between their base sets [23, Theorem 2.10]. Here \(\varphi : S \to T\) is \(\omega\)-preserving as a function from \((S, S)\) to \((T, T)\) if \(s_1 \omega_S s_2 \Rightarrow \varphi(s_1) \omega_T \varphi(s_2)\) (often referred to as “condition (a)” in earlier papers). Note that this condition does not guarantee \(\varphi^{-1}[B] \in \mathcal{S}\) for \(B \in \mathcal{T}\), so the inverse image \(\varphi^{-1}[B]\), inherited from the inverse image for the corresponding difunction is used in its place. Hence \(\varphi : (S, S, \tau_S, \kappa_S) \to (T, T, \tau_T, \kappa_T)\) is \textit{bicontinuous} if \(G \in \tau_T \Rightarrow \varphi^{-1}[G] \in \tau_S\) and \(K \in \kappa_T \Rightarrow \varphi^{-1}[K] \in \kappa_S\). We note that when these textures are plain we have \(\varphi^{-1}[B] = \varphi^{-1}[B]\), and we will then use whichever notation seems the most appropriate in a given situation.

It will transpire that, as for dicom pact \(a\)-\(T_2\) spaces, real dicom pact \(a\)-\(T_2\) spaces are nearly plain, so it suffices to consider \(\omega\)-preserving point functions in place of difunctions. Such functions are also of interest in a wider context and in particular we recall from [23] the following characterization of complete biregularity [8].

\textbf{Proposition 1.1.} Let \((S, S, \tau, \kappa)\) be a ditopological texture space. Then \((\tau, \kappa)\) is completely biregular if and only if the following conditions hold:

1. Given \(G \in \tau, a \in S\) with \(G \not\subseteq Q_a\) there exists an \(\omega\)-preserving bicontinuous point function \(\varphi : (S, S, \tau, \kappa) \to (\mathbb{R}, \mathbb{R}, \tau_R, \kappa_R)\) satisfying \(\varphi^{-1}[1] \subseteq \varphi \subseteq 1\) and for which \(P \subseteq \varphi^{-1}[P] \subseteq G\).
2. Given \(K \in \kappa, a \in S\) with \(P \not\subseteq K\) there exists an \(\omega\)-preserving bicontinuous point function \(\varphi : (S, S, \tau, \kappa) \to (\mathbb{R}, \mathbb{R}, \tau_R, \kappa_R)\) satisfying \(\varphi^{-1}[K] \subseteq \varphi \subseteq 1\) and for which \(\varphi^{-1}[Q] \subseteq K\) and \(K \subseteq \varphi^{-1}[P]\).

Our attention will be focused on the set \(BA(S)\) of bicontinuous \(\omega\)-preserving point functions from \((S, S, \tau, \kappa)\) to the real ditopological texture space \((\mathbb{R}, \mathbb{R}, \tau_R, \kappa_R)\) which is defined by \(\mathbb{R} = [(\infty, r) \mid r \in \mathbb{R}] \cup [(r, \infty) \mid r \in \mathbb{R}] \cup \mathbb{R} \cup \emptyset\), \(\tau_R = [(\infty, r) \mid r \in \mathbb{R}] \cup \mathbb{R} \cup \emptyset\) and \(\mathbb{R} = \{(\infty, r) \mid r \in \mathbb{R} \cup \mathbb{R}\}\). This is a \(T\)-space over \(T_r(\varphi) = \varphi + (-r)\), where \(r\) denotes the constant function with value \(r \in \mathbb{R}\). This will be the \(T\)-space we use in place of \(P(X)\), although we will show that it is equivalent, as far as real compactness is concerned, to consider \(BA(S_p)\) instead.

For \(\varphi \in BA(S)\) we recall from [23] the functions \(\varphi^*_a, \varphi^*_s \in BA(S)\) given by

\[
\varphi_a(s) = \sup \{\varphi(v) \mid v \omega_S s\}, \quad \varphi^*_a(s) = \inf \{\varphi(u) \mid s \omega_S u\}, \quad \forall s \in S.
\]

In general we have \(\varphi_a \leq \varphi \leq \varphi^*_a\). We note also the equalities

\[
(\varphi \lor \psi)_a = \varphi_a \lor \psi_a \quad \text{and} \quad (\varphi \land \psi)_a = \varphi^*_a \land \psi^*_a, \quad \forall \varphi, \psi \in BA(S),
\]

\textbf{References:}

the inequalities
\[ \varphi^* \lor \psi^* \leq (\varphi \lor \psi)^* \quad \text{and} \quad (\varphi \land \psi)_* \leq \varphi_* \land \psi_* , \quad \forall \varphi, \psi \in \text{BA}(S), \] (1.3)
with equality if \( \psi \) is constant, and the relation
\[ \varphi_* \leq \psi \iff \varphi \leq \psi^* , \quad \forall \varphi, \psi \in \text{BA}(S) \] (1.4)
for future reference. We note that \((S, \emptyset, \tau, \kappa)\) is called a \(\ast\)-space [23] if \(\varphi_* = \varphi^*\) for all \(\varphi \in \text{BA}(S)\). A plain ditopological space is clearly a \(\ast\)-space, so for a nearly plain space \((S, \emptyset, \tau, \kappa), (S_p, \emptyset, \tau_p, \kappa_p)\) is always a \(\ast\)-space.

Finally, we recall the basic constructs \texttt{fTex} and \texttt{fdItop} from [6,7], respectively.

Bi-ideals in \(T\)-lattices

We will require a little more of the theory of bi-ideals in \(T\)-lattices from [2] than is given in [23].

We recall that a distributive lattice \([14] A\) with a distinguished element \(0\) is called a \(T\)-\textit{lattice} if there exists a mapping \(T : \mathbb{R} \times A \to A\) given by \(T_r(a) = T(r, a) \forall a \in A, r \in \mathbb{R}\), satisfy:

(i) \(T_r : A \to A\) is a lattice homomorphism for each \(r \in \mathbb{R}\).
(ii) \(T_r \circ T_s = T_{s \circ r} = T_{r+s} \) for all \(r, s \in \mathbb{R}\).
(iii) \(T_r(a) = a \iff r = 0\), for all \(a \in A\).
(iv) \(T_r(a) \leq a \) for all \(a \in A\) and \(r > 0\).

We note in particular that the mapping \(r \mapsto T_{-r}(0)\) is an injection of \(\mathbb{R}\) into \(A\) which takes \(0 \in \mathbb{R}\) to the distinguished element \(0 \in A\). For this reason, \(T_{-r}(0)\) may be denoted by \(r\).

Definition 1.2.

(1) The binary relation \(\rho\) on \(A\) is a \textit{dispersion} if for all \(a, b, a', b' \in A\) it satisfies:
   (i) \(a \rho b, a \leq a' \text{ and } b \geq b' \implies a' \rho b'\), and
   (ii) \(a \rho b, a' \rho b' \text{ and } a' \rho b' \implies (a \land a') \rho (b \lor b')\).

(2) A \textit{bi-ideal} in \(A\) is a pair \((L, M)\) consisting of a lattice ideal \(L\) and a lattice dual ideal \(M\) with \(0 \in L \cap M\).

(3) If \(\rho\) is a dispersion on \(A\), the bi-ideal \((L, M)\) is \textit{\(\rho\)-regular} if \((L \times M) \cap \rho = \emptyset\).

Bi-ideals are partially ordered by \((L, M) \leq (L', M') \iff L \subseteq L' \text{ and } M \subseteq M'\). By Zorn’s Lemma each \(\rho\)-regular bi-ideal has a maximal \(\rho\)-regular refinement.

Definition 1.3. On the \(T\)-lattice A the dispersions \(\rho_e\) and \(\rho_b\) are given by
\[
\rho_e = \{(a, b) \in A \times A \mid \exists r \in \mathbb{R} \text{ with } b \leq r \leq 0 \text{ or } 0 < r \leq a]\],
\[
\rho_b = \{(a, b) \in A \times A \mid \exists r > 0 \text{ with } T_r(a) \lor 0 > b \land 0\}.
\]

Note that, since \(\rho_e \leq \rho_b\), a \(\rho_b\)-regular bi-ideal is also \(\rho_e\)-regular. Also, \(a \in L \Rightarrow T_r(a) \notin M \forall r > 0\) and \(a \in M \Rightarrow T_r(a) \notin L \forall r < 0\) are both necessary and sufficient conditions for \((L, M)\) to be \(\rho_b\)-regular.

For a fixed bi-ideal \((L, M)\) the equivalence relation \(\sim\) on \(A\) is defined by
\[
a \sim b \iff (T_r(a) \in L \iff T_r(b) \in L) \quad \text{and} \quad (T_r(a) \in M \iff T_r(b) \in M).
\]
The quotient set \(A/(L, M)\) is partially ordered by
\[
[a] \leq [b] \iff (T_r(b) \in L \Rightarrow T_r(a) \in L) \quad \text{and} \quad (T_r(a) \in M \Rightarrow T_r(b) \in M),
\]
and if \((L, M)\) is \(\rho_b\)-regular, \(r \mapsto [r]\) is an order preserving injection of \(\mathbb{R}\) into \(A/(L, M)\). In case the image of \(\mathbb{R}\) under this mapping is the whole of \(A/(L, M)\) the bi-ideal is called real. For a \(\rho_e\) regular bi-ideal \((L, M)\), \((L, M)\) is real if and only if given \(a \in A\) there exists a unique real number \(\alpha\) such that \(T_r(a) \in L \cap M \iff r = \alpha\). Each real bi-ideal is maximal \(\rho_b\)-regular. It is also \textit{prime}, that is \(L\) is a prime ideal and \(M\) a prime dual ideal.

If \((L, M)\) is a bi-ideal the bi-ideal \((L^+, M^+)\) is defined by
\[
L^+ = \{a \in A \mid T_r(a) \in L \forall r > 0\} \quad \text{and} \quad M^+ = \{a \in A \mid T_r(a) \in M \forall r < 0\}.
\] (1.5)
We note that \((L, M) \preceq (L^+, M^+)\), and that \((L, M)\) is \(\rho_e\)-regular or \(\rho_b\)-regular if and only if the same is true of \((L^+, M^+)\). Hence, if \((L, M)\) is real, \((L, M) = (L^+, M^+)\).
The element \([a]\) of \(A/(L, M)\) is called finite if \([r_1] \leq [a] \leq [r_2]\) for some \(r_1, r_2 \in \mathbb{R}\), it is infinite if \([r] \leq [a]\) for all \(r \in \mathbb{R}\) or \([a] \leq [r]\) for all \(r \in \mathbb{R}\). The element \(a \in A\) is finite or infinite at \((L, M)\) when \([a]\) has the corresponding property in \(A/(L, M)\). Finally, \((L, M)\) is called finite if every element of \(A\) is finite at \((L, M)\).

According to [2, Lemma 3.1.7, Corollary], if \((L, M)\) is \(\rho_b\)-regular then the only infinite elements of \(A/(L, M)\) are the greatest and least elements, when these exist. We shall also need the following result:

**Proposition 1.4.** (See [2, Proposition 3.1.7, Corollary 2]) Any maximal \(\rho_b\) regular refinement of a finite bi-ideal is real.

Now let \(B\) and \(C\) be sub-\(T\)-lattices of \(A\) and suppose that \(B \subseteq C\). Then any dispersion \(\rho\) on \(A\) induces a dispersion on \(B\) and \(C\) which we continue to denote by \(\rho\).

If \((L, M)\) is a bi-ideal in \(C\) then clearly \((L \cap B, M \cap B)\) is a bi-ideal in \(B\). Moreover, if \((L, M)\) is maximal \(\rho\)-regular in \(C\), \((L \cap B, M \cap B)\) is maximal \(\rho\)-regular in \(B\).

On the other hand suppose that \((L, M)\) is a \(\rho\)-regular bi-ideal in \(B\) and assume that \(\rho_b \leq \rho\). Let

\[
L_C = \{c \in C \mid \exists b \in L \text{ and } \epsilon > 0 \text{ with } c \land \epsilon \leq b\},
\]

\[
M_C = \{c \in C \mid \exists b \in M \text{ and } \epsilon > 0 \text{ with } b \leq c \lor (-\epsilon)\}. \tag{1.6}
\]

Then \((L_C, M_C)\) is a \(\rho\)-regular bi-ideal in \(C\) which is contained in every prime \(\rho\)-regular bi-ideal in \(C\) whose restriction to \(B\) is \((L, M)\). Hence if \((L, M)\) is maximal \(\rho\)-regular in \(B\) there exists at least one maximal \(\rho\)-regular bi-ideal in \(C\) whose restriction to \(B\) is \((L, M)\).

With \(B, C\) as above, \(C\) is called a (finite) \(\rho\)-refinement of \(B\) if every (finite) maximal \(\rho\)-regular bi-ideal in \(B\) has a unique extension to a (finite) \(\rho\)-regular bi-ideal in \(C\).

\(B\) is called (finitely) \(\rho\)-complete in \(A\) if it has no proper (finite) \(\rho\)-refinement in \(A\). \(A\) (finitely) \(\rho\)-complete (finite) \(\rho\)-refinement of \(B\) will be called a (finite) \(\rho\)-completion of \(B\).

It is shown in [2, Theorem 3.1.3] that every sub-\(T\)-lattice of \(A\) has a unique \(\rho_b\)-completion and a unique finite \(\rho_b\)-completion in \(A\).

An element \(a \in A\) is bounded if \(r_1 \leq a \leq r_2\) for some \(r_1, r_2 \in \mathbb{R}\). The set of bounded elements of \(A\) is denoted by \(A^*\).

Clearly, \(A^*\) is a sub-\(T\)-lattice of \(A\). It is shown in [2] that \(A^*\) is finitely-\(\rho_b\)-complete, and that the \(\rho_b\)-completion of \(A^*\) is \(A\).

Now let \(B\) be a subset of \(A\) containing \(0\). We denote by \((B)\) the smallest sub-\(T\)-lattice of \(A\) containing \(B\). Its elements are obtained from those of \(A\) by a finite number of applications of the operations \(\lor, \land\) and \(T_r\).

Let \(g : B \rightarrow \mathbb{R}\) be a function satisfying \(g(0) = 0\), and define

\[
L^g = \left\{ a \in (B) \mid \exists b_1, \dotsc, b_n \in B, \ r > 0 \text{ with } a \land r \leq \left( \bigvee_{i=1}^n T_g(b_i) b_i \right) \lor 0 \right\},
\]

\[
M^g = \left\{ a \in (B) \mid \exists b_1, \dotsc, b_n \in B, \ r > 0 \text{ with } a \lor -r \geq \left( \bigwedge_{i=1}^n T_g(b_i) b_i \right) \land 0 \right\}. \tag{1.7}
\]

Then \((L^g, M^g)\) is a bi-ideal. If this bi-ideal is \(\rho_b\)-regular, \(g\) is called a \(B\)-resolution and \((L^g, M^g)\) the corresponding \(B\)-derivative. The set of all \(B\)-resolutions is denoted by \(R_B\).

A real bi-ideal \((L, M)\) in \((B)\) determines a \((B)\)-resolution \(g : (B) \rightarrow \mathbb{R}\) by \(T_{g(a)}(a) \in L \cap M\), and \((L, M)\) is the \((B)\)-derivative of \(g\). It is shown in [2, Proposition 3.2.1] that a \((B)\)-resolution \(g\) has a real derivative if and only if belongs to \(H_{(B)}\), the set of \(T\)-lattice homomorphisms of \((B)\) to \(\mathbb{R}\) considered as a \(T\)-lattice under \(T_r(\xi) = x - r, x, r \in \mathbb{R}\). Hence, \(H_{(B)}\) is in one to one correspondence with the real bi-ideals in \((B)\).

If \(g \in H_{(B)}\) we have the equalities

\[
L^g = \left\{ a \in (B) \mid g(a) \leq 0 \right\} \quad \text{and} \quad M^g = \left\{ a \in (B) \mid g(a) \geq 0 \right\},
\]

and we also note that in this case \((L^g, M^g) = (L^g, M^g)\).

Finally we mention some concepts from [23] specific to the \(T\)-lattice \(BA(S)\). For \(S\) we define \(L(S) = \{s \in BA(S) \mid \varphi(s) \leq 0\} \quad \text{and} \quad M(S) = \{s \in BA(S) \mid \varphi(s) \geq 0\} \quad \text{as well as} \quad (L(S), M(S))\) is a real bi-ideal in \(BA(S)\) and a \(\rho_b\) regular bi-ideal \((L, M)\) in \(BA(S)\) is called difixed if there is a (necessarily unique) element \(s \in S_b\) satisfying \((L, M) = (L(S), M(S))\).

A bi-ideal \((L, M)\) in \(BA(S)\) is called a \(*\)-bi-ideal if \(L\) and \(M\) are closed under the operations \(\ast\) and \(\ast\).

There is an important link between real bi-ideals in \(BA(S)\) and diffilters [21] in the space \((S, S, \tau, \kappa)\). In particular, if \((L, M)\) is a \(\rho_b\)-regular \(*\)-bi-ideal then \(Z_0(L, M)\) defined by \(Z_0(L, M) = T_1 \times T_M\),

\[
T_1 = \{A \in S \mid \exists \varphi \in L, \ r > 0 \text{ with } \varphi^+ P_r \subseteq A\},
\]

\[
T_M = \{A \in S \mid \exists \varphi \in M, \ r > 0 \text{ with } A \subseteq \varphi^+ Q_r\},
\]

is a regular difilter. Moreover, if \((L, M)\) is a real \(*\)-bi-ideal, then \(Z_0(L, M)\) is disconvergent if and only if \((L, M)\) is difixed.

Naturally, in a \(*\)-space these properties hold for a \(*\)-ideal \((L, M)\) with the appropriate property.
2. Real dicompactness of ditopological texture spaces

Throughout the remainder of this paper, $(S, S, \tau, \kappa)$ will denote a completely biregular bi-$T_2$ ditopological texture space, unless stated otherwise.

We will be interested in subsets of $BA(S)$ which are “bigenerating” in the sense that they contain enough functions to determine the topology $\tau$ and cotopology $\kappa$. The following lemma and the definition that follows will make this concept exact.

**Lemma 2.1.** For $B \subseteq BA(S)$ the following are equivalent:

1. (i) the family $\{\varphi^r Q_r | \varphi \in B, r \in \mathbb{R}\}$ is a subbase for $\tau$, and
2. (ii) the family $\{\varphi^{-1} P_r | \varphi \in B, r \in \mathbb{R}\}$ is a subbase of $\kappa$.

**Proof.** For each $s \in S^+$,

1. (i) $\eta^+(s) = \{A \in S | \exists \varphi_1, \ldots, \varphi_n \in B, \epsilon_1, \ldots, \epsilon_n > 0$ with $T_{\varphi_i}(s)(\varphi_i)^{-1} Q_{\epsilon_i} \not\subseteq Q_s$ and $\bigcap_{i=1}^n T_{\varphi_i}(s)(\varphi_i)^{-1} Q_{\epsilon_i} \subseteq A\}$, and
2. (ii) $\mu^+(s) = \{A \in S | \exists \varphi_1, \ldots, \varphi_n \in B, \epsilon_1, \ldots, \epsilon_n > 0$ with $P_i \not\subseteq T_{\varphi_i}(s)(\varphi_i)^{-1} P_{-\epsilon_i}$ and $A \subseteq \bigcup_{i=1}^n T_{\varphi_i}(s)(\varphi_i)^{-1} P_{-\epsilon_i}\}$.

**Definition 2.2.** A set $B \subseteq BA(S)$ which satisfies $0 \in B$ will be called bigenerating if it satisfies one, and hence both, of the conditions (1) and (2) in Lemma 2.1.

The set $BA(S)$ itself is bigenerating. Indeed, if $0 \in BA(S)$, then take $G \subseteq \tau$ and $s \in S$ with $G \nsubseteq Q_s$. By Proposition 1.1 there exists $\varphi \in BA(S)$ with $P_s \subseteq \varphi^{-1} P_{-\epsilon} \subseteq \varphi^{-1} Q_{-\epsilon} \subseteq \varphi^{-1} Q_1 \subseteq G$. This shows that the family $\{\varphi^r Q_r | \varphi \in BA(S), r > 0\}$ is a base for $\tau$ and Lemma 2.1(1)(i) holds, and (1)(ii) may be proved likewise.

**Definition 2.3.** For $s \in S$ the mapping $\hat{s} : BA(S) \to \mathbb{R}$ is defined by $\hat{s}(\varphi) = \varphi(s)$.

Clearly, $\hat{s} \in H_{BA(S)}$. Take $B \subseteq BA(S)$ with $0 \in B$. To give a necessary and sufficient condition for $B$ to be bigenerating let us define:

$$L_{BA(S)}^{[1]} = \left\{ \varphi \in BA(S) \mid \exists \varphi_1, \ldots, \varphi_n \in B, \delta > 0 \text{ with } \varphi_\ast \wedge \delta \leq \bigvee_{i=1}^n T_{\hat{s}(\varphi_i)}(\varphi_i) \vee 0 \right\},$$

$$M_{BA(S)}^{[1]} = \left\{ \varphi \in BA(S) \mid \exists \varphi_1, \ldots, \varphi_n \in B, \delta > 0 \text{ with } \varphi_\ast \vee -\delta \geq \bigwedge_{i=1}^n T_{\hat{s}(\varphi_i)}(\varphi_i) \wedge 0 \right\}.$$

See (1.1) for the definitions of $\varphi_\ast, \varphi_\ast$. Using (1.2) and (1.3) it is easy to see that $(L_{BA(S)}^{[1]}, M_{BA(S)}^{[1]})$ is a bi-ideal. Moreover, by (1.6) and (1.7), in a $\ast$-space this bi-ideal coincides with $(L_{BA(S)}^{[1]}, M_{BA(S)}^{[1]})$.

**Proposition 2.4.** The following are equivalent for $0 \in B \subseteq BA(S)$.

1. (1) $B$ is bigenerating.
2. (2) For $s, t \in S$ we have $(L(s), M(t)) \preceq (L_{BA(S)}^{[1]} [1], M_{BA(S)}^{[1]} [1])$.

**Proof.** Necessity. To prove the first inclusion take $P_s \nsubseteq Q_t$, $\varphi \in L(s)$ and $\epsilon > 0$. Then $\varphi(s) \in Q_\epsilon$ and so $P_s \nsubseteq \varphi^{-1} Q_\epsilon$, whence $\varphi^{-1} Q_\epsilon \nsubseteq Q_t$. Since $B$ is bigenerating we now have $\varphi_1, \ldots, \varphi_n \in B, r_1, \ldots, r_n \in \mathbb{R}$ satisfying

$$\bigcap_{i=1}^n \varphi_i^{-1} Q_{r_i} \subseteq \varphi^{-1} Q_\epsilon \quad \text{and} \quad \bigcap_{i=1}^n \varphi_i^{-1} Q_{r_i} \nsubseteq Q_t.$$

From the second relation we have $\varphi_i(t) \in Q_{r_i}$, hence $\varphi_i(t) < r_i$ and we may choose $\delta > 0$ for which $\varphi_i(t) + \delta < r_i$ for $1 \leq i \leq n$. It is now easy to verify that

$$(\varphi_\ast - \epsilon) \wedge \delta \leq \bigvee_{i=1}^n (\varphi_i - \varphi_i(t)) \vee 0,$$

whence $T_{\varphi}(\varphi_\ast) \wedge \delta \leq \bigvee_{i=1}^m T_{\hat{s}(\varphi_i)}(\varphi_i) \vee 0$. This gives $T_{\varphi}(\varphi) \in L_{BA(S)}^{[1]}$, and since $\epsilon > 0$ is arbitrary, $\varphi \in L_{BA(S)}^{[1]}$, as required. The proof of $M(t) \subseteq M_{BA(S)}^{[1]}$ is dual to this, and is omitted.
Sufficiency. Since the sets \( \varphi^- Q_r \) form a base for \( \tau \), take \( s \in S \) with \( \varphi^- Q_r \not\subseteq Q_s \). We must show the existence of \( \varphi_1, \ldots, \varphi_n \in B \) and \( r_1, \ldots, r_n \in \mathbb{R} \) satisfying
\[
P_s \subseteq \bigcap_{i=1}^n \varphi_i^- Q_{r_i} \subseteq \varphi^- Q_r. \quad (2.1)
\]
Choose \( t \in S \) with \( \varphi^- Q_r \not\subseteq Q_t \) and \( P_t \not\subseteq Q_s \). By hypothesis \( L(t) \subseteq [L_{BA(S)}^+]^* \), and \( \varphi(t) < r \). Choose \( \alpha \in \mathbb{R} \) with \( \varphi(t) < \alpha < r \).

Then \( T_\alpha(\varphi) \in \overline{L_{BA(S)}^*} \), and so there exist \( \varphi_1, \ldots, \varphi_n \in B \) and \( \delta > 0 \) with \( T_\alpha(\varphi)_* \land \delta \leq \bigvee_{i=1}^n T_\alpha(\varphi_i) \lor 0 \). If we choose \( \theta \in \mathbb{R} \) satisfying \( 0 < \theta < \min(\delta, r - \alpha) \) it is not difficult to verify \((2.1)\) for these \( \varphi_i \in B \) and \( r_j = \varphi_i(s) + \theta \in \mathbb{R}, i = 1, \ldots, n \).

This verifies that \( B \) generates the topology \( \tau \), and likewise it generates the cotopology \( \kappa \). \( \square \)

**Definition 2.5.** Let \((S, S, \tau, \kappa)\) be a ditopological texture space and \( B \subseteq BA(S)\) bigenerating.

(i) \((B)^*\) denotes the smallest \( T\)-lattice containing \( B \) which is also closed under the operations \( \land \) and \( \lor \). That is, the elements of \((B)^*\) are obtained from \( B \) by a finite number of applications of the operations \( \lor, \land, T_\tau \), and \( \lor \).

(ii) \((S, S, \tau, \kappa)\) will be called \( B\)-real dicompact if every real bi-ideal in \((B)^*\) is fixed.

In particular a \( BA(S)\)-real dicompact space will be called real dicompact.

**Example 2.6.** Consider the space \((\mathbb{R}, \mathbb{R}, \tau_\mathbb{R}, \kappa_\mathbb{R})\) and put \( B = \{t_\mathbb{R}, 0\} \), where \( t_\mathbb{R} \) is the identity function on \( \mathbb{R} \). Lemma 2.1(1) is trivially satisfied by \( B \), so it is a bigenerating subset of \( BA(\mathbb{R}) \). We show that \((\mathbb{R}, \mathbb{R}, \tau_\mathbb{R}, \kappa_\mathbb{R})\) is \( B\)-real dicompact.

We begin by noting that \((\mathbb{R}, \mathbb{R}, \tau_\mathbb{R}, \kappa_\mathbb{R})\) is plain and hence a \( *\)-space. Hence \((B)^* = (B)\), and we take a real bi-ideal \((L, M)\) in \((B)\). For some \( \alpha \in \mathbb{R} \) we have \( T_\alpha(\varphi) \subseteq L \land M \). We show that \((L, M)\) is fixed by \( \alpha \). However for \( \varphi = t_\mathbb{R} \in B \) and \( \varphi = 0 \in B \) we have \( T_\alpha(\varphi) \varphi \in L \land \varphi(\alpha) \leq r \) and \( T_\alpha(\varphi) \varphi \in M \iff \varphi(\alpha) \geq r \), while a simple induction argument on the form of the elements of \((B)\) shows the same to be true for all \( \varphi \in (B) \), so we deduce \((L, M) = (L(\alpha), M(\alpha))\), as required. This establishes that \((\mathbb{R}, \mathbb{R}, \tau_\mathbb{R}, \kappa_\mathbb{R})\) is \( B\)-real dicompact.

**Proposition 2.7.** Let \( B, C \subseteq BA(S) \) be bigenerating and suppose that \((B)^* \subseteq (C)^*\). Then if \((S, S, \tau, \kappa)\) is \( B\)-real dicompact it is \( C\)-real dicompact.

**Proof.** Let \((L, M)\) be a real bi-ideal in \((C)^*\). Then \((L \land (B)^* \subseteq C)^* \) is a real bi-ideal in \((B)^*\), and hence fixed by some \( s \in S \). Hence \((L \land (B)^*, M \land (B)^*) = (L(s) \land (B)^*), (M(s) \land (B)^*)\), while as \( s \) varies \((L(s), M(s)) = ([L_{BA(S)}^{(B)}]^*, [M_{BA(S)}^{(B)}]^*)\) by Proposition 2.4 and the definitions. Set \( L_{\beta(C)} = L_{BA(S)}^{(B)} \land (C)^*\). We wish to show that \( L_{\beta(C)} \subseteq L \). Suppose on the contrary that there exists \( \varphi \in L_{\beta(C)}^* \), with \( \varphi \notin L \). Then \( L = \{ \psi \in (C)^* \mid \exists \mu \in L \land \psi \leq \mu \land \varphi \} \) is an ideal in \((C)^*\) properly containing \( L \).

Suppose that \((L', M')\) is not \( \rho_0\)-regular. Then we have \( \psi \in L', \theta \in M \) and \( r > 0 \) for which \( T_r(\psi \lor 0) \geq \theta \land 0 \). Take \( \mu \in L \) with \( \psi \leq \mu \land \varphi, \varphi_1, \ldots, \varphi_n \in B \) and \( \delta > 0 \) satisfying \( \varphi_\land \delta \leq \bigvee_{i=1}^n T_\delta(\varphi_i) \lor 0 \). Finally, choose \( r' \in \mathbb{R} \) with \( 0 < r' < \min(r, \delta) \).

**Corollary 2.8.** A \( B\)-real dicompact space is real dicompact.

**Proof.** For a bigenerating set \( B \) we have \((B)^* \subseteq BA(S) = (BA(S))^*\), so the result follows from Proposition 2.7. \( \square \)

In particular, it follows from Example 2.6 and the above corollary that the real ditopological texture space \((\mathbb{R}, \mathbb{R}, \tau_\mathbb{R}, \kappa_\mathbb{R})\) is real dicompact.

**Proposition 2.9.** If \((S, S, \tau, \kappa)\) is \( B\)-real dicompact then it is a nearly plain \( *\)-space.
Proof. By Corollary 2.8 we may assume that $(S, S, \tau, k)$ is real dicipact. Take $s \in S$. Then $(L(s), M(s))$ is a real bi-ideal in $\text{BA}(S)$ and hence difixed by some $w \in S_p$. Since $L(s) = L(w)$ we have $\varphi(s) = \varphi(w)$ for all $\varphi \in \text{BA}(S)$. Hence, since $\varphi_s \in \text{BA}(S)$ and $w \circ \varphi_s \text{BA}(S) w$ we have $\varphi(s) = \varphi_s(w) = \varphi(w) = \varphi(s)$. Since $s$ is arbitrary $\varphi_s = \varphi$, and likewise $\varphi^* = \varphi$, so $(S, S, \tau, k)$ is a $*$-space.

To show that $(S, S)$ is nearly plain it will suffice to show that $Q_4 = Q_w$. Suppose that $Q_w \not\subseteq Q_s$. Then since $(S, S, \tau, k)$ is bi-$T_2$ we have $H \in \tau$, $K \in \kappa$ with $H \subseteq K$, $P_w \not\subseteq K$ and $H \not\subseteq Q_s$ by [8, Theorem 4.17]. By complete biregularity we have $\varphi, \psi \in \text{BA}(S), r_1, r_2 \in \mathbb{R}$ with $\varphi \subseteq Q_r \subseteq H$, $\psi \subseteq Q_r \not\subseteq K$, and $K \subseteq \psi^\perp P_{r_2}$, $P_w \subseteq \psi^\perp P_{r_2}$. Now $\varphi(s) < r_1$, $\psi(w) > r_2$ so we may choose $\delta > 0$ with $\varphi(s) + \delta < r_1$, $\psi(w) - \delta > r_2$, and it is then easy to verify that

$$(\psi - \varphi(w) + \delta) \wedge \delta \leq (\varphi - \varphi(s))^* \vee 0.$$ 

Hence, $\delta = (\psi(w) - \varphi(w) + \delta) \leq (\varphi - \varphi(s))^*(w) \vee 0 = (\varphi - \varphi(s))(w) \vee 0 = 0$ since $\varphi(w) = \varphi(s)$, and we have a contradiction. Hence $Q_w \subseteq Q_s$, and the opposite inclusion is proved likewise. \[\square\]

Remark 2.10. If for a bigenerating set $B$ we assume only that the real ideals in $B$ are difixed the latter part of the above proof may easily be modified to show that $(S, S)$ is nearly plain. However, to conclude that $(S, S, \tau, k)$ is a $*$-space would seem to require the stronger assumption used in the definition of $B$-real dicipactness, namely that every real bi-ideal in $B^*$ is difixed. Naturally, if it is known in advance that $(S, S, \tau, k)$ is a completely biregular bi-$T_2$ $*$-space then $(B^*)^* = (B)$ for all bigenerating sets $B$, so the $B$-real dicipact spaces are characterized by the condition that all the real bi-ideals in $(B)$ are difixed. Moreover, since all bi-ideals are then $*$-bi-ideals, it is equivalent to require all the real $*$-bi-ideals in $(B)$ are difixed.

Proposition 2.11. Let $B, C \subseteq \text{BA}(S)$ be bigenerating sets with $(B) \subseteq (C)$ and suppose that $(S, S, \tau, k)$ is $C$-real dicipact. Then the following are equivalent:

(1) The space $(S, S, \tau, k)$ is $B$-real dicipact.
(2) The bi-ideal $(L_C, M_C)$ is finite for each real bi-ideal $(L, M)$ in $(B)$.
(3) The $T$-lattice $(C)$ is a finite $p_b$-refinement of $(B)$.

Here, $(L_C, M_C)$ is defined as in (1.6).

Proof. (1) $\Rightarrow$ (2). Let $(L, M)$ be a real bi-ideal in $(B)$. Then $(L, M)$ is difixed by some $s \in S_p$, so $(L, M) = (L(s), M(s)) = (L_{s}^* s, M_{s}^* s)$ and we obtain

$$(L_C, M_C) = (L_{s}^* s)_C, (M_{s}^* s)_C.$$ 

Since $B$ is bigenerating we may apply Proposition 2.4 to give $(L(s), M(s)) = ([L_{BA(s)}^* s]^+, [M_{BA(s)}^* s]^+)$ since $s \circ s s$. Hence, bearing in mind that we are dealing with a $*$-space we have $(L(s) \cap C, M(s) \cap C) = ([L_{s}^* s]_C]^+, ([M_{s}^* s]_C]^+)$. Comparing this with (2.2) now gives

$$(L_C^+, M_C^+) = (L(s) \cap C, M(s) \cap C).$$ 

It follows that $(L_C^+, M_C^+)$ is real in $(C)$, and hence in particular finite. Hence $(L_C, M_C)$ is finite also.

(2) $\Rightarrow$ (3). Let $(L, M)$ be a real bi-ideal in $(B)$, and $(L', M')$ any maximal $p_b$-regular extension of $(L, M)$ to $(C)$. Since $(L_C, M_C) \subseteq (L', M')$, and $(L_C, M_C)$ is finite by hypothesis, we have from Proposition 1.4 that $(L', M')$ is real. Since $(S, S, \tau, k)$ is $C$-real dicipact, $(L', M')$ is difixed by some $s \in S_p$, and so $(L', M') = (L(s) \cap C, M(s) \cap C)$. In particular, $(L, M) = (L(s) \cap B, M(s) \cap (B))$, so $(L, M)$ is difixed by $s$.

Now if $(L, M)$ is also difixed by $s'$ then since the points $s$ and $s'$ are plain, the argument used in the proof of [23, Lemma 4.9] gives $(L(s) \cap (B), M(s) \cap (B)) = (L(s') \cap (B), M(s') \cap (B)) \supseteq s \omega s'$ and $s' \omega s \supseteq s = s'$. This means that $(L(s) \cap (C))$ is the unique maximal $p_b$-regular extension of $(L, M)$ to $(C)$, and hence $(C)$ is a finite $p_b$-refinement of $(B)$.

(3) $\Rightarrow$ (1). Let $(L, M)$ be a real bi-ideal in $(B)$ and $(L', M')$ its unique real extension to $(C)$. By hypothesis $(L', M')$ is difixed, and clearly $(L, M)$ is difixed by the same element of $S_p$. Hence, $(S, S, \tau, k)$ is $B$-real dicipact. \[\square\]

Corollary 2.12. Let $(S, S, \tau, k)$ be real dicipact and $B \subseteq \text{BA}(S)$ bigenerating. Then $(S, S, \tau, k)$ is $B$-real dicipact if and only if the finite $p_b$-completion of $(B)$ is $\text{BA}(S)$.

Lemma 2.13. Let $(S, S, \tau, k)$ be nearly plain, $(S_p, S_p, \tau_p, p_p)$ the associated plain space and $\varphi_p : S \to S_p$ the bicontinuous surjection. Then:

(1) For $\varphi \in \text{BA}(S_p)$ we have $(\varphi \circ \varphi_p)|_{S_p} = \varphi$.
(2) For $\varphi \in \text{BA}(S)$ we have $\varphi_s \supseteq (\varphi|_{S_p}) \circ \varphi_p \subseteq \varphi^*$. 

Proof. (1) is trivial since $\varphi_p$ is the identity on $S_p$, so we prove (2). If the first inequality is false we have $s \in S$ with $\varphi_s(s) > (\varphi|_{S_p})(\varphi_p(s))$. Let $w = \varphi_p(s) \in S_p$. Then $\varphi_s(s) \triangleright (\varphi(w))$ so we have $P_s \subseteq Q_v$ with $\varphi(v) > \varphi(w)$. However $P_w \nsubseteq Q_w = Q_s$, so $P_w \nsubseteq Q_v$ and we have the contradiction $\varphi(v) \leq \varphi(w)$ because $\varphi$ is $\omega$-preserving. Hence the first inequality is established, and the proof of the second inequality is dual and hence omitted. □

Proposition 2.14. Let $(S, S, \tau, \kappa)$ be a completely biregular bi-$T_2$ space and $B$ a subset of $BA(S)$.

(1) Let $B$ be bigenerating and $(S, S, \tau, \kappa)$, $B$-real dicipact. Then $(S, S, \tau, \kappa)$ is a nearly plain $*$-space. Denote by $\alpha$ the mapping $\varphi \mapsto \varphi|_{S_p}$ from $BA(S)$ to $BA(S_p)$. Then $\alpha$ is a $T$-lattice isomorphism, $\alpha(B)$ is bigenerating in $(S_p, S_p, \tau_p, \kappa_p)$ and $(S, S, \tau, \kappa)$ is $\alpha(B)$-real dicipact.

(2) Let $(S, S, \tau, \kappa)$ be a nearly plain $*$-space and define the isomorphism $\alpha$ as in (1). Let $\alpha(B)$ be bigenerating and $(S_p, S_p, \tau_p, \kappa_p)$, $\alpha(B)$-real dicipact. Then $(S, S, \tau, \kappa)$ is $B$-real dicipact.

Proof. (1) Firstly, $(S, S, \tau, \kappa)$ is a nearly plain $*$-space by Proposition 2.9. The mapping $\alpha : BA(S) \rightarrow BA(S_p)$ given by $\alpha(\varphi) = \varphi|_{S_p}$ is clearly a $T$-lattice homomorphism which is onto by Lemma 2.13(1). Since $(S, S, \tau, \kappa)$ is a $*$-space we have $\varphi = (\varphi|_{S_p}) \circ \varphi_p$ for each $\varphi \in BA(S)$ by Lemma 2.13(2), so $\alpha$ is injective and hence a $T$-lattice isomorphism. Moreover, the inverse $\alpha^{-1}$ is given by $\varphi \mapsto \varphi \circ \varphi_p$, $\varphi \in BA(S_p)$.

It is straightforward to show that $\alpha(B) \subseteq BA(S_p)$ is bigenerating, either directly or by verifying $\alpha(L(s)) = L_p(s)$, $\alpha(L_{BA(S)}(\varphi)) = L_{BA(S_p)}(\varphi|_{S_p})$, and corresponding results for the dual ideals, for all $s \in S_p$ and using Proposition 2.4. The space $(S_p, S_p, \tau_p, \kappa_p)$, being plain, is a $*$-space so by Remark 2.10 we must show every real bi-ideal $(L, M)$ in $(\alpha(B))$ is difixed. However, $(\alpha^{-1}L, \alpha^{-1}M)$ is a real bi-ideal in $(B)$ and so difixed by some $s \in S_p$, again by Remark 2.10, and it is easy to verify that $(L, M)$ is also difixed by $s$. Hence, $(S_p, S_p, \tau_p, \kappa_p)$ is $\alpha(B)$-real dicipact.

(2) Under the given hypothesis the properties of $\alpha$ stated above still hold. In this case for $s \in S$ we may verify $\alpha(L(s)) = L_p(\varphi_p(s))$, $\alpha(L_{BA(S)}(\varphi)) = L_{BA(S_p)}(\varphi|_{S_p})$, etc., and the proof then follows similarly to that of (1). □

We now present an important characterization of $B$-real dicipact spaces. In view of Proposition 2.14 there will be no loss of generality in considering a nearly plain $*$-space $(S, S, \tau, \kappa)$, and in restricting our attention to the plain space $(S_p, S_p, \tau_p, \kappa_p)$ and the $T$-lattice $BA(S_p)$. Throughout the discussion below, therefore, $B$ will be a bigenerating subset of $BA(S_p)$ and we seek a necessary and sufficient condition for $(S_p, S_p, \tau_p, \kappa_p)$ to be $B$-real dicipact.

Denote by $(\mathbb{R}^{(B)}, \mathcal{R}^{(B)}, \tau^{(B)}_B, \kappa^{(B)}_B)$ the product of $(B)$ copies of the ditopological texture space $(\mathbb{R}, \mathcal{R}, \tau^{(R)}_\mathbb{R}, \kappa^{(R)}_\mathbb{R})$. Since a product of plain textures is plain, this is a plain texture and hence its restriction to the subset $H^{(B)}$ of $\mathbb{R}^{(B)}$ is also a plain ditopological texture space. We denote this space by $(H^{(B)}, \mathcal{H}^{(B)}, \tau^{(B)}_H, \kappa^{(B)}_H)$, and define the mapping

$\xi^{(B)} : S_p \rightarrow H^{(B)}$

by $\xi^{(B)}(s) = \xi^{(B)}(s)$ for all $s \in S_p$. Note that in $(H^{(B)}, \mathcal{H}^{(B)}, \tau^{(B)}_H, \kappa^{(B)}_H)$ we have

$$P_h = H^{(B)} \cap \bigcap_{\varphi \in (B)} E(\varphi, P_h(\varphi)), \quad Q_h = H^{(B)} \cap \bigcup_{\varphi \in (B)} E(\varphi, Q_h(\varphi)).$$

for $h \in H^{(B)}$ by [6, Proposition 1.3] and the fact that the texture is plain, and the p-sets and q-sets in the subspace $\xi^{(B)}(S_p)$ are given by the same formulæ with $H^{(B)}$ replaced by $\xi^{(B)}(S_p)$. We have:

Lemma 2.15. For $s_1, s_2 \in S_p$ the following are equivalent.

1. $s_1 \omega s_2$.
2. $\varphi(s_1) \subseteq \varphi(s_2) \forall \varphi \in (B)$.
3. $\xi^{(B)}(s_1) \omega \xi^{(B)}(s_2)$.

Proof. (1) $\Rightarrow$ (2) is immediate since the elements of $(B)$ are $\omega$-preserving, and (2) $\Leftrightarrow$ (3) follows from the formulæ (2.3). It remains to prove (2) $\Rightarrow$ (1). Suppose $\varphi(s_1) \subseteq \varphi(s_2) \forall \varphi \in (B)$ but that $P_{s_1} \nsubseteq Q_{s_1}$. Then $P_{s_2} \nsubseteq Q_{s_2}$ gives $Q_{s_1} \nsubseteq Q_{s_2}$. Since $(S_p, S_p, \tau_p, \kappa_p)$ is bi-$T_2$ we have $H \in \tau_p$, $K \in \kappa_p$ with $P_{s_2} \subseteq H \subseteq K$ and $P_{s_1} \nsubseteq K$, so since $B$ is bigenerating we have $\varphi_i \in B$, $\epsilon_i, \delta_i > 0$ with

$$\bigcap_{i=1}^n T_{\varphi_i}(\varphi_i(-\delta_i)) \subseteq H \subseteq K \subseteq \bigcup_{j=1}^m T_{\varphi_j}(\varphi_j(\epsilon_i))^\uparrow P_{-\delta_j}.$$

However this implies $s_2 \in T_{\varphi_j}(\varphi_j)^\uparrow P_{-\delta_j}$ for some $j$, and hence we obtain the contradiction $0 \leq \varphi_j(s_2) - \varphi_j(s_1) < -\delta_j$ since $\varphi_j \in B \subseteq (B)$. □
Corollary 2.16. \( \xi_B \) is a f\textbf{i}tex isomorphism between \((S_p, S_p)\) and the texture induced by \((H_B, \mathcal{K}_B)\) on \( \xi_B(S_p) \).

Proof. By Lemma 2.15, \( \xi_B \) is a bijection between \( S_p \) and the subset \( \xi_B(S_p) \) of \( H_B \); since in a plain texture \((T, J)\), \( t_1 = t_2 \Leftrightarrow P_{t_1} \subseteq P_{t_2} \) and \( P_{t_1} \subseteq P_{t_2} \Leftrightarrow t_1 \omega T t_2 \) and \( t_2 \omega T t_1 \). Moreover, the same lemma shows that both \( \xi_B \) and its inverse are \( \omega \)-preserving. For plain textures this is sufficient to ensure that \( \xi_B \) is a f\textbf{i}tex isomorphism [6]. \( \square \)

The following proposition considerably strengthens the above result.

Proposition 2.17.

(1) \( \xi_B \) is an f\textbf{Di}top isomorphism between \((S_p, S_p, \tau_p, \kappa_p)\) and its image in \((H_B, \mathcal{K}_B, \tau_B, \kappa_B)\). Moreover, \( \xi_B(S_p) \) is a jointly dense subset of \( H_B \).

(2) \( H_B \) is jointly closed in \((\mathbb{R}^B, \mathcal{R}_B, \tau^B, \kappa^B)\).

Proof. (1) The bicontinuity of \( \xi_B \) follows from the evident equality

\[
(\xi_B^{-1}(A) \cap E(\varphi, A)) = \varphi^{-}A, \quad \forall A \in \mathcal{R} \text{ and } \forall \varphi \in \langle B \rangle,
\]

and the fact that \( \{\xi_B^{-1}(A) \cap E(\varphi, G) \mid \varphi \in \langle B \rangle, G \in \tau_G\} \) is a subbase for the topology and \( \{\xi_B^{-1}(A) \cap E(\varphi, K) \mid \varphi \in \langle B \rangle, K \in \kappa_K\} \) a subbase for the copology on \( \xi_B(S_p) \). Likewise, the bicontinuity of the inverse mapping \( \xi_B^{-1} \) follows from

\[
(\xi_B^{-1})^{-1} \varphi^{-}A = \{\xi_B^{-1}(A) \cap E(\varphi, A) \mid \forall A \in \mathcal{R} \text{ and } \forall \varphi \in \langle B \rangle
\]

and Lemma 2.1(1).

Finally, we must show that if \( \xi_B^{-1}(A) \subseteq W \in \tau^B_{\xi_B(S_p)} \) then \( W = H_B \). Suppose on the contrary that there exists such a set \( W \) with \( W \neq H_B \), and take \( h \in H_B \setminus W \) with \( h \notin W \). Then we have \( G \in \eta^*(h) \), \( K \in \mu^*(h) \) satisfying \( G \cap W \subseteq K \), and hence \( G \cap \xi_B^{-1}(A) \subseteq K \).

By the definition of the product ditopology we have \( \varphi_i \in \langle B \rangle \), \( r_i \in \mathbb{R} \), \( i = 1, \ldots, n \) with \( P_{r_i} \subseteq H_B \cap \bigcap_{i=1}^n (\varphi_i, Q_{r_i}) \subseteq G \) and \( \varphi_j \in \langle B \rangle \), \( k_j \in \mathbb{R} \), \( j = 1, \ldots, m \) with \( K \subseteq H_B \cap \bigcap_{j=1}^m (\varphi_j, P_{k_j}) \subseteq Q_h \). This gives \( h(\varphi_i) < r_i \), \( h(\varphi_j) > k_j \) and so we may choose \( \epsilon > 0 \) with \( h(\varphi_i) + 2\epsilon < r_i \), \( h(\varphi_j) - 2\epsilon > k_j \) for all \( i \) and \( j \). Let \( r_j' = h(\varphi_i) + \epsilon, k_j' = h(\varphi_j) - \epsilon \) and consider the real bi-ideal \((L^h, M^h)\), where \( L^h = \langle \varphi_i \in \langle B \rangle \mid h(\varphi_i) \leq 0 \rangle, M^h = \langle \varphi_j \in \langle B \rangle \mid h(\varphi_j) \geq 0 \rangle \) since \( h \in H_B \). We may apply [23, Proposition 3.5(2)] to deduce that \( Z_{g}(L^h, M^h) \) is a regular difilter since we are working in the plain texture \((S_p, S_p, \tau_p, \kappa_p)\). However, we clearly have \( \varphi_i - r_i' \in L^h, i = 1, \ldots, n \) and \( \varphi_j - k_j' \in M^h, j = 1, \ldots, m \), so

\[
\bigcap_{i=1}^n (\varphi_i - r_i')^{-} P_{r_i} \subseteq \bigcup_{j=1}^m (\varphi_j - k_j')^{-} Q_{\epsilon} = Q_{-\epsilon},
\]

and taking \( s \in \bigcap_{i=1}^n (\varphi_i - r_i')^{-} P_{r_i}, s \notin \bigcup_{j=1}^m (\varphi_j - k_j')^{-} Q_{-\epsilon} \) leads easily to \( \xi_B^{-1}(s) \in G, \xi_B^{-1}(s) \notin K \), which contradicts (2.4).

(2) By the discussion in [23, 2.3] for the joint topology on the plain space \((\mathbb{R}^B, \mathcal{R}_B, \tau^B, \kappa^B)\) is

\[
\left\{ E(\varphi, Q_r) \mid \varphi \in \langle B \rangle, r \in \mathbb{R} \right\} \cup \left\{ \mathbb{R}^B \setminus E(\varphi, P_k) \mid \varphi \in \langle B \rangle, k \in \mathbb{R} \right\}.
\]

Take \( g \) in the closure of \( H_B \) for this topology. We must show that \( g \in H_B \).

If \( g(0) < 0 \) then \( E(0, Q_0) \) is a nhd. of \( g \) for the joint topology, so we may find \( h \in E(0, Q_0) \cap H_B \). However, \( h(0) = 0 \) for \( h \in H_B \), which gives the contradiction \( h \notin E(0, Q_0) \). Likewise, \( g(0) > 0 \) leads to a contradiction and we have established \( g(0) = 0 \).

Let us verify that

\[
g(\varphi \vee \psi) = g(\varphi) \vee g(\psi)
\]

for all \( \varphi, \psi \in \langle B \rangle \). Suppose that \( g(\varphi \vee \psi) > g(\varphi) \vee g(\psi) \) and let \( \epsilon > 0 \) satisfy \( g(\varphi \vee \psi) - \epsilon > g(\varphi) + g(\psi) + \epsilon \). Now

\[
E(\varphi, Q_{g(\varphi)+\epsilon}) \cap E(\psi, Q_{g(\psi)+\epsilon}) \subseteq \mathbb{R}^B \setminus E(\varphi \vee \psi, P_{g(\varphi \vee \psi)-\epsilon})
\]

is a nhd. of \( g \) in the joint topology on \( \mathbb{R}^B \), and so meets \( H_B \) in some element \( h \). Hence \( h(\varphi) < g(\varphi) + \epsilon, h(\psi) < g(\psi) + \epsilon \) and \( h(\varphi \vee \psi) > g(\varphi \vee \psi) - \epsilon \), so

\[
g(\varphi \vee \psi) - \epsilon < h(\varphi \vee \psi) = h(\varphi) + h(\psi) < g(\varphi) \vee g(\psi) + \epsilon,
\]

which is a contradiction. Likewise, \( g(\varphi \vee \psi) < g(\varphi) \vee g(\psi) \) leads to a contradiction, and we have shown that \( g(\varphi \vee \psi) = g(\varphi) \vee g(\psi) \). The equalities \( g(\varphi \wedge \psi) = g(\varphi) \wedge g(\psi) \) and \( g(\varphi - r) = g(\varphi) - r, r \in \mathbb{R} \) may be proved likewise, and the details are left to the interested reader. We deduce that \( g \in H_B \), so \( H_B \) is closed in \( \mathbb{R}^B \) for the joint topology, as required. \( \square \)
Corollary 2.18.

(1) For a real diconpact ditopological texture space \((S, S, \tau, \kappa), (S_p, S_p, \tau_p, \kappa_p)\) can be embedded as a jointly closed subspace of a product of copies of the space \((\mathbb{R}, \mathbb{R}, \tau_\mathbb{R}, \kappa_\mathbb{R})\).

(2) The joint topology on \(S_p, S_p, \tau_p, \kappa_p\) of a real diconpact ditopological texture space \((S, S, \tau, \kappa)\) is real compact.

Proof. (1) If \((S, S, \tau, \kappa)\) is real diconpact then by Proposition 2.14(1), \((S_p, S_p, \tau_p, \kappa_p)\) is also. Hence, by Proposition 2.17(2) it will be sufficient to show that whenever \((S_p, S_p, \tau_p, \kappa_p)\) is \(B\)-real diconpact we have \(\xi_{(B)}(S_p) = H_{(B)}\). For \(h \in H_{(B)}\) the bi-ideal \((L^h, M^h)\) is real in \((B)\), and we have the equalities \(L^h = \{\varphi \in (B) \mid h(\varphi) \leq 0\}, M^h = \{\varphi \in (B) \mid h(\varphi) \geq 0\}\). By hypothesis \((L^h, M^h)\) is diffixed by some \(s \in S_p\), whence \((L^h, M^h) = (L(s) \cap (B), M(s) \cap (B))\) and we easily deduce that \(h = \hat{s} \in \xi_{(B)}(S_p)\).

(2) By (2.5) it is easy to see that the joint topology of \((\mathbb{R}^{(B)} \times \mathbb{R}^{(B)}, \tau_\mathbb{R}^{(B)}, \tau_\mathbb{R}^{(B)}, \kappa_\mathbb{R}^{(B)})\) coincides with the product topology of the spaces \(\mathbb{R}\) with their standard topology. It follows that the joint topological space of \((S, S, \tau, \kappa)\) on \(S_p\) may be embedded as a closed subspace of a product of copies of the space \(\mathbb{R}\) under its usual topology and hence is real compact by [15].

We now consider the converse of Corollary 2.18(1).

Theorem 2.19. A completely bi-regular bi-\(T_2\) nearly plain \(+\)-space \((S, S, \tau, \kappa)\) can be diconpact if and only if the space \((S_p, S_p, \tau_p, \kappa_p)\) can be embedded as a jointly closed subspace of a product of the spaces \((\mathbb{R}, \mathbb{R}, \tau_\mathbb{R}, \kappa_\mathbb{R})\).

Proof. Necessity is just Corollary 2.18(1), so we prove sufficiency. To simplify the notation we assume without loss of generality that \((S_p, S_p, \tau_p, \kappa_p)\) is a jointly closed subspace of some product \((\mathbb{R}^J, \mathbb{R}^J, \tau_\mathbb{R}^J, \kappa_\mathbb{R}^J)\) of copies of \((\mathbb{R}, \mathbb{R}, \tau_\mathbb{R}, \kappa_\mathbb{R})\). Denoting the projection mappings by \(\pi_j\) it is clear that \(B = \{\pi_j \mid j \in J\} \cup \{0\}\) is a diconpacting subset of \(\text{BA}(S_p)\) and so by Proposition 2.14(2) and Corollary 2.8 it will be sufficient to show that \((S_p, S_p, \tau_p, \kappa_p)\) is \(B\)-real diconpact.

Let \((L, M)\) be a real bi-ideal in \((B)\), and let \(h \in H_{(B)}\) be the \((B)\)-resolution of \((L, M)\). For \(j \in J\) let \(s_j = h(\pi_j | S_p)\) and consider \(s = (s_j) \in \mathbb{R}^J\). We first show that \(s \in S_p\). Suppose this is not so. Then, since \(S_p\) is jointly closed in \(\mathbb{R}^J\) we have a joint nh
d

\[
\bigcap_{\alpha = 1}^{m} E(j_{\alpha}, Q_{s_{\alpha}}) \cap \bigcap_{\beta = 1}^{n} (\mathbb{R}^J \setminus E(j_{\beta}, P_{p_{\beta}})) \tag{2.6}
\]

of \(s\) which does not meet \(S_p\). Now \(s_{j_{\alpha}} < r_{s_{\alpha}}, s_{j_{\beta}} > k_{\beta}\) so we may choose \(\epsilon > 0\) for which \(s_{j_{\alpha}} + \epsilon < r_{s_{\alpha}}, s_{j_{\beta}} - \epsilon > k_{\beta}\) for all \(1 \leq \alpha \leq m\) and \(1 \leq \beta \leq n\). By definition \(L = L^h = \{\varphi \in (B) \mid h(\varphi) \leq 0\}\), and \(h(\pi_{j_{\alpha}} | S_p - s_{j_{\alpha}}) = h(\pi_{j_{\alpha}} | s_{j_{\alpha}} - s_{j_{\alpha}}) = 0\), so \(\bigvee \{\pi_{j_{\alpha}} | s_{j_{\alpha}} - s_{j_{\alpha}} \mid 1 \leq \alpha \leq m\} \vee 0 \in L\). Likewise, \(\bigwedge \{\pi_{j_{\alpha}} | s_{j_{\alpha}} - s_{j_{\alpha}} \mid 1 \leq \beta \leq n\} \vee 0 \in M\). On the other hand, since \((L, M)\) is \(\rho_0\)-regular we have \(T_\epsilon \left(\bigvee \{\pi_{j_{\alpha}} | s_{j_{\alpha}} - s_{j_{\alpha}} \mid 1 \leq \alpha \leq m\} \vee 0\right) \neq M\), so

\[
\bigwedge \{\pi_{j_{\alpha}} | s_{j_{\alpha}} - s_{j_{\alpha}} \mid 1 \leq \beta \leq n\} \vee 0 \subset T_\epsilon \left(\bigvee \{\pi_{j_{\alpha}} | s_{j_{\alpha}} - s_{j_{\alpha}} \mid 1 \leq \alpha \leq m\} \vee 0\right).
\]

We now have \(t \in S_p\) for which \(\pi_{j_{\alpha}}(t) - s_{j_{\alpha}} \supset 0 > (\pi_{j_{\alpha}} - s_{j_{\alpha}}) \cup 0 = \pi_{j_{\alpha}} - s_{j_{\alpha}}\) for all \(\alpha\) and \(\beta\). However, this implies that \(t\) is in the nhd. (2.6) and this contradiction shows that \(s \in S_p\).

For \(\varphi \in B\) it is immediate that \(\varphi(s) = h(\varphi)\), and a simple induction argument of the form of the elements in \((B)\) shows that \(\varphi(s) = h(\varphi)\) for all \(\varphi \in (B)\). Hence

\[
(L(s) \cap (B), M(s) \cap (B)) = (L^h, M^h) = (L, M),
\]

and so \((L, M)\) is diffixed by \(s \in S_p\) as required.

An important special choice for \(B\) is \(\text{BA}^+*(S)\), the set of bounded elements of \(\text{BA}(S)\). It is easy to verify that this is a diconpacing sub-\(T\)-lattice of \(\text{BA}(S)\). In parallel with the topological and bitopological cases we refer to a space \((S, S, \tau, \kappa)\) with \(\text{BA}^+*(S) = \text{BA}(S)\) as pseudo diconpact.

Proposition 2.20. The following are equivalent for \((S, S, \tau, \kappa)\).

(i) Pseudo diconpact and real diconpact.

(ii) \(\text{BA}^+*(S)\)-real diconpact.

(iii) \(B\)-real diconpact for all diconpacting \(B \subseteq \text{BA}^+*(S)\).

(iv) Diconpact.

Proof. (i) \(\Rightarrow\) (ii). Immediate from the definitions.

(ii) \(\Rightarrow\) (iii). This follows from Proposition 2.11 since in \(\text{BA}^+*(S)\) all \(\rho_0\)-regular bi-ideals are finite.
(iii) ⇒ (iv). Let $B \subseteq BA^*(S)$ be a bigenerating sub-$T$-lattice of $BA^*(S)$. By the proof of Corollary 2.18(1) we may embed $(S, S_p, T_p, \kappa_p)$ as a jointly closed subspace of $(R^B, R^B, \tau_{R^B}, \kappa_{R^B})$. Moreover, each $\varphi \in B$ is bounded and if we take $a_\varphi \leq \varphi \leq b_\varphi$, it is not difficult to see that we may replace the $\varphi^{th}$ factor in this product with $(\bar{a}_\varphi, b_\varphi, \bar{a}_\varphi, b_\varphi, \tau_{R^B}, \kappa_{R^B})$ (see [8, Notes 5.4(4)]). These are dicompact plain spaces and hence their product is a dicompact plain space. By [23, Proposition 4.6(2)] the joint topology of this space is compact, and hence the jointly closed subspace corresponding to $(S, S_p, T_p, \kappa_p)$ is also compact. This proves that the joint topology of $(S, S_p, T_p, \kappa_p)$ is compact. Hence $(S, S_p, T_p, \kappa_p)$ is dicompact by [23, Proposition 4.6(2)], and so $(S, S_p, T_p, \kappa_p)$ is dicompact since these spaces are isomorphic in $dfDitop$.

(iv) ⇒ (i) This is just (1) ⇒ (2) in [23, Theorem 4.2].

**Corollary 2.21.** A completely biregular bi-$T_2$ nearly plain *-space $(S, S_p, T_p, \kappa_p)$ is dicompact if and only if $(S, S_p, T_p, \kappa_p)$ may be embedded as a jointly closed subspace of a product of spaces of the form $(\bar{a}_b, \bar{a}_b, a_b, a_b)$.

3. Categorical results

We begin by recalling the construct $ifNPdtop$ of nearly plain ditopological spaces and $\omega$-preserving mappings, the adjoint functor $\exists : ifNPdtop \to Top$ given by

$$\exists((S, S_p, T_p, \kappa_p)) \overset{\psi}{\longrightarrow} (T, T_p, \kappa_p) \overset{\psi|_p}{\rightarrow} (T_p, \kappa_p),$$

and its co-adjoint

$$\Xi((X, \tau) \overset{\psi}{\longrightarrow} (Y, \nu)) = (X, \mathcal{P}(X), \tau, \mathcal{P}^c) \overset{\psi}{\rightarrow} (Y, \mathcal{P}(Y), \nu, \mathcal{P}^c)$$

from $Top$ to $ifNPdtop$ [23, Theorem 4.4, Corollary 4.5]. By Corollary 2.18(2) we see that $\exists$ may be restricted to a functor $\exists : ifRdComp_2 \to RComp$, where $ifRdComp_2$ denotes the construct of real dicompact bi-$T_2$ spaces and $RComp$ the construct of (Hausdorff) real compact topological spaces.

We wish to show that if $(X, \tau)$ is real compact then $\Xi(X, \tau)$ is real dicompact. Clearly $BA(X) = C(X)$, and denoting by $B$ the characterization theorem for real compact spaces [15, 11.12] says that $X$ may be embedded as a closed subspace of $R^B$ under the mapping $\xi_B : X \to R^B$, where the spaces $R$ have their usual topology. We have already noted that the product topology on $R^B$ coincides with the joint topology of the ditopological texture space $(R^B, R^B, \tau_{R^B}, \kappa_{R^B})$, so to establish the real dicompactness of $(X, \mathcal{P}(X), \tau, \mathcal{P}^c)$ by Theorem 2.19 it will suffice to show that for $Y \in \mathcal{P}(X)$ there exists a set $Y_1 \subseteq R^B$ satisfying $Y_1 \cap \xi_B(X) = \xi_B(Y)$. However

$$Y_1 = \bigcup_{y \in Y} \left( \bigcap_{\varphi \in B} E(\varphi, P\varphi(y)) \right)$$

is easily seen to have the required properties and we have established:

**Theorem 3.1.** The functor $\exists : ifRdComp_2 \to RComp$ is an adjoint and $\Xi : RComp \to ifRdComp_2$ its co-adjoint. In particular, $\Xi$ embeds $RComp$ as a coreflective subcategory of $ifRdComp_2$.

Next we extend [23, Theorem 4.8] to the real dicompact case. The names used for the categories are an obvious modification of the ones used for the dicompact case. The proof follows the same lines as the proof of [23, Theorem 4.8], and is omitted.

**Theorem 3.2.** The functor $\mathbb{D} : ifRdComp_2/\sim_p \to Tlat_{df}$ is faithful and creates isomorphisms.

This leads at once to analogues of the Hewitt Isomorphism Theorem [15].

**Corollary 3.3 (Hewitt Isomorphism Theorem).** Let $(S_k, S_k, \tau_k, \kappa_k), k = 1, 2$, be real dicompact bi-$T_2$ spaces. Then these spaces are isomorphic in $dfRdComp_2$ (resp., in $ifRdComp_2/\sim_p$) if and only if the T-lattices $BDF(S_k)$ (resp., $BA(S_k)$), $k = 1, 2$, are isomorphic.

As mentioned in the introduction, the theory of T-lattices and bi-ideals was first conceived in a bitopological context. We conclude this paper by looking in greater detail at the relationship between the ditopological and bitopological cases insofar as real compactness is concerned.

We begin by defining a functor $\Xi$ from $ifNPdtop$ to $Bitop$ by

$$\Xi((S, S_p, T_p, \kappa_p)) \overset{\psi}{\longrightarrow} (T, T_p, \kappa_p) \overset{\psi}{\longrightarrow} (T_p, \kappa_p).$$

It is trivial to verify that this is indeed a functor and we omit the details. Clearly, it generalizes the functor with the same name from $fPDitop$ to $Bitop$ given in [7]. To define a suitable functor in the opposite direction we restrict ourselves to
weakly pairwise $T_0$ bitopological spaces $(X, u, v)$, and consider the smallest subset $K_{uv}$ of $P(X)$ which contains $u \cup v^c$ and is closed under arbitrary intersections and unions. Clearly the elements of $K_{uv}$ have the form

$$A = \bigcap_{j \in J} A_j,$$

where $A_j = U_j \cup \bigcup_{i \in I_j} \{(V_i^j)^c \mid v_i^j \in v\}$, $U_j \in u$, $j \in J$. (3.1)

To show that $K_{uv}$ is a (plain) texturing of $X$ it remains only to verify that it separates points. For $x_1, x_2 \in X$ with $x_1 \neq x_2$ we have $x_1 \not\in K_{uv}^u \cap K_{uv}^v$ or $x_2 \not\in K_{uv}^u \cap K_{uv}^v$ by the weak pairwise $T_0$ axiom, whence $x_1, x_2$ are separated by a set in $u \cup v^c \subseteq K_{uv}$, as required. We consider the plain bitopological space $(X, K_{uv}, u, v^c)$ and define $\mathcal{R}$ by

$$\mathcal{R}(X, u, v) \xrightarrow{\varphi} (Y, u_Y, v_Y) = (\{X, K_{uv} \cap u, u_X \} \xrightarrow{\varphi} (Y, K_{uv} \cap u_Y, v_Y)).$$

Now when $(X, u, v) \xrightarrow{\varphi} (Y, u_Y, v_Y)$ is pairwise continuous, $(X, K_{uv} \cap u, u_X) \xrightarrow{\varphi} (Y, K_{uv} \cap u_Y, v_Y)$ is $\omega$-preserving and bicontinuous. Indeed, take $x_1 \varphi X x_2$. Now from (3.1) we clearly have, since $(X, K_{uv} \cap u)$ is plain,

$$x_1 \varphi X x_2 \iff x_1 \in P_{x_2}$$

$$\iff (X_2 \in U \in u_X \Rightarrow x_1 \in U) \land (x_1 \in V \in v_X \Rightarrow x_2 \in V). (3.2)$$

Hence, $\varphi (x_2) \in U \in u_X \Rightarrow x_2 \in \varphi^{-1}[U] \in u_X \Rightarrow x_1 \in \varphi^{-1}[U] \Rightarrow \varphi(x_1) \in U$, and likewise $\varphi (x_1) \in V \in v_Y \Rightarrow \varphi(x_2) \in V$, whence $\varphi (x_1) \varphi X \varphi (x_2)$. This shows $\varphi$ is $\omega$-preserving, and bicontinuity is an immediate consequence of the pairwise continuity of $(X, u, v) \xrightarrow{\varphi} (Y, u_Y, v_Y)$ and the fact that for $A \in K_{uv} \cap u$ we have $\varphi^{-1} A = \varphi^{-1}[A]$ since the textures are plain. Hence $\mathcal{R}$ is a functor.

We consider the preservation of certain properties under the functors $\mathcal{U}$ and $\mathcal{R}$.

Lemma 3.4. With the notation as above:

1. If $(X, u, v)$ is weakly pairwise $T_2$ then $\mathcal{U}(X, u, v)$ is bi-$T_2$.
2. If $(S, \mathcal{R}, \tau, \kappa)$ is bi-$T_2$ then $\mathcal{U}(S, \mathcal{R}, \tau, \kappa)$ is weakly pairwise $T_2$.
3. If $(S, \mathcal{R}, \tau, \kappa)$ is completely biregular then $\mathcal{U}(S, \mathcal{R}, \tau, \kappa)$ is pairwise completely regular.
4. If $(X, u, v)$ is pairwise completely regular then $\mathcal{R}(X, u, v)$ is completely biregular.

Proof. (1) Take $x_1, x_2 \in X$ with $Q_{x_1} \not\subseteq Q_{x_2}$ in $(X, K_{uv})$. Now $P_{x_1} \not\subseteq P_{x_2}$, that is $x_1 \not\in P_{x_2}$ and from (3.2) we deduce that $x_1 \not\in K_{uv}^u$ or $x_2 \not\in K_{uv}^v$. Since $(X, u, v)$ is weakly pairwise $T_2$ it is pairwise $R_1$ and hence in both cases we have $U \in u \subseteq V \subseteq v$ with $x_1 \in U$, $x_2 \in V$ and $U \cap V = \emptyset$. Now $U \not\subseteq Q_{x_1}$, $U \subseteq K = X \setminus V \subseteq v$ and $P_{x_1} \not\subseteq K$, so $\mathcal{U}(X, u, v)$ is bi-$T_2$.

(2) Left to the interested reader.

(3) First we note that $\mathcal{U}(\mathbb{R}, \mathcal{R}, \tau_{\mathbb{R}}, \kappa_{\mathbb{R}}) = (\mathbb{R}, s, t)$ and that the elements of $BA(S)$ are precisely the $\text{ifNpDitop}$ morphisms from $(S, \mathcal{R}, \tau, \kappa)$ to $(\mathbb{R}, \mathcal{R}, \tau_{\mathbb{R}}, \kappa_{\mathbb{R}})$.

$$\begin{array}{ccc}
(S, \mathcal{R}, \tau, \kappa) & \xrightarrow{\mathcal{U}} & (S_p, \tau_p, \kappa_p^c) \\
\varphi & \downarrow & \varphi_{|S_p} \\
(\mathbb{R}, \mathcal{R}, \tau_{\mathbb{R}}, \kappa_{\mathbb{R}}) & \xrightarrow{\mathcal{U}} & (\mathbb{R}, s, t)
\end{array}$$

It will clearly suffice to show that $B = \{\varphi_{|S_p} \mid \varphi \in BA(S)\}$ is a bigenerating subset of $P(S_p)$. Certainly $B \subseteq P(S_p)$ since $\mathcal{U}$ is a functor. Let $F \subseteq S_p$ be $\tau_p$-closed and take $u \in S_p$ with $u \not\in F$. Then $u \in H = S_p \setminus F \in \tau_p$ so we have $H_1 \in \tau$ with $H = H_1 \cap S_p$, and clearly $H_1 \not\subseteq u \in (S, \mathcal{R}, \tau, \kappa)$. Since $(S, \mathcal{R}, \tau, \kappa)$ is completely biregular, by Proposition 1.1(1) we have $\varphi \in BA(S)$ satisfying $-1 < \varphi \leq 1$, $P_u \subseteq \varphi^{-1} P_{-1}$ and $\varphi^{-1} Q_{1} \subseteq H_1$. Now $\varphi_{|S_p} \in B$, $\varphi_{|S_p}(u) = -1$ and $\varphi_{|S_p}(F) \subseteq \{1\}$, so the functions in $B$ generate the topology $\tau_p$ on $S_p$. Likewise they generate the topology $\kappa_p^c$, so $B$ is bigenerating and in particular $(S_p, \tau_p, \kappa_p^c)$ is pairwise completely regular.

(4) On noting that $\mathcal{R}(\mathbb{R}, s, t) = (\mathbb{R}, \mathcal{R}, \tau_{\mathbb{R}}, \kappa_{\mathbb{R}})$ we obtain the diagram below.

$$\begin{array}{ccc}
(X, u, v) & \xrightarrow{\mathcal{R}} & (X, K_{uv}, u, v^c) \\
\varphi & \downarrow & \varphi \\
(\mathbb{R}, s, t) & \xrightarrow{\mathcal{R}} & (\mathbb{R}, s, t)
\end{array}$$

Clearly $\mathcal{R}$ is full, so $P(X) = BA(X)$. The remainder of the proof is straightforward, and is omitted. □

The above lemma shows that with respect to these functors our basic assumption that $(S, \mathcal{R}, \tau, \kappa)$ is a completely biregular bi-$T_2$ space is fully consistent with the assumption in [2, Chapter 3] that $(X, u, v)$ is pairwise completely regular and weakly pairwise Hausdorff. Moreover, the additional requirement that $(S, \mathcal{R}, \tau, \kappa)$ should be a $\omega$-space will not cause a problem as the image of $(X, u, v)$ under $\mathcal{R}$ is plain and hence also a $\omega$-space.
For the purposes of this paper we denote the construct of pairwise completely regular weakly Hausdorff bitopological spaces and pairwise continuous functions by \(p\text{Reg}_{\omega_2}\), and the category of completely biregular bi-\(T_2\) nearly plain \((\omega\times\omega)\)-spaces and \(\omega\)-preserving bicontinuous functions by \(i\text{fNpCbiR}_2\) (respectively, \(i\text{fNpCbiR}_2^\ast\)).

**Theorem 3.5.** \(\mathcal{U}: i\text{fNpCbiR}_2^\ast \to p\text{Reg}_{\omega_2}\) is an adjoint functor and \(\mathcal{R}: p\text{Reg}_{\omega_2}\to i\text{fNpCbiR}_2^\ast\) a co-adjoint of \(\mathcal{U}\).

**Proof.** Take \((X, u, v)\in\text{Ob}p\text{Reg}_{\omega_2}\). We show that \((t_X, (X, X_{uv}, u, v^c))\) is a \(\mathcal{U}\)-universal arrow with domain \((X, u, v)\). It is clearly \(\mathcal{U}\)-structured, so take \((S, S, \tau, \kappa)\in\text{Ob}i\text{fNpCbiR}_2^\ast\) and \(\varphi\in p\text{Reg}_{\omega_2}(X, u, v, (S, \tau, \kappa))\). We verify first that \(\varphi: (X, X_{uv}, u, v^c)\to (S, \tau, \kappa)\) is an \(i\text{fNpTex}\) morphism, whence it is the unique such morphism for which the following diagram commutative.

\[
\begin{align*}
(X, u, v) & \xrightarrow{t_X} \Omega(X, X_{uv}, u, v^c) = (X, u, v) \\
\varphi & \downarrow \\
\Omega(\varphi) & \downarrow \\
(S, S, \tau, \kappa) & = (S, \tau, \kappa)
\end{align*}
\]

Suppose on the contrary that \(\varphi\) is not \(\omega\)-preserving. Then we have \(x_1, x_2 \in X\) with \(x_1\omega x_2\) and \(P_{\varphi(x_2)} \subseteq Q_{\varphi(x_1)}\). Since \(\varphi(x_2) \in S_p\) we have \(P_{\varphi(x_2)} \subseteq Q_{\varphi(x_2)}\), whence \(Q_{\varphi(x_1)} \subseteq Q_{\varphi(x_2)}\). Since \((S, S, \tau, \kappa)\) is bi-\(T_2\) we have \(H \in \tau, \kappa \in K\) with \(H \subseteq K\), \(P_{\varphi(H)} \subseteq K\) and \(H \subseteq Q_{\varphi(x_2)}\). Now \(U = \varphi^{-1}[H \cap S_p] \in u\) by pairwise continuity, and \(x_2 \in U\) so by (3.2) we have \(x_1 \in U\) and hence the contradiction \(\varphi(x_1) \in H \subseteq K\).

On the other hand, for \(A \subseteq S\) we have \(\varphi^{-1}[A \cap S_p] = \varphi^{-1}A\) since \((X, X_{uv})\) is plain and \(\varphi\) maps into \(S_p\). Hence \(\varphi \in i\text{fNpCbiR}_2^\ast((X, X_{uv}, u, v^c), (S, \tau, \kappa))\), and since \((X, X_{uv}, u, v^c) = \mathcal{R}(X, u, v)\) the proof is complete by [1, Remark 19.2].

We next note that by the proof of Lemma 3.4(3) the set \[\varphi|_{S_p} \mid \varphi \in \text{BA}(S)\] is a bigenerating subset of \(\text{P}(S_p)\). However this set is just \(\text{BA}(S_p)\), and so is a \(T\)-lattice, while a bi-ideal in \(\text{BA}(S_p)\) is diffixed if and only if it is fixed by the same point of \(S_p\) in the sense of [2]. Hence, by Proposition 2.14 we see that if \((S, S, \tau, \kappa)\) is real dicompact every real bi-ideal in \(\text{BA}(S_p)\) is diffixed, whence \((S, t_p, \kappa_p)\) is \(\text{BA}(S_p)\)-bireal compact, and hence bireal compact in the sense of [2] (equivalently, in view of the characterization theorem [2, Theorem 3.3.2], real compact in the sense of Brümmer and Salbany [10]).

A similar argument for the functor \(\mathcal{R}\) shows that if \((X, u, v)\) is bireal compact, \(\mathcal{R}(X, u, v)\) is real dicompact. Hence, using an obvious notation:

**Corollary 3.6.** \(\mathcal{U}: i\text{fRdiComp}_2 \to \text{biRComp}\) is an adjoint functor and \(\mathcal{R}: \text{biRComp} \to i\text{fRdiComp}_2\) the co-adjoint of \(\mathcal{U}\).

Making a restriction to plain textures we have the following stronger results.

**Theorem 3.7.** \(\mathcal{U}: \text{fPCbiR}_2 \to p\text{Reg}_{\omega_2}\) is an isomorphism with inverse \(\mathcal{R}\).

**Proof.** We wish first to show that if \((S, S, \tau, \kappa)\) is plain and \(\varphi: S \to \mathbb{R}\) is pairwise continuous then \(\varphi: (S, S, \tau, \kappa) \to (\mathbb{R}, \mathbb{R}, \tau |_{\mathbb{R}}, \kappa |_{\mathbb{R}})\) is \(\omega\)-preserving.

Take \(u, u' \in S\) with \(u \leq u'\) and suppose that \(P_{\varphi(u')} \subseteq Q_{\varphi(u)}\). Then \(\varphi(u') < \varphi(u)\) and we take \(k \in \mathbb{R}\) with \(\varphi(u') < k < \varphi(u)\).

Setting \(V = \{r \in \mathbb{R} \mid r > k\} \subseteq \kappa_{\mathbb{R}}^k\) gives \(\varphi^{-1}[V] \subseteq \kappa^k\), whence \(\varphi^{-1}[V^c] \subseteq \kappa \subseteq S\). But \(P_w \subseteq \varphi^{-1}[V^c]\), so \(\varphi^{-1}[V^c] \subseteq Q_u\) and we obtain the contradiction \(\varphi(u) \leq k\). This establishes that \(\varphi\) is \(\omega\)-preserving and hence a \(\text{FPDtop}\) morphism.

Now take \((S, S, \tau, \kappa) \in\text{Ob}f\text{PCbiR}_2\). Then \(U(S, S, \tau, \kappa) = (S, \tau, \kappa^k)\), and so \((\mathcal{U} \circ \mathcal{U})(S, S, \tau, \kappa) = (S, X_{\tau^k}, \tau, \kappa)\). We must prove that \(\hat{S} = X_{\tau^k}\). By the proof of Lemma 3.4 and the above result we have \(\text{BA}(S, S) = \text{P}(S) = \text{BA}(S, X_{\tau^k}, \tau, \kappa)\), where these sets are taken for \((S, S, \tau, \kappa), (S, \tau, \kappa^k)\) and \((S, X_{\tau^k}, \tau, \kappa)\), respectively, and we denote this \(T\)-lattice by \(\mathcal{B}\) for short.

By Corollary 2.16, \(\xi_B\) is an \(\text{fReg}\) isomorphism between both \((S, S)\) and \((S, X_{\tau^k})\) and the texture induced on \(\xi_B[S]\) by \((H_B, \mathcal{H}_B)\). It is clear that \(X_{\tau^k} \subseteq S\), so take \(A_0 \in S\). By [6, Proposition 3.15], the mapping \(A \mapsto \xi_B[A]\) from \(X_{\tau^k}\) to \(\mathcal{H}_B|_{\xi_B[S]}\) is onto, and \(\xi_B[A_0] \in \mathcal{H}_B|_{\xi_B[S]}\), so there exists \(A_1 \in X_{\tau^k}\) with \(\xi_B[A_1] = \xi_B[A_0]\). However, \(A_0, A_1\) in \(S\) and be the same proposition the mapping \(A \mapsto \xi_B[A]\) from \(S\) to \(\mathcal{H}_B|_{\xi_B[S]}\) is one to one, so \(A_0 = A_1 \in X_{\tau^k}\), as required.

This gives \(\mathcal{R} \circ \mathcal{U} = \text{id}_{f\text{PCbiR}_2}\), and \(\mathcal{U} \circ \mathcal{R} = \text{id}_{p\text{Reg}_{\omega_2}}\) is trivial so the proof is complete.

**Corollary 3.8.** For \((S, S, \tau, \kappa) \in \text{Ob}f\text{PCbiR}_2\) we have \(\hat{S} = X_{\tau^k}\).

**Corollary 3.9.** \(\mathcal{U}: \text{fPRdiComp}_2 \to \text{biRComp}\) is an isomorphism with inverse \(\mathcal{R}\).

We now turn to the question of characterizing those plain real dicompact spaces whose image under \(\mathcal{U}\) is a trivial bitopology, that is a bitopology of the form \((X, u, v)\).

**Lemma 3.10.** For \((S, S, \tau, \kappa) \in \text{Ob}f\text{PRdiComp}_2\), \(\mathcal{U}(S, S, \tau, \kappa)\) is trivial if and only if \((S, S, \tau, \kappa) = (\mathcal{P}(S), \tau, \tau^c)\).
**Proof.** If \( \mathcal{U}(S, S, \tau, \kappa) = (S, \tau, \kappa^c) \) is a trivial bitopology we have \( \tau = \kappa^c \) by definition, so \( \kappa = \tau^c \). By Corollary 3.8 we have \( S = \mathcal{K}^c \) and, by (3.1) we deduce that \( S \) has the property \( A \in S \Rightarrow A^c \in S \). However it is now clear that for \( s \in S \) we must have \( P_s = \{ s \} \), so since \( (S, S) \) is plain we obtain \( S = \mathcal{P}(S) \).  

**Corollary 3.11.** The functor \( \mathcal{U} \) induces an isomorphism from the full subcategory of \( \text{fPRdiComp}_2 \) whose objects are complemented ditopologies on a discrete texture to the full subcategory of \( \text{biRComp} \) whose objects have trivial bitopologies. Likewise, \( \mathcal{R} \) induces the inverse of this isomorphism.

It will be noted that the full subcategory of \( \text{fPRdiComp}_2 \) mentioned in the above Corollary is the same as the coreflective subcategory of \( \text{fRdiComp}_2 \) found in Theorem 3.1 by considering the joint topology.

We mention finally that if instead of restricting to plain textures we work with categories in which the morphisms are ditopifications we can raise Theorem 3.5 and Corollary 3.6 to an equivalence. More specifically, defining \( \mathcal{R}_d : \text{pCReg}_{gw2} \rightarrow \text{dfNpCbiR}_2^c \) by

\[
\mathcal{R}_d((X, u_X, v_X) \xrightarrow{\psi} (Y, u_Y, v_Y)) = (X, \mathcal{K}^c_{u_Xv_X, u_X}, v_X^c) \xrightarrow{(F_2, F_2)} (Y, \mathcal{K}^c_{u_Yv_Y, u_Y}, v_Y^c)
\]

we have, using an obvious notation:

**Theorem 3.12.** \( \mathcal{R}_d : \text{pCReg}_{gw2} \rightarrow \text{dfNpCbiR}_2^c \) is an equivalence.

**Proof.** Clearly \( \mathcal{R}_d \) may be written as the composition of the functors

\[
\text{pCReg}_{gw2} \xrightarrow{\mathcal{R}} \text{fPCbiR}_2 \xrightarrow{\mathcal{D}} \text{dfPbiR}_2^c \hookrightarrow \text{dfNpCbiR}_2^c.
\]

Here \( \mathcal{R} \) is an isomorphism by Theorem 3.5, \( \mathcal{D} \) is a restriction of the isomorphism between \( \text{fP Ditop} \) and \( \text{dfP Ditop} \) given in [7, Theorem 2.7] and \( \mathcal{C} \) is a restriction of the embedding of \( \text{dfP Ditop} \) in \( \text{dfNp Ditop} \) which was shown to be an equivalence in [23, Theorem 4.3]. Hence \( \mathcal{R}_d \) is an equivalence.

**Corollary 3.13.** \( \mathcal{R}_d : \text{biRComp} \rightarrow \text{dfRdiComp}_2 \) is an equivalence.

**References**


[22] F. Yıldız, Spaces of bicontinuous real difunctions and real compactness, PhD thesis (in Turkish), Hacettepe University, 2006.