The Kumaraswamy Exponential–Weibull Distribution: Theory and Applications

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Abstract

Significant progress has been made towards the generalization of some well–known lifetime models, which have been successfully applied to problems arising in several areas of research. In this paper, some properties of the new Kumaraswamy exponential-Weibull (KwEW) distribution are provided. This distribution generalizes a number of well-known special lifetime models such as the Weibull, exponential, Rayleigh, modified Rayleigh, modified exponential and exponentiated Weibull distributions, among others. The beauty and importance of the new distribution lies in its ability to model monotone and non-monotone failure rate functions, which are quite common in environmental studies. We derive some basic properties of the KwEW distribution including ordinary and incomplete moments, skewness, kurtosis, quantile and generating functions, mean deviations and Shannon entropy. The method of maximum likelihood and a Bayesian procedure are used for estimating the model parameters. By means of a real lifetime data set, we prove that the new distribution provides a better fit than the Kumaraswamy Weibull, Marshall-Olkin exponential-Weibull, extended Weibull, exponential-Weibull and Weibull models. The application indicates that the proposed model can give better fits than other well-known lifetime distributions.

Keywords: Exponential–Weibull distribution, Fox–Wright generalized ${}_p\Psi_q$ function, generalized distribution, lifetime data, maximum likelihood, moment.

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1. Introduction

In many applied areas like lifetime analysis, finance, insurance and biology, there is a clear need for extended forms of the classical distributions, i.e., new distributions more flexible to model real data that present a high degree of skewness and kurtosis in these areas. Recent developments focus on new techniques by adding parameters to existing distributions for building classes of more flexible distributions. Following this idea, Cordeiro et al. [6] introduced an interesting method by adding two new parameters to a parent distribution to model data with a high degree of skewness and kurtosis. The generated family can provide more flexibility to model various types of data. If $G(x)$ is the cumulative distribution function (cdf) of a baseline model, then the *Kumaraswamy generalized* (Kw-G) family has cdf given by

$$F(x) = 1 - \{1 - G^{\alpha}(x)\}^{\gamma}. \tag{1.1}$$

The probability density function (pdf) corresponding to (1.1) is given by

$$f(x) = \alpha\, \gamma\, g(x)\, G^{\alpha-1}(x)\, \{1 - G^{\alpha}(x)\}^{\gamma-1}\,. \tag{1.2}$$

Each new Kw-G distribution can be obtained from a specified G distribution. For $\alpha = \gamma = 1$, the G distribution is a basic exemplar of the Kw-G family with a continuous crossover towards cases with different shapes (e.g., a particular combination of skewness and kurtosis). One major benefit of equation (1.2) is its ability of fitting skewed data that can not be properly fitted by existing distributions. Further, it allows for greater flexibility of its tails and can be widely applied in many areas of reliability and biology.

The Weibull distribution is a very popular distribution for modeling lifetime data. When modeling monotone hazard rates, it may be an initial choice because of its skewed density shapes. However, it does not have a bathtub or upside–down bathtub shaped hazard rate function (hrf) and can not be used to model the lifetime of certain systems. Such bathtub hazard curves have nearly flat middle portions and the corresponding densities have a positive anti-mode. An example of the bathtub-shaped failure rate is the human mortality experience with a high infant mortality rate which reduces rapidly to reach a low level. Unimodal failure rates can be observed in course of a disease whose mortality reaches a peak after some finite period and then declines gradually. Thus, it cannot be used to model lifetime data with a bathtub shaped hazard function, such as human mortality and machine life cycles. Therefore, several researchers have developed various extensions and modified forms of the Weibull distribution having a number of parameters ranging from two to five parameters.

In the last few years, new classes of distributions aim to define generalized Weibull distributions to cope with bathtub shaped failure rates. Mudholkar and Srivastava [17] and Mudholkar et al. [18] pioneered and studied the exponentiated Weibull (ExpW) distribution to analyze bathtub failure data. A good review of some of these extended models is presented in Pham and Lai [25]. Also, the additive Weibull distribution was proposed by Xie and Lai [27], the modified Weibull distribution by Lai et al. [12] and the generalized modified Weibull distribution by Carrasco et al. [2]. Further, Lee et al. [13] and Silva et al. [23] defined two extensions of the Weibull model called the beta Weibull (BW) and beta modified Weibull (BMW) distributions, respectively.

The *exponential–Weibull* (EW) distribution proposed by Cordeiro *et al.* [5] has cdf and pdf given by

$$G(x) = 1 - \mathrm{e}^{-\lambda\,\mathrm{x} - \beta\,\mathrm{x}^{\mathrm{k}}}\,1_{\mathbb{R}_+}(\mathrm{x}),\;\; \lambda > 0,\; \beta > 0,\; \mathrm{k} > 0 \tag{1.3}$$

and

$$g(x) = (\lambda + \beta\,k\,x^{k-1})\,\mathrm{e}^{-\lambda\,\mathrm{x} - \beta\,\mathrm{x}^{\mathrm{k}}}\,1_{\mathbb{R}_+}(\mathrm{x}), \tag{1.4}$$

respectively, where $\lambda > 0$ and $k > 0$ are shape parameters, $\beta > 0$ is a scale parameter and $1_A(x)$ denotes the characteristic function of the set A, i.e. $1_A(x) = 1$ when $x \in A$ and equals 0 elsewhere.

We generalize the EW model by defining the *Kumaraswamy exponential–Weibull* (KwEW) distribution. The cdf and pdf of the KwEW distribution, for which the EW is the baseline model, are given by

$$F(x) = 1 - \left\{1 - \left(1 - \mathrm{e}^{-\lambda\,\mathrm{x} - \beta\,\mathrm{x}^{\mathrm{k}}}\right)^{\alpha}\right\}^{\gamma}\,1_{\mathbb{R}_+}(x) \tag{1.5}$$

and

$$\begin{aligned} f(x) &= \alpha\gamma\left(\lambda + k\beta x^{k-1}\right)\mathrm{e}^{-\lambda\,\mathrm{x} - \beta\,\mathrm{x}^{\mathrm{k}}}\left(1 - \mathrm{e}^{-\lambda\,\mathrm{x} - \beta\,\mathrm{x}^{\mathrm{k}}}\right)^{-1+\alpha} \\ &\quad \times \left\{1 - \left(1 - \mathrm{e}^{-\lambda\,\mathrm{x} - \beta\,\mathrm{x}^{\mathrm{k}}}\right)^{\alpha}\right\}^{-1+\gamma}\,1_{\mathbb{R}_+}(x), \end{aligned} \tag{1.6}$$

respectively, where $\lambda > 0$, $\beta > 0$, $k > 0$, $\alpha > 0$ and $\gamma > 0$. Hereafter, we denote by $X \sim \mathrm{KwEW}_{\alpha,\gamma}(\lambda, \beta, k)$ a random variable having the pdf (1.6).

The density (1.6) is much more flexible than the EW density and can allow for greater flexibility of the tails. It can exhibit different behavior depending on the parameter values. In fact, Figure 1 (a,c) and Figure 2 (d) reveal that the mode of the pdf increases as α and λ increases, respectively. Figure 2 (e) also shows that the mode of the pdf increases as k increases. The new parameter γ behaves somewhat as a scale parameter as shown in Figure 1(b). The structure of the density function (1.6) can be motivated as it provides more flexible distribution than the two-parameter Weibull and many other extended Weibull distributions (see Table 1).

The rest of the paper is organized as follows. In Section 2, twelve widely-known special models of the proposed distribution are presented. A useful expansion for the KwEW density and explicit expressions for certain mathematical quantities of X are obtained in Section 3. We demonstrate in Section 4 that the KwEW density is an infinite mixture of EW densities. Further, we obtain alternative expressions for the moments and generating function. The estimation of the model parameters by maximum likelihood and a Bayesian procedure are addressed in Section 5. We prove in Section 6 the flexibility of the new distribution for modeling lifetime data by means of a real data set. A bivariate extension is given in Section 7. The paper is concluded in Section 8.

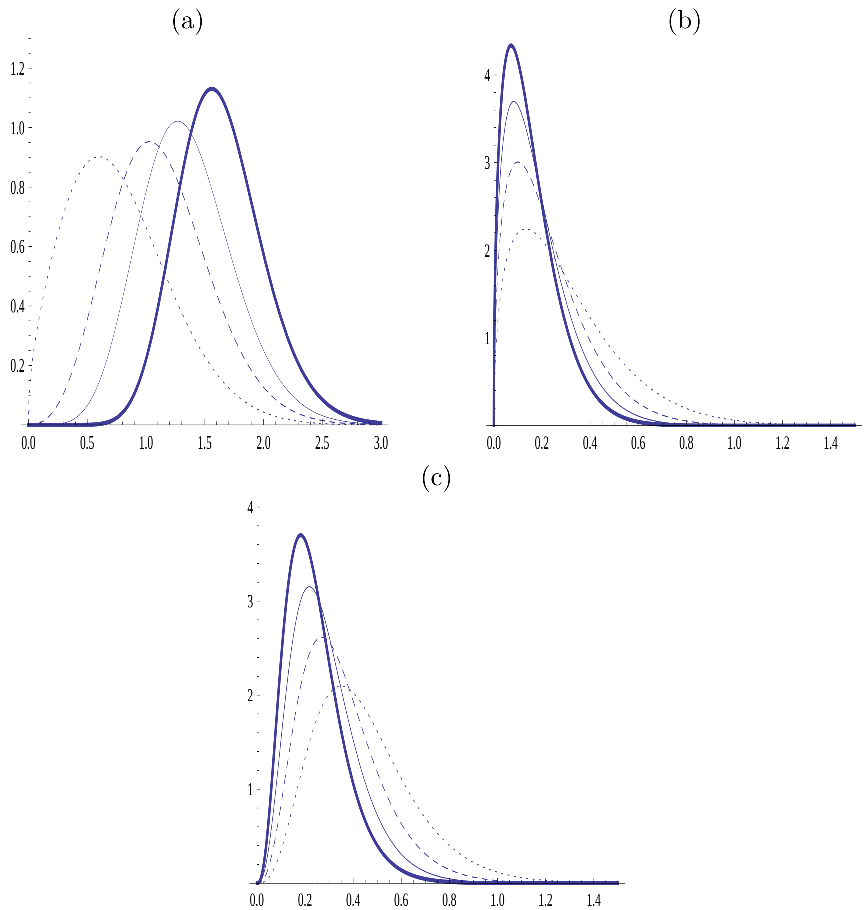

Figure 1. Plots of the KwEW density function. (a) $\lambda = 0.5$, $\beta = 0.6$, $k = 2$, $\gamma = 1.5$ and $\alpha = 1.4$ (dotted line), $\alpha = 3$ (dashed line), $\alpha = 5$ (solid line), $\alpha = 10$ (thick line). (b) $\lambda = 3.5$, $\beta = 1.6$, $k = 2$, $\alpha = 1.5$ and $\gamma = 1$ (dotted line), $\gamma = 1.5$ (dashed line), $\gamma = 2$ (solid line), $\gamma = 2.5$ (thick line). (c) $\beta = 2.6$, $k = 1.2$, $\alpha = 3.5$, $\gamma = 1.7$ and $\lambda = 1$ (dotted line) $\lambda = 2$, (dashed line), $\lambda = 3$ (solid line), $\lambda = 4$ (thick line).

2. Special Distributions

We point out some special cases of the $\mathrm{KwEW}_{\alpha,\gamma}(\lambda, \beta, k)$ distribution by specifying its parameters values. Table 1 lists twelve important special models of the new distribution. For example, the $\mathrm{KwEW}_{\alpha,\gamma}(0, \beta, k)$ model reduces to the *Kw-modified Weibull* [12], the $\mathrm{KwEW}_{1,1}(\lambda, \beta, k)$ refers to the *exponential–Weibull* [5], the $\mathrm{KwEW}_{1,1}(\lambda, \beta, 2)$ is the *modified Rayleigh*, the $\mathrm{KwEW}_{1,1}(\lambda, \beta, 1)$ is the *modified exponential* and the $\mathrm{KwEW}_{1,1}(0, \beta, k)$ becomes the classical two-parameter Weibull. If $k = 1$ and $k = 2$ in addition to $\alpha = 1$, $\gamma = 1$ and $\lambda = 0$, it coincides with the exponential and Rayleigh distributions, respectively. Finally, the $\mathrm{KwEW}_{1,\gamma}(0, \beta, k)$ model becomes the ExpW distribution pioneered by [17, 18].

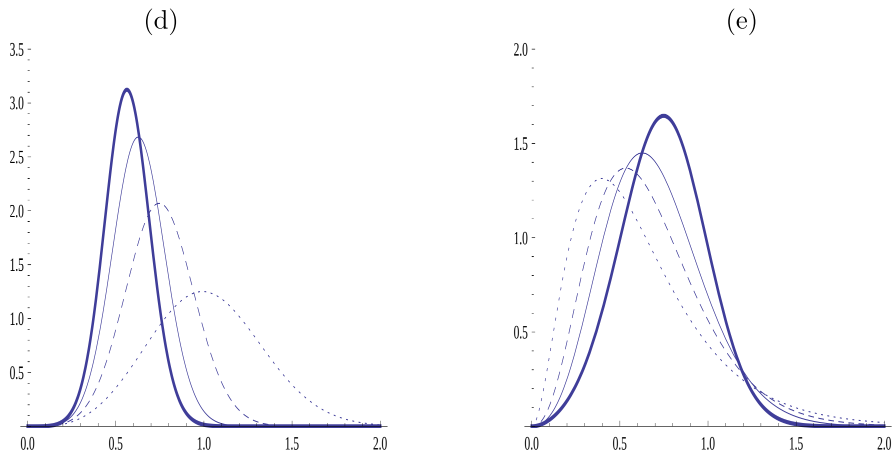

Figure 2. Plots of the KwEW density function. (d) $\lambda = 1.3$, $k = 3$, $\alpha = 5$, $\gamma = 1.3$ and $\beta = 0.5$ (dotted line), $\beta = 2$ (dashed line), $\beta = 4$ (solid line), $\beta = 6$ (thick line). (e) $\lambda = 1$, $\beta = 1.5$, $\alpha = 3$, $\gamma = 1.3$ and $k = 1$ (dotted line), $k = 1.5$ (dashed line), $k = 2$ (solid line), $k = 3$ (thick line).

Table 1. Some special distributions

Model	λ	β	k	α	γ
Kw-Modified Weibull	0	–	–	–	–
Kw-Exponential	–	0	–	–	–
Kw-Rayleigh	0	–	2	–	–
Exponentiated Weibull	0	–	–	1	–
Kw-Linear Failure Rate	–	–	2	–	–
Exponential Weibull	–	–	–	1	1
Two Parameter Weibull	0	–	–	1	1
Exponential	0	–	1	1	1
Rayleigh	0	–	2	1	1
Modified Rayleigh	–	–	2	1	1
Modified Exponential	–	–	1	1	1
Linear Failure Rate	–	–	2	1	1

3. Main Properties

We derive computational sum–representations and explicit expressions for the ordinary and central moments, skewness, kurtosis, generating and quantile functions, Shannon entropy and mean deviations of X. These expressions can be evaluated analytically or numerically using packages such as *Mathematica*, *Matlab* and *Maple*. In numerical applications, the infinite sums can be truncated whenever convergence is observed.

3.1. A Useful Expansion. Here, we provide a useful expansion for the KwEW pdf (1.6). By using the power series

$$(1-z)^{\beta-1}=\sum_{n=0}^{\infty} a_n\, z^n,\ |z|<1,\ \beta>0, \tag{3.1}$$

we obtain

$$f(x)=\alpha\,\gamma\,\left(\lambda+k\beta x^{k-1}\right)\sum_{m=0}^{\infty} W_m\,\left(\mathrm{e}^{-\lambda\,\mathrm{x}-\beta\,\mathrm{x}^{\mathrm{k}}}\right)^{m+1}, \tag{3.2}$$

where

$$a_n=\frac{(-1)^n\Gamma(\beta)}{\Gamma(\beta-n)n!},\quad W_m=\sum_{n=0}^{\infty}\frac{(-1)^{n+m}\,\Gamma(\gamma)\,\Gamma\{(n+1)\gamma\}}{\Gamma(\gamma-n)\Gamma\{(n+1)\gamma-m\}m!\,n!}.$$

3.2. Moments. Some key features of a distribution such as skewness and kurtosis can be studied through its moments. We derive closed-form expressions for the ordinary and central moments, generating function, skewness and kurtosis of X.

First, we introduce the Fox-Wright function ${}_p\Psi_q$, which is an extension of the usual generalized hypergeometric function ${}_pF_q$, with $p\in\mathbb{N}_0$ numerator parameters $a_1,\cdots,a_p\in\mathbb{C}$ and $q\in\mathbb{N}_0$ denominator parameters $b_1,\cdots,b_q\in\mathbb{C}\setminus\mathbb{Z}_0^-$, defined by

$${}_p\Psi_q\left[\begin{array}{c}(a_1,A_1),\ \cdots,(a_p,A_p)\\ \\ (b_1,B_1),\ \cdots,(b_q,B_q)\end{array}\middle|\, z\right]=\sum_{n\geq 0}\frac{\Gamma(a_1+A_1n)\cdots\Gamma(a_p+A_pn)}{\Gamma(b_1+B_1n)\cdots\Gamma(b_q+B_qn)}\,\frac{z^n}{n!},$$

where the empty products are conventionally taken to be equal to one, and

$$A_j>0,\ j=\overline{1,p},\ B_k>0,\ k=\overline{1,q},\quad \Delta=1+\sum_{j=1}^{q}B_j-\sum_{j=1}^{p}A_j\geq 0,$$

(see, for instance [11, p. 56]). The convergence will occur for suitably bounded values of $|z|$ such that

$$|z|<\nabla=\left(\prod_{j=1}^{p}A_j^{-A_j}\right)\left(\prod_{j=1}^{q}B_j^{B_j}\right).$$

We derive closed-form expressions for the real order moments of X. We have

$$\begin{aligned}\mathsf{E}(X^r)&=\alpha\,\gamma\sum_{m=0}^{\infty}W_m\int_0^\infty x^r\,\left(\lambda+\beta\,k\,x^{k-1}\right)\,\mathrm{e}^{-\lambda\,(\mathrm{m}+1)\,\mathrm{x}}\,\mathrm{e}^{-\beta\,(\mathrm{m}+1)\,\mathrm{x}^{\mathrm{k}}}\,\mathrm{dx}\\ &=\alpha\,\gamma\,\lambda\sum_{m=0}^{\infty}W_m\int_0^\infty x^r\,\mathrm{e}^{-\lambda\,(\mathrm{m}+1)\,\mathrm{x}}\,\mathrm{e}^{-\beta\,(\mathrm{m}+1)\,\mathrm{x}^{\mathrm{k}}}\,\mathrm{dx}\\ &\quad+\alpha\gamma\beta k\sum_{m=0}^{\infty}W_m\int_0^\infty x^{r+k-1}\mathrm{e}^{-\lambda(\mathrm{m}+1)\mathrm{x}}\mathrm{e}^{-\beta(\mathrm{m}+1)\mathrm{x}^{\mathrm{k}}}\mathrm{dx}.\end{aligned}$$

The rth moment is a linear combination of integrals of the type $\mathcal{I}(\omega)$ based on a similar approach by [19, Eq. (2.1)], where $\omega = (\kappa, \mu, a, \eta)$ and all components are positive parameters,

$$\mathcal{I}(\omega) = \int_0^\infty x^{\kappa-1}\, \mathrm{e}^{-(\mu\mathrm{x}+\mathrm{ax}^\eta)}.$$

A representation for this integral is given by [21, p. 515, Corollary 1.1]:

$$\mathcal{I}(\omega) = \begin{cases} \mu^{-\kappa}\, {}_1\Psi_0\left[\begin{array}{c} (\kappa,\eta) \\ \text{---} \end{array} \middle| -\frac{a}{\mu^\eta}\right], & 0<\eta<1, \\ \frac{\Gamma(\kappa)}{(\mu+a)^\kappa}, & \eta = 1, \\ \frac{1}{\eta a^{\kappa/\eta}}\, {}_1\Psi_0\left[\begin{array}{c} \left(\frac{\kappa}{\eta},\frac{1}{\eta}\right) \\ \text{---} \end{array} \middle| -\frac{\mu}{a^{1/\eta}}\right], & \eta > 1. \end{cases} \tag{3.3}$$

Thus, for all $k \in (0,1)$, we can write

$$\begin{aligned} \mathsf{E}(X^r) &= \alpha\,\gamma\,\lambda \sum_{m=0}^\infty W_m\, \mathcal{I}(r+1, \lambda\,(m+1), \beta\,(m+1), k) \\ &\quad + \alpha\gamma\beta k \sum_{m=0}^\infty W_m\, \mathcal{I}(r+k-1, \lambda(m+1), \beta(m+1), k) \\ &= \sum_{m=0}^\infty W_m \frac{\gamma\,\alpha}{\lambda^r\,(m+1)^{r+1}}\, {}_1\Psi_0\left[\begin{array}{c} (r+1,k) \\ \text{---} \end{array} \middle| -\frac{\beta}{\lambda^k\,(m+1)^{k-1}}\right] \\ &\quad + \sum_{m=0}^\infty W_m \frac{\alpha\,\gamma\,\beta\,k}{(\lambda\,(m+1))^{r+k}}\, {}_1\Psi_0\left[\begin{array}{c} (r+k,k) \\ \text{---} \end{array} \middle| -\frac{\beta}{\lambda^k(m+1)^{k-1}}\right]. \end{aligned} \tag{3.4}$$

For $k = 1$, we have

$$\mathsf{E}(X^r) = \frac{\lambda\,\alpha\,\gamma\,\Gamma(r+1)}{(\lambda+\beta)^{r+1}} \sum_{m=0}^\infty \frac{W_m}{(m+1)^{r+1}}. \tag{3.5}$$

The remaining values of the parameter $k > 1$ lead to

$$\begin{aligned} \mathsf{E}(X^r) &= \sum_{m=0}^\infty W_m \frac{\alpha\,\gamma\,\lambda}{k\{\beta(m+1)\}^{\frac{r+1}{k}}}\, {}_1\Psi_0\left[\begin{array}{c} \left(\frac{r+1}{k},\frac{1}{k}\right) \\ \text{---} \end{array} \middle| -\frac{(m+1)^{1-\frac{1}{k}}\,\lambda}{\beta^{\frac{1}{k}}}\right] \\ &\quad + \sum_{m=0}^\infty W_m \frac{\alpha\,\gamma\,\beta\,k}{\{\beta(m+1)\}^{\frac{r+k}{k}}}\, {}_1\Psi_0\left[\begin{array}{c} \left(\frac{r}{k}+1,\frac{1}{k}\right) \\ \text{---} \end{array} \middle| -\frac{(m+1)^{1-\frac{1}{k}}\,\lambda}{\beta^{\frac{1}{k}}}\right]. \end{aligned} \tag{3.6}$$

Hence, we have the following result:

3.1. Theorem. *If $X \sim \mathrm{KwEW}_{\alpha,\gamma}(\lambda,\beta,k)$, then (for all $r>-1$) we have*

$$
(3.7)\qquad \mathsf{E}(X^r) = \begin{cases}
\sum_{m=0}^{\infty} W_m \dfrac{\gamma\,\alpha}{\lambda^r\,(m+1)^{r+1}} \\
\quad \times {}_1\Psi_0\left[\begin{array}{c} (r+1,k) \\ \hline \end{array} \middle| -\dfrac{\beta}{\lambda^k\,(m+1)^{k-1}} \right] \\
\quad + \sum_{m=0}^{\infty} W_m \dfrac{\alpha\,\gamma\,\beta\,k}{\{\lambda\,(m+1)\}^{r+k}} \\
\quad \times \ {}_1\Psi_0\left[\begin{array}{c} (r+k,k) \\ \hline \end{array} \middle| -\dfrac{\beta}{\lambda^k(m+1)^{k-1}} \right], & 0<k<1, \\[2ex]
\sum_{m=0}^{\infty} W_m \dfrac{\lambda\,\alpha\,\gamma\,\Gamma(r+1)}{(\lambda+\beta)^{r+1}\,(m+1)^{r+1}}, & k=1, \\[2ex]
\sum_{m=0}^{\infty} W_m \dfrac{\alpha\,\gamma\,\lambda}{k\{\beta(m+1)\}^{\frac{r+1}{k}}} \\
\quad \times {}_1\Psi_0\left[\begin{array}{c} \left(\dfrac{r+1}{k}, \dfrac{1}{k}\right) \\ \hline \end{array} \middle| -\dfrac{(m+1)^{1-\frac{1}{k}}\,\lambda}{\beta^{\frac{1}{k}}} \right] \\
\quad + \sum_{m=0}^{\infty} W_m \dfrac{\alpha\,\gamma\,\beta\,k}{\{\beta(m+1)\}^{\frac{r+k}{k}}} \\
\quad \times {}_1\Psi_0\left[\begin{array}{c} \left(\dfrac{r}{k}+1, \dfrac{1}{k}\right) \\ \hline \end{array} \middle| -\dfrac{(m+1)^{1-\frac{1}{k}}\,\lambda}{\beta^{\frac{1}{k}}} \right], & k>1.
\end{cases}
$$

Proof. It only remains to verify the convergence conditions of the Fox–Wright series, which depends only on the parameter k. Note that, when $k \in (0,1)$, $\Delta = 1-k > 0$, so that both series in (3.4) converge. So, it does when $k=1$. Finally, for $k>1$, the value $\Delta = 1-\frac{1}{k} > 0$ ensures that the moment $\mathsf{E}(X^r)$ is finite for any $r>-1$. $\square$

3.2. Remark. For certain integer and rational values of the parameter k, we adopt a representation of the Fox–Wright ${}_1\Psi_0$ function in terms of the generalized hypergeometric ${}_pF_q$ functions, which is discussed in detail in [16]. By their Eq. (3.3), for all positive rational $A = \frac{m}{M}$, one has

$$
{}_1\Psi_0\left[\begin{array}{c} (a, \frac{m}{M}) \\ \hline \end{array} \middle| z \right] = \Gamma(a) + \sum_{j=1}^{M} \frac{\Gamma(a+\frac{m}{M}j)\,z^j}{j!}
$$
$$
\times \ {}_{m+1}F_M\left[\begin{array}{c} 1, \frac{j}{M}+\frac{a}{m}, \cdots, \frac{j}{M}+\frac{a+m-1}{m} \\ \hline \end{array} \middle| \frac{m^m\,z^M}{M^M} \right],
$$

where ${}_pF_q$ stands for the generalized hypergeometric function which is a built–in *Mathematica* function specified by

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HypergeometricPFQ[{a_1,\ldots, a_p},{b_1,\ldots, b_q},z].
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On the other hand, the same authors also give an insight into transforming Fox–Wright Ψ functions into Meijer G–functions for rational arguments. Referring to [16, Eq. (5.1)], one has

$$ {}_1\Psi_0\left[\begin{array}{c} (a, \frac{m}{M}) \\ \text{---} \end{array}\middle| z\right] = \frac{2\sqrt{M}\, m^a}{\Gamma(a)\,\sqrt{m}\,\pi^{\frac{M+m-1}{2}}} \\ \times\, G^{M,m}_{m,M}\left(\frac{m^m\,(-z)^M}{M^M}\,\middle|\, \begin{array}{c} 1-\frac{a}{m},\cdots, 1-\frac{a+m-1}{m} \\ 0, \frac{1}{M},\cdots,\frac{M-1}{M}\end{array}\right). $$

See, for example, the monographs [14, Ch. V] and [11] for an introduction to the G–function. ■

3.3. Remark. The nth factorial moment of order of X is given by

$$ \Phi_n = \mathsf{E}[X(X-1)(X-2)\cdots(X-n+1)] = \left.\frac{\mathrm{d}^n\left[\mathsf{E}(t^X)\right]}{\mathrm{d}t^n}\right|_{t=1}. $$

Based on the Viète–Girard formula for expanding the polynomial $X(X-1)(X-2)\cdots(X-n+1)$, we obtain

$$ \Phi_n = \sum_{r=1}^n (-1)^{n-r}\left\{\sum_{1\le \ell_1<\cdots<\ell_r\le n-1} \ell_1\cdots\ell_r\right\}\mathsf{E}(X^r), $$

where the second sum represents elementary symmetric polynomials:

$$ e_r = e_r(\ell_1,\cdots,\ell_r) = \sum_{1\le \ell_1<\cdots<\ell_r\le n-1} \ell_1\cdots\ell_r, \qquad r = \overline{0, n-1}. $$

This in conjunction with positive integer rth order moment expression given in equation (3.7) provides an exact power series for the fractional order moments. ■

3.4. Remark. The moment generating function (mgf) $M(t) = E(\mathrm{e}^{\mathrm{t\,X}})$ of X can be obtained by setting $r = 0$ and replacing $[\lambda\,(m+1)]$ by $[\lambda\,(m+1)-t]$ in equation (3.7).■

3.5. Remark. The central moments (μ_n) and cumulants (κ_n) of X are easily obtained from (3.7) as

$$ \mu_n = \sum_{k=0}^n (-1)^k \binom{n}{k}\mu_1'^k\,\mu_{n-k}' \qquad \text{and} \qquad \kappa_n = \mu_n' - \sum_{k=1}^{n-1}\binom{n-1}{k-1}\kappa_k\,\mu_{n-k}', $$

respectively, where $\kappa_1 = \mu_1'$. Thus, $\kappa_2 = \mu_2' - \mu_1'^2$, $\kappa_3 = \mu_3' - 3\mu_2'\mu_1' + 2\mu_1'^3$, etc. Clearly, the skewness and kurtosis measures can be calculated from the ordinary moments using well-known relationships.■

Some numerical values for the skewness and kurtosis of X are listed in Table 2. The figures in this table indicate a large range for the skewness of X, although the kurtosis does not vary much.

Table 2. Skewness and kurtosis of the KwEW distribution for selected parameter values.

λ	β	k	α	γ	Skewness	Kurtosis
1.0	2.6	1.5	10	30	-0.001	1.229
2.0	2.6	1.5	10	30	-0.001	1.229
3.0	2.6	1.5	10	30	-0.001	1.229
4.0	2.6	1.5	10	30	-0.001	1.229
–	–	–	–	–	–	–
1.3	0.5	2.5	25	18	-0.002	1.234
1.3	2.0	2.5	25	18	-0.002	1.234
1.3	4.0	2.5	25	18	-0.002	1.234
1.3	6.0	2.5	25	18	-0.002	1.234
–	–	–	–	–	–	–
0.2	3.4	1.0	2.0	3.0	0.150	1.251
0.2	3.4	1.5	2.0	3.0	0.150	1.251
0.2	3.4	2.0	2.0	3.0	0.150	1.251
0.2	3.4	3.0	2.0	3.0	0.150	1.251
–	–	–	–	–	–	–
0.7	0.7	2.0	0.2	5.0	0.914	5.283
0.7	0.7	2.0	1.2	5.0	0.218	1.275
0.7	0.7	2.0	1.8	5.0	0.149	1.245
0.7	0.7	2.0	10	5.0	0.049	1.238
–	–	–	–	–	–	–
3.5	1.6	3.0	5.0	0.5	0.190	1.306
3.5	1.6	3.0	5.0	1.0	0.146	1.277
3.5	1.6	3.0	5.0	1.5	0.123	1.263
3.5	1.6	3.0	5.0	2.0	0.108	1.254

Next, we discuss some other structural properties of X, i.e., survival, hazard rate, mean residual life, entropy, mean deviations and quantile function (qf).

3.3. Survival, Hazard rate, Quantile function, Skewness and Kurtosis. Central role is playing in the reliability theory by the ratio of the pdf and survival function. The survival function of X is given by

$$(3.8)\qquad S(x) = \left\{1 - \left(1 - \mathrm{e}^{-\lambda\,\mathrm{x} - \beta\,\mathrm{x}^{\mathrm{k}}}\right)^{\alpha}\right\}^{\gamma} 1_{\mathbb{R}_+}(x)\,.$$

Then, the hrf of X reduces to

$$(3.9)\qquad h(x) = \frac{\alpha\gamma\left(kx^{-1+k}\beta + \lambda\right)\mathrm{e}^{-\lambda\,\mathrm{x} - \beta\,\mathrm{x}^{\mathrm{k}}}\left(1 - \mathrm{e}^{-\lambda\,\mathrm{x} - \beta\,\mathrm{x}^{\mathrm{k}}}\right)^{-1+\alpha}}{\left\{1 - \left(1 - \mathrm{e}^{-\lambda\,\mathrm{x} - \beta\,\mathrm{x}^{\mathrm{k}}}\right)^{\alpha}\right\}^{\gamma}} 1_{\mathbb{R}_+}(x)\,.$$

Figures 3 (a), (b) and (c) display some plots of $h(x)$.

The qf of X is determined by inverting (1.5) as

$$(3.10)\qquad Q(u) = F^{-1}(u) = -\frac{\log[1 - \{1 - (1-u)^{1/\gamma}\}^{1/\alpha}]}{\lambda + \beta}.$$

Simulating KwEW random variable is straightforward. Let U be a uniform variable on the unit interval $(0,1)$. Thus, by means of the inverse transformation method, the random variable X given by

$$(3.11)\qquad X = -\frac{\log[1 - \{1 - (1-U)^{1/\gamma}\}^{1/\alpha}]}{(\lambda + \beta)}$$

follows the density (1.6). In particular, the median of X is

$$M = -\frac{\log[1-\{1-0.5^{1/\gamma}\}^{1/\alpha}]}{(\lambda+\beta)}.$$

Further, the mode of $f(x)$ is obtained as

$$MO = -\frac{\log\left\{1-\left(\frac{2-\alpha}{1-\alpha\,\gamma}\right)^{1/\alpha}\right\}}{(\lambda+\beta)}$$

The shortcomings of the classical kurtosis measure are well-known. There are many heavy tailed distributions for which this measure is infinite. So, it becomes uninformative precisely when it needs to be. Indeed, our motivation to use quantile-based measures stemmed from the non-existence of the classical kurtosis for many of the Kw-G distributions. The Bowley's skewness is based on quartiles

$$S = \frac{Q(3/4) - 2Q(1/2) + Q(1/4)}{Q(3/4) - Q(1/4)}$$

and the Moors' kurtosis is based on octiles

$$K = \frac{\{Q(7/8) - Q(5/8)\} + \{Q(3/8) - Q(1/8)\}}{Q(6/8) - Q(2/8)},$$

where $Q(\cdot)$ is given by (3.10).

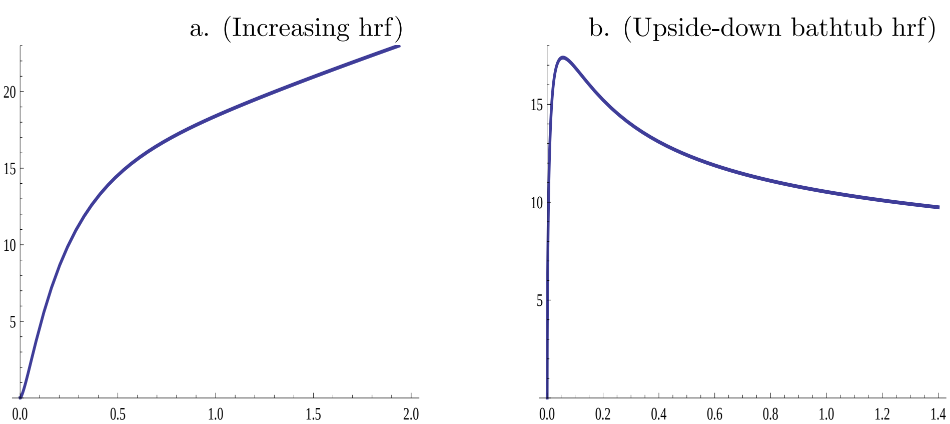

Figure 3. (a) $\alpha = 2.5$, $\gamma = 3$, $\lambda = 4$, $\beta = 1.2$, $k = 1.8$ (b) $\alpha = 2.3$, $\gamma = 1.3$, $\lambda = 1.8$, $\beta = 9$, $k = 0.7$.

3.4. Mean residual life function. The mean residual life function (mrlf) is defined by

$$K(x) = \frac{1}{S(x)}\,[E(X) - m_1(x)] - x,$$

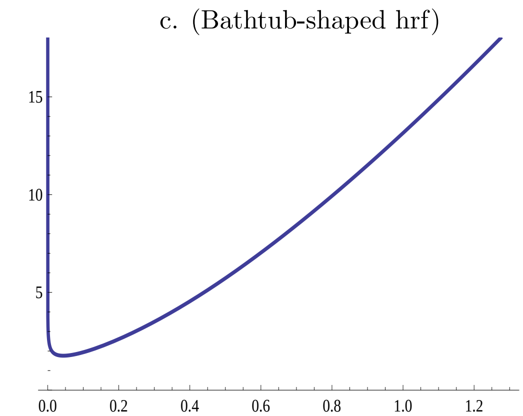

Figure 4. (c) $\alpha = 0.8$, $\gamma = 0.5$, $\lambda = 2.3$, $\beta = 10$, $k = 2.4$.

where $f(x)$, $E(X)$ and $S(x)$ are given in (1.6), (3.7) and (3.8), respectively, and

$$m_1(x) = \int_0^x y\, f(y)\, \mathrm{d}y = \alpha\, \gamma \sum_{m=0}^{\infty} W_m \times \int_0^x y\, \left(\lambda + \beta\, k\, y^{k-1}\right)\, \mathrm{e}^{-\lambda\,(\mathrm{m}+1)\,\mathrm{y}}\, \mathrm{e}^{-\beta\,(\mathrm{m}+1)\,\mathrm{y}^{\mathrm{k}}}\, \mathrm{dy}$$

is the first incomplete moment of X. By expanding the exponential in the last expression, we obtain

$$\begin{aligned} m_1(x) &= \alpha\, \gamma \sum_{m=0}^{\infty} W_m \sum_{j=0}^{\infty} \frac{(-1)^j\, [\lambda\,(m+1)]^j}{j!} \\ &\quad \times \int_0^x y^{j+1}\, \left(\lambda + \beta\, k\, y^{k-1}\right)\, e^{-\beta\,(m+1)\,y^k}\, \mathrm{d}y \\ &= \alpha\, \gamma \sum_{m=0}^{\infty} W_m \sum_{j=0}^{\infty} \frac{(-1)^j\, [\lambda\,(m+1)]^j}{j!} \\ &\quad \times \left(\lambda \int_0^x y^{j+1}\, G_{0,1}^{1,0}\left(\beta\,(m+1) y^{p/q} \,\middle|\, \begin{matrix} - \\ 0 \end{matrix}\right) \mathrm{d}y \right. \\ &\quad \left. + \beta\, \frac{p}{q} \int_0^x y^{j+p/q}\, G_{0,1}^{1,0}\left(\beta\,(m+1)\, y^{p/q} \,\middle|\, \begin{matrix} - \\ 0 \end{matrix}\right) \mathrm{d}y \right), \end{aligned} \tag{3.12}$$

where $\mathrm{e}^{-\mathrm{g(x)}} = \mathrm{G}_{0,1}^{1,0}\left(\mathrm{g(x)} \,\middle|\, \begin{matrix} - \\ 0 \end{matrix}\right)$, $k = p/q$ and $p \geq 1$ and $q \geq 1$ are natural co-prime numbers and

$$\begin{aligned}
&\int_0^x y^t\, G_{0,1}^{1,0}\left(\beta\,(m+1)\,y^{p/q} \,\middle|\, \begin{matrix} - \\ 0 \end{matrix}\right) \mathrm{d}y \\
&= \frac{q\,x^{p\,(t+1)}}{p(2\,\pi)^{(q-1)/2}}\, G_{p,p+q}^{q,p}\left(\frac{\{\beta\,(m+1)\}^q\,x^p}{q^q} \,\middle|\, \begin{matrix} \frac{-t}{p}\,, \frac{1-t}{p}\,, \ldots, \frac{p-t-1}{p}\,, - \\ 0\,, \frac{-t-1}{p}\,, \frac{t}{p}\,, \ldots, \frac{p-t-2}{p} \end{matrix}\right).
\end{aligned} \tag{3.13}$$

Equation (3.13) is obtained by using (13) of [5]. So, the first incomplete moment of X is easily obtained from (3.12) and (3.13).

Some applications of $m_1(x)$ refer to the Bonferroni and Lorenz curves of X defined, for a given probability π, by $B(\pi) = m_1(q)/(\pi\mu_1')$ and $L(\pi) = m_1(q)/\mu_1'$, respectively, where $\mu_1' = E(X)$ and $q = Q(\pi)$ is the value of (3.10) at $u = \pi$.

3.5. Entropy. An entropy is a concept encountered in physics and engineering. It is a measure of variation or uncertainty of a random variable X. An extension of Shannon's entropy for the continuous case can be defined as follows:

$$H(f) = -\int_0^\infty \log[f(x)]\, f(x)\, \mathrm{d}x\,. \tag{3.14}$$

Combining (1.6) and (3.14), we can write

$$\begin{aligned}
H(f) = &-\alpha\,\gamma \sum_{m=0}^\infty W_m \,\log\left(\alpha\,\gamma \sum_{m=0}^\infty W_m\right) \\
&\times \int_0^\infty \left(\lambda + \beta\,k\,x^{k-1}\right)\, \mathrm{e}^{-\lambda\,(\mathrm{m}+1)\,\mathrm{x}}\, \mathrm{e}^{-\beta\,(\mathrm{m}+1)\,\mathrm{x}^{\mathrm{k}}}\, \mathrm{dx} \\
&-\alpha\,\gamma \sum_{m=0}^\infty W_m \\
&\times \int_0^\infty \left(\lambda + \beta\,k\,x^{k-1}\right) \log\left(\lambda + \beta\,k\,x^{k-1}\right) \mathrm{e}^{-\lambda\,(\mathrm{m}+1)\,\mathrm{x}}\, \mathrm{e}^{-\beta\,(\mathrm{m}+1)\,\mathrm{x}^{\mathrm{k}}} \mathrm{dx} \\
&+\lambda\,\alpha\,\gamma \sum_{m=0}^\infty (m+1)\,W_m \\
&\times \int_0^\infty x\,\left(\lambda + \beta\,k\,x^{k-1}\right)\, \mathrm{e}^{-\lambda\,(\mathrm{m}+1)\,\mathrm{x}}\, \mathrm{e}^{-\beta\,(\mathrm{m}+1)\,\mathrm{x}^{\mathrm{k}}}\, \mathrm{dx} \\
&+\beta\,\alpha\,\gamma \sum_{m=0}^\infty (m+1)\,W_m \\
&\times \int_0^\infty x^k\,\left(\lambda + \beta\,k\,x^{k-1}\right)\, \mathrm{e}^{-\lambda\,(\mathrm{m}+1)\,\mathrm{x}}\, \mathrm{e}^{-\beta\,(\mathrm{m}+1)\,\mathrm{x}^{\mathrm{k}}}\, \mathrm{dx}.
\end{aligned} \tag{3.15}$$

Note that the first, third and fourth integrals on the right–hand side of (3.15) can be determined by using (3.7) for $r = 0$, 1 and k, respectively. The second one can be evaluated by numerical integration.

3.6. Order statistics. Let $X_1, X_2, \ldots, X_n$ be a random sample from the KwEW distribution and $X_{1:n} \leq X_{2:n} \leq \ldots \leq X_{n:n}$ denote the corresponding order statistics. Let $f_{i:n}(x)$ and $F_{i:n}(x)$ denote, respectively, the pdf and the cdf of the ith order statistic $X_{i:n}$. We can write

$$f_{i:n}(x) = \frac{n!f(x)}{(i-1)!\,(n-i)!}\sum_{l=0}^{n-i}\binom{n-i}{l}(-1)^l\;F(x)^{i-1+l},$$

and

$$F_{i:n}(x) = \frac{n!}{(i-1)!\,(n-i)!}\sum_{l=0}^{n-i}\frac{(-1)^l}{i+l}\binom{n-i}{l}F(x)^{i+l},$$

where $F(x)$ and $f(x)$ are given by equations (1.5) and (1.6), respectively. Using (3.1) and after some algebra, we obtain

$$f_{i:n}(x) = \frac{n!\,\alpha\,\gamma\,\left(\lambda+\beta\,k\,x^{k-1}\right)}{(i-1)!\,(n-i)!}\sum_{l=0}^{n-i}\sum_{u=0}^{\infty}\binom{n-i}{l}W_u\,\mathrm{e}^{-\lambda(\mathrm{u}+1)\,\mathrm{x}}\,\mathrm{e}^{-\beta(\mathrm{u}+1)\,\mathrm{x}^{\mathrm{k}}}$$

and

$$F_{i:n}(x) = \frac{n!}{(i-1)!\,(n-i)!}\sum_{l=0}^{n-i}\sum_{s=0}^{\infty}\binom{n-i}{l}\frac{\Gamma(i+l)\,(-1)^{l+s}}{\Gamma(i+l-s)s!\,(i+l)}$$
$$\times\left\{1-\left(1-\mathrm{e}^{-\lambda\,\mathrm{x}-\beta\,\mathrm{x}^{\mathrm{k}}}\right)^{\alpha}\right\}^{\gamma\,s},$$

where

$$W_u = \sum_{t=0}^{\infty}\sum_{s=0}^{\infty}\frac{(-1)^{l+s+t+u}\,\Gamma(i+l)\,\Gamma\{(s+1)\gamma\}\Gamma\{(t+1)\alpha\}}{\Gamma(i+l-s)\,\Gamma\{(s+1)\gamma-t\}\Gamma\{(t+1)\alpha-u\}s!\,t!\,u!}.$$

The sth moment of $X_{i:n}$ is given by

$$E\left(X_{i:n}^s\right) = \int_0^{\infty} x^s\,f_{i:n}(x)\,\mathrm{d}x.$$

By using $f_{i:n}(x)$ and equation (3.3), the moments of $X_{i:n}$ can be easily obtained.

3.7. Mean deviations. The mean deviations provide important information about characteristics of a population and they can be calculated from the first incomplete moment. Indeed, the amount of dispersion in a population may be measured to some extent by the deviations from the mean and median. The mean deviations of X about the mean $\mu_1' = E(X)$ and about the median M can be expressed as $\delta_1 = 2\mu F(\mu_1') - 2m_1(\mu_1')$ and $\delta_2 = \mu_1' - 2m_1(M)$, where $F(\mu_1')$ is calculated from (1.5) and $m_1(z) = \int_0^z x\,f(x)dx$ can be determined from (3.12) and (3.13).

4. Alternative Properties

In this section, we provide an alternative mixture representation for the pdf of X. By combining (1.4) and (3.2), we can write

$$f(x) = \sum_{m=0}^{\infty} V_m\,g_{m+1}(x), \tag{4.1}$$

where (for $m \geq 0$) $V_m = \alpha\gamma\, W_m/(m+1)$ and $g_{m+1}(x)$ is the pdf of the EW model with parameters $\lambda^\star = (m+1)\lambda$, $\beta^\star = (m+1)\beta$ and k. So, the KwEW density function is a mixture of EW densities.

Based on equation (4.1) and the results by Cordeiro *et al.* [5], we can obtain the following properties of X.

4.1. Moments. The calculations in this section involve some special functions. In particular, the gamma function $\Gamma(r) = \int_0^\infty w^{r-1}\mathrm{e}^{-\mathrm{w}}\mathrm{dw}$ ($r > 0$), and other functions given in Appendices A and B. In order to obtain μ'_s, we require an integral of the type

$$I(s;\lambda^\star,\beta^\star,k) = \int_0^\infty x^s\,\mathrm{e}^{-(\lambda^\star \mathrm{x} + \beta^\star \mathrm{x}^\mathrm{k})}\,\mathrm{dx}. \tag{4.2}$$

We provide four representations for (4.2). First, by expanding $\mathrm{e}^{-\lambda^\star \mathrm{x}}$ in Taylor series, we obtain

$$\begin{aligned} I(s;\lambda^\star,\beta^\star,k) &= \sum_{j=0}^\infty \frac{(-\lambda^\star)^j}{j!} \int_0^\infty x^{s+j}\,\mathrm{e}^{-\beta^\star \mathrm{x}^\mathrm{k}}\mathrm{dx} \\ &= \frac{1}{k\beta^{\star(s+1)/k}} \sum_{j=0}^\infty \frac{(-1)^j}{j!}\left(\frac{\lambda^\star}{\beta^{\star 1/k}}\right)^j \Gamma\left(\frac{s+1+j}{k}\right). \end{aligned}$$

The above sum can be expressed in a simple form for $k > 1$ using the Fox–Wright generalized hypergeometric function defined in Appendix A. We have

$$I(s;\lambda^\star,\beta^\star,k) = \frac{1}{k\beta^{\star(s+1)/k}}\,{}_1\Psi_0\left[\begin{array}{c}\left(\frac{s+1}{k},\frac{1}{k}\right)\\ -\end{array};-\frac{\lambda^\star}{\beta^{\star 1/k}}\right]. \tag{4.3}$$

Applying (4.3) to (4.1), we can write

$$\mu'_s = E(X^s) = \sum_{m=0}^\infty V_m\left[\lambda^\star\, I(s;\lambda^\star,\beta^\star,k) + \beta^\star\, k\, I(s+\lambda^\star-1;\lambda^\star,\beta^\star,k)\right]. \tag{4.4}$$

Secondly, we offer two formulae for the integral (4.2) provided that $k = p/q$, where $p \geq 1$ and $q \geq 1$ are relatively natural co-prime numbers. We use equation (2.3.2.13) in [26, p. 321] to obtain formulae for $I(s;\lambda^\star,\beta^\star,k)$ when $0 < k < 1$ and $k > 1$. We exclude the case $k = 1$ since the model is non-identifiable. For irrational k, an approximation of vanishingly small error can be made using increasingly accurate rational approximations for k. Let $z = (p^p\beta^{\star q})/(q^q\lambda^{\star p})$, ${}_pF_q(a_1,\ldots,a_p;b_1,\ldots,b_q;x)$ be the well-known generalized hypergeometric function and $\Delta(\tau,a) = (a/\tau,(a+1)/\tau,\ldots,(a+\tau-1)/\tau)$. The generalized hypergeometric functions are available in *Mathematica*. For $0 < k < 1$, we obtain

$$\begin{aligned} I(s;\lambda^\star,\beta^\star,k) = \sum_{j=0}^{q-1} &\frac{(-\beta^\star)^j\,\Gamma(s+1+jp/q)}{\lambda^{\star(s+1+jp/q)}\,j!} \\ &\times {}_{p+1}F_q\left(1,\Delta(p,s+1+jp/q);\Delta(q,1+j);(-1)^q z\right). \end{aligned} \tag{4.5}$$

For $\gamma > 1$, we have

$$
\begin{aligned}
I(s;\lambda^\star,\beta^\star,k) = &\sum_{j=0}^{p-1} \frac{(-1)^j\, q\,\Gamma\left([s+1+j]q/p\right)}{p\,\beta^{\star(s+1+j)q/p}\; j!} \\
&\times {}_{q+1}F_p\left(1, \Delta(q,[s+1+j]q/p); \Delta(p, 1+j); \frac{(-1)^p}{z}\right).
\end{aligned}
\tag{4.6}
$$

A fourth representation for the integral (4.2) also holds when $k = p/q$, where $p \geq 1$ and $q \geq 1$ are natural co-prime numbers. It follows in terms of the Meijer $G^{m,n}_{p,q}$ function defined in Appendix B and also available in *Mathematica*. For an arbitrary function $g(\cdot)$, we use the result

$$
\exp\{-g(x)\} = G^{1,0}_{0,1}\left(g(x)\Big|\begin{matrix} - \\ 0 \end{matrix}\right),
\tag{4.7}
$$

and then equation (4.2) can be expressed in the same form of equation (2.24.3.1) given by [26, p. 350]. Hence, we obtain

$$
I(s;\lambda^\star,\beta^\star,k) = \frac{p^{s+1/2}}{(2\pi)^{(p+q)/2-1}\,\lambda^{\star s+1}}\; G^{q,p}_{p,q}\left(\frac{\beta^{\star q}\,p^p}{\lambda^{\star p}\,q^q}\Bigg|\begin{matrix} \frac{-s}{p}, \frac{1-s}{p}, \ldots, \frac{p-s-1}{p} \\ 0 \end{matrix}\right).
\tag{4.8}
$$

Further, if $q = 1$, using equation (9.31.2) in [10]

$$
G^{m,n}_{p,q}\left(z^{-1}\Big|\begin{matrix} a_r \\ b_s \end{matrix}\right) = G^{n,m}_{q,p}\left(z\Big|\begin{matrix} 1-b_s \\ 1-a_r \end{matrix}\right),
$$

we have, as a special case of (4.8), the following result [3]

$$
I(s;\lambda^\star,\beta^\star,k) = \frac{p^{s+1/2}}{(2\pi)^{(p-1)/2}\,\lambda^{\star s+1}}\; G^{p,1}_{1,p}\left(\frac{\lambda^{\star p}}{\beta^\star\,p^p}\Bigg|\begin{matrix} 1 \\ \frac{(s+1)}{p}, \frac{(s+2)}{p}, \ldots, \frac{(s+p)}{p} \end{matrix}\right).
$$

Equations (4.3), (4.4), (4.5), (4.6) and (4.8) are the main results of this section.

4.2. Incomplete Moments. For lifetime models, it is useful to obtain the sth incomplete moment of X given by $T_s(y) = \int_0^y x^s\, f(x) dx$. We define $J(s,a) = J(s,a;\beta,\gamma) = \int_0^a x^s\, \mathrm{e}^{-\beta \mathrm{x}^\gamma} \mathrm{dx}$. Moreover, it is simple to verify from (1.6) that $T_s(y)$ can be expressed as

$$
T_s(y) = \int_0^y x^s\,(\lambda^\star + \beta^\star\, k\, x^{k-1})\, \mathrm{e}^{-(\lambda^\star \mathrm{x} + \beta^\star \mathrm{x}^\mathrm{k})} \mathrm{dx}.
$$

By expanding the exponential in the last expression, we have

$$
T_s(y) = \sum_{j=0}^{\infty} \frac{(-1)^j \lambda^{\star j}}{j!}\left[\lambda^\star\, J(s+j, y) + \beta^\star k\, J(s+k-1, y)\right].
\tag{4.9}
$$

We now provide a formula for $T_s(y)$ in terms of the Meijer $G^{m,n}_{p,q}$ function (see Appendix B) which holds only when $k = p/q$, where $p \geq 1$ and $q \geq 1$ are natural co-prime numbers. By using (4.7), we can write

$$
J(s,a) = \int_0^a x^s\, G^{1,0}_{0,1}\left(\beta^\star x^{p/q}\,\Big|\begin{matrix} - \\ 0 \end{matrix}\right) \mathrm{d}x.
$$

By using equation (2.24.2.2) in [26, p. 348], we can express $J(s,a)$ as

$$J(s,a) = \frac{q\, a^{p(s+1)}}{p\,(2\pi)^{(q-1)/2}}\, G^{q,p}_{p,p+q}\left(\frac{\beta^{\star q}\, a^{p}}{q^{q}} \middle| \begin{array}{l} \frac{-s}{p}, \frac{1-s}{p}, \ldots, \frac{p-s-1}{p}, - \\ 0, \frac{-s-1}{p}, \frac{s}{p}, \ldots, \frac{p-s-2}{p} \end{array} \right). \tag{4.10}$$

Combining equations (4.9) and (4.10), we obtain the incomplete moments of X.

4.3. Generating Function. For $t < \lambda^{\star}$, the mgf of X follows from (4.1) as

$$M(t) = \sum_{m=0}^{\infty} V_m\, I(s; \lambda^{\star} - t, \beta^{\star}, k).$$

Thus, we can use the results in Section 4.1 to obtain an explicit expression for $M(t)$

$$M(t) = \sum_{m=0}^{\infty} V_m \left[\frac{1}{k\beta^{\star (s+1)/k}} \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} \left(\frac{\lambda^{\star} - t}{\beta^{\star 1/k}} \right)^{j} \Gamma\left(\frac{s+1+j}{k} \right) \right].$$

5. Parameter Estimation

5.1. Maximum likelihood estimation. Several approaches for parameter estimation were proposed in the literature but the maximum likelihood method is the most commonly employed. The maximum likelihood estimates (MLEs) enjoy desirable properties and can be used when constructing confidence intervals and test statistics. Large sample theory for these estimates delivers simple approximations that work well in finite samples. However, we can approximate quantities such as the density of test statistics that depend on the sample size in order to obtain better approximation for the MLEs, which can be easily handled either analytically or numerically.

Let $\theta = (\lambda, \beta, k, \alpha, \gamma)$ be the parameter vector of the KwEW distribution. The log-likelihood for θ given the data set $x_1, \ldots, x_n$ obtained from (1.6) is given by

$$\begin{aligned} \ell(\theta) = {} & n\,[\log(\alpha) + \log(\gamma)] + \sum_{i=1}^{n} \log\left(\mathrm{e}^{-\,\mathrm{x_i^k}\beta - \mathrm{x_i}\lambda} \right) \\ & - (1-\alpha) \sum_{i=1}^{n} \log\left(1 - \mathrm{e}^{-\mathrm{x_i^k}\beta - \mathrm{x_i}\lambda} \right) + \sum_{i=1}^{n} \log\left(k x_i^{k-1} \beta + \lambda \right) \\ & - (1-\gamma) \sum_{i=1}^{n} \log\left\{ 1 - \left(1 - \mathrm{e}^{-\mathrm{x_i^k}\beta - \mathrm{x_i}\lambda} \right)^{\alpha} \right\}. \end{aligned} \tag{5.1}$$

The associated nonlinear log-likelihood equations $\frac{\partial \ell(\theta)}{\partial \theta} = 0$ are given by

$$\begin{aligned}
\frac{\partial \ell(\theta)}{\partial \lambda} &= \sum_{i=1}^{n} -x_i + (\alpha-1)\sum_{i=1}^{n} \frac{\mathrm{e}^{-\lambda \mathrm{x_i} - \beta \mathrm{x_i^k}}\mathrm{x_i}}{1-\mathrm{e}^{-\lambda \mathrm{x_i} - \beta \mathrm{x_i^k}}} + \sum_{i=1}^{n} \frac{1}{\lambda + k\beta x_i^{-1+k}} \\
&\quad - (\gamma-1)\sum_{i=1}^{n} \frac{\mathrm{e}^{-\lambda \mathrm{x_i} - \beta \mathrm{x_i^k}}\left(1-\mathrm{e}^{-\lambda \mathrm{x_i} - \beta \mathrm{x_i^k}}\right)^{-1+\alpha}\alpha \mathrm{x_i}}{1-\left(1-\mathrm{e}^{-\lambda \mathrm{x_i} - \beta \mathrm{x_i^k}}\right)^{\alpha}} = 0, \\
\frac{\partial \ell(\theta)}{\partial \beta} &= \sum_{i=1}^{n} -x_i^k + (\alpha-1)\sum_{i=1}^{n} \frac{\mathrm{e}^{-\lambda \mathrm{x_i} - \beta \mathrm{x_i^k}}\mathrm{x_i^k}}{1-\mathrm{e}^{-\lambda \mathrm{x_i} - \beta \mathrm{x_i^k}}} + \sum_{i=1}^{n} \frac{k x_i^{-1+k}}{\lambda + k\beta x_i^{-1+k}} \\
&\quad - (\gamma-1)\sum_{i=1}^{n} \frac{\mathrm{e}^{-\lambda \mathrm{x_i} - \beta \mathrm{x_i^k}}\left(1-\mathrm{e}^{-\lambda \mathrm{x_i} - \beta \mathrm{x_i^k}}\right)^{-1+\alpha}\alpha \mathrm{x_i^k}}{1-\left(1-\mathrm{e}^{-\lambda \mathrm{x_i} - \beta \mathrm{x_i^k}}\right)^{\alpha}} = 0, \\
\frac{\partial \ell(\theta)}{\partial k} &= \sum_{i=1}^{n} -\beta \log(x_i)\, x_i^k + (\alpha-1)\sum_{i=1}^{n} \frac{\mathrm{e}^{-\lambda \mathrm{x_i} - \beta \mathrm{x_i^k}}\beta \log(\mathrm{x_i})\,\mathrm{x_i^k}}{1-\mathrm{e}^{-\lambda \mathrm{x_i} - \beta \mathrm{x_i^k}}} \\
&\quad - (\gamma-1)\sum_{i=1}^{n} \frac{\mathrm{e}^{-\lambda \mathrm{x_i} - \beta \mathrm{x_i^k}}\left(1-\mathrm{e}^{-\lambda \mathrm{x_i} - \beta \mathrm{x_i^k}}\right)^{-1+\alpha}\alpha\beta \log(\mathrm{x_i})\,\mathrm{x_i^k}}{1-\left(1-\mathrm{e}^{-\lambda \mathrm{x_i} - \beta \mathrm{x_i^k}}\right)^{\alpha}} \\
&\quad + \sum_{i=1}^{n} \frac{\beta\, x_i^{-1+k} + k\,\beta\,\log(x_i)\; x_i^{-1+k}}{\lambda + k\,\beta\, x_i^{-1+k}} = 0, \\
\frac{\partial \ell(\theta)}{\partial \alpha} &= \frac{n}{\alpha} + \sum_{i=1}^{n} \log\left(1-\mathrm{e}^{-\lambda \mathrm{x_i} - \beta \mathrm{x_i^k}}\right) - (\gamma-1) \\
&\quad \times \sum_{i=1}^{n} \frac{\left(1-\mathrm{e}^{-\lambda \mathrm{x_i} - \beta \mathrm{x_i^k}}\right)^{\alpha}\log\left(1-\mathrm{e}^{-\lambda \mathrm{x_i} - \beta \mathrm{x_i^k}}\right)}{1-\left(1-\mathrm{e}^{-\lambda \mathrm{x_i} - \beta \mathrm{x_i^k}}\right)^{\alpha}} = 0\,,
\end{aligned}$$

$$\frac{\partial \ell(\theta)}{\partial \gamma} = \frac{n}{\gamma} + \sum_{i=1}^{n} \mathrm{Log}\left\{1-\left(1-\mathrm{e}^{-\lambda \mathrm{x_i} - \beta \mathrm{x_i^k}}\right)^{\alpha}\right\} = 0. \tag{5.2}$$

For estimating the model parameters, numerical iterative techniques should be employed to solve these equations. We can investigate the global maximum of the log-likelihood by setting different starting values for the parameters. The information matrix will be required for interval estimation. The elements of the 5×5 total observed information matrix $J(\theta) = \{J_{rs}(\theta)\}$ (for $r, s = \lambda, \beta, k, \alpha, \gamma$) can be obtained from the authors upon request. The asymptotic distribution of $(\hat{\theta} - \theta)$ is $N_5(O, K(\theta)^{-1})$, under standard regularity conditions, where $K(\theta) = E\{J(\theta)\}$ is the expected information matrix and $J(\hat{\theta})$ is the observed information matrix evaluated at $\hat{\theta}$. The multivariate normal $N_5(O, J(\hat{\theta})^{-1})$ distribution can be used to construct approximate confidence intervals for the individual parameters.

5.2. Bayesian analysis. In the Bayesian approach, the information referring to the model parameters is obtained through a posterior marginal distribution. Here, we use the simulation method of Markov Chain Monte Carlo (MCMC) by the

Metropolis-Hastings algorithm. Since we have no prior information from historical data or from previous experiment, we assign conjugate but weakly informative prior distributions to the parameters. We assume informative (but weakly) prior distribution and then the posterior distribution is a well-defined proper distribution. We also assume that the elements of the parameter vector are independent and that the joint prior distribution for all unknown parameters has a pdf given by

$$\pi(\lambda, \beta, k, \alpha, \gamma) \propto \pi(\lambda) \times \pi(\beta) \times \pi(k) \times \pi(\alpha) \times \pi(\gamma). \tag{5.3}$$

Here, $\lambda \sim \Gamma(a_1, b_1)$, $\beta \sim \Gamma(a_2, b_2)$, $k \sim \Gamma(a_3, b_3)$, $\alpha \sim \Gamma(a_4, b_4)$ and $\gamma \sim \Gamma(a_5, b_5)$, where $\Gamma(a_i, b_i)$ denotes a gamma distribution with mean a_i/b_i, variance a_i/b_i^2 and density function given by

$$f(\upsilon; a_i, b_i) = \frac{b_i^{a_i} \upsilon^{a_i - 1} \exp(-\upsilon b_i)}{\Gamma(a_i)},$$

where $\upsilon > 0$, $a_i > 0$ and $b_i > 0$. All hyper-parameters are specified. Combining the likelihood function (5.1) and the prior distribution (5.3), the joint posterior distribution for λ, β, k, α and γ reduces to

$$\begin{aligned}\pi(\lambda, \beta, k, \alpha, \gamma | x) \propto (\alpha\gamma)^n \, \mathrm{e}^{-\lambda \sum_{\mathrm{i}=1}^{\mathrm{n}} \mathrm{x_i} - \beta \sum_{\mathrm{i}=1}^{\mathrm{n}} \mathrm{x_i^k}} \prod_{\mathrm{i}=1}^{\mathrm{n}} \Bigg\{ & \left(\lambda + \mathrm{k}\beta \mathrm{x_i^{k-1}}\right) \\ \times \left(1 - \mathrm{e}^{-\lambda \mathrm{x_i} - \beta \mathrm{x_i^k}}\right)^{-1+\alpha} & \left\{1 - \left(1 - \mathrm{e}^{-\lambda \mathrm{x_i} - \beta \mathrm{x_i^k}}\right)^{\alpha}\right\}^{-1+\gamma} \Bigg\} \\ \times \pi(\lambda, \beta, k, \alpha, \gamma). &\end{aligned} \tag{5.4}$$

The joint posterior density above is analytically intractable because the integration of the joint posterior density is not easy to perform. In this direction, we first obtain the full conditional distributions of the unknown parameters given by

$$\begin{aligned}\pi(\lambda | x, \beta, k, \alpha, \gamma) \propto \mathrm{e}^{-\lambda \sum_{\mathrm{i}=1}^{\mathrm{n}} \mathrm{x_i}} \prod_{\mathrm{i}=1}^{\mathrm{n}} \Bigg\{ & \left(\lambda + \mathrm{k}\beta \mathrm{x^{k-1}}\right) \left(1 - \mathrm{e}^{-\lambda \mathrm{x_i} - \beta \mathrm{x_i^k}}\right)^{-1+\alpha} \\ \times & \left\{1 - \left(1 - \mathrm{e}^{-\lambda \mathrm{x_i} - \beta \mathrm{x_i^k}}\right)^{\alpha}\right\}^{-1+\gamma} \Bigg\} \times \pi(\lambda),\end{aligned}$$

$$\pi(\beta|x,\lambda,k,\alpha,\gamma) \propto \mathrm{e}^{-\beta\sum_{\mathrm{i}=1}^{\mathrm{n}}\mathrm{x}_{\mathrm{i}}^{\mathrm{k}}} \prod_{\mathrm{i}=1}^{\mathrm{n}} \Bigg\{ \left(\lambda + \mathrm{k}\beta\mathrm{x}_{\mathrm{i}}^{\mathrm{k}-1}\right)\left(1-\mathrm{e}^{-\lambda\mathrm{x}_{\mathrm{i}}-\beta\mathrm{x}_{\mathrm{i}}^{\mathrm{k}}}\right)^{-1+\alpha}$$
$$\times\left\{1-\left(1-\mathrm{e}^{-\lambda\mathrm{x}_{\mathrm{i}}-\beta\mathrm{x}_{\mathrm{i}}^{\mathrm{k}}}\right)^{\alpha}\right\}^{-1+\gamma}\Bigg\} \times \pi(\beta),$$
$$\pi(k|x,\lambda,\beta,\alpha,\gamma) \propto \mathrm{e}^{-\beta\sum_{\mathrm{i}=1}^{\mathrm{n}}\mathrm{x}_{\mathrm{i}}^{\mathrm{k}}} \prod_{\mathrm{i}=1}^{\mathrm{n}} \Bigg\{ \left(\lambda + \mathrm{k}\beta\mathrm{x}_{\mathrm{i}}^{\mathrm{k}-1}\right)\left(1-\mathrm{e}^{-\lambda\mathrm{x}_{\mathrm{i}}-\beta\mathrm{x}_{\mathrm{i}}^{\mathrm{k}}}\right)^{-1+\alpha}$$
$$\times\left\{1-\left(1-\mathrm{e}^{-\lambda\mathrm{x}_{\mathrm{i}}-\beta\mathrm{x}_{\mathrm{i}}^{\mathrm{k}}}\right)^{\alpha}\right\}^{-1+\gamma}\Bigg\} \times \pi(k),$$
$$\pi(\alpha|x,\lambda,\beta,k,\gamma) \propto \alpha^{n} \prod_{i=1}^{n} \Bigg\{ \left(1-\mathrm{e}^{-\lambda\mathrm{x}_{\mathrm{i}}-\beta\mathrm{x}_{\mathrm{i}}^{\mathrm{k}}}\right)^{\alpha}$$
$$\times\left\{1-\left(1-\mathrm{e}^{-\lambda\mathrm{x}_{\mathrm{i}}-\beta\mathrm{x}_{\mathrm{i}}^{\mathrm{k}}}\right)^{\alpha}\right\}^{-1+\gamma}\Bigg\} \times \pi(\alpha)$$

and

$$\pi(\gamma|x,\lambda,\beta,k,\alpha) \propto \gamma^{n} \prod_{i=1}^{n} \left\{1-\left(1-\mathrm{e}^{-\lambda\mathrm{x}_{\mathrm{i}}-\beta\mathrm{x}_{\mathrm{i}}^{\mathrm{k}}}\right)^{\alpha}\right\}^{\gamma} \times \pi(\gamma).$$

Since the full conditional distributions for λ, β, k, α and γ do not have explicit expressions, we require the use of the Metropolis-Hastings algorithm.

5.3. Simulation study. We also assess the performance of the MLEs in terms of the sample size n. The simulation is performed using the Ox matrix programming language. The number of Monte Carlo replications is $10,000$. For maximizing the log-likelihood function, we use the MaxBFGS subroutine with analytical derivatives. The evaluation of the estimates is performed based on the following quantities for each sample size: the empirical mean squared errors (MSEs) and the root MSEs (RMSEs) using the R package from the Monte Carlo replications. The inversion method is used to generate samples, i.e., the variates having the KwEW distribution are generated using (3.10). The MLEs are evaluated for each simulated data, say $(\hat{\lambda}_i, \hat{\beta}_i, \hat{k}_i, \hat{\alpha}_i, \hat{\gamma}_i)$ (for $i = 1, \ldots, 10,000$) and the biases and MSEs are computed by

$$bias_h(n) = \frac{1}{10000}\sum_{i=1}^{10000}(\hat{h_i} - h) \quad \text{and} \quad MSE_h(n) = \frac{1}{10000}\sum_{i=1}^{10000}(\hat{h_i} - h)^2$$

for $h = \lambda,\ \beta,\ k,\ \alpha,\ \gamma$.
Let the sample size be $n = 250$, 350 and 450 and consider different values for the shape parameters λ, k, α and γ, whereas the scale parameter β is fixed at one. The empirical results are given in Table 3.

The figures in this table indicate that the estimates are quite stable and, more importantly, are close to the true values for these sample sizes. Additionally, as the sample size increases, the RMSEs decrease as expected. We can conclude that the MLEs are robust.

Table 3. Empirical means and the RMSEs in parentheses for $\beta = 1$

n	$\hat{\lambda}$	$\hat{\beta}$	$\hat{k}$	$\hat{\alpha}$	$\hat{\gamma}$
		$\lambda = 2.3$	$k = 1.6$	$\alpha = 1.5$	$\gamma = 1$
250	1.218	1.689	1.467	1.702	1.346
	(0.954)	(0.517)	(0.978)	(1.597)	(1.510)
350	1.214	1.574	1.428	1.503	1.335
	(0.897)	(0.501)	(0.834)	(1.453)	(1.478)
450	1.213	1.572	1.346	1.548	1.217
	(0.895)	(0.498)	(0.740)	(1.404)	(1.156)
		$\lambda = 3.4$	$k = 1.8$	$\alpha = 2$	$\gamma = 2.3$
250	1.414	1.023	1.101	2.471	2.601
	(1.221)	(0.742)	(0.456)	(2.102)	(2.102)
350	1.367	1.367	1.084	2.495	2.495
	(0.918)	(0.904)	(0.285)	(2.104)	(2.104)
450	1.278	1.278	1.053	2.348	2.348
	(1.012)	(0.843)	(0.324)	(1.945)	(1.945)
		$\lambda = 0.4$	$k = 2$	$\alpha = 2.5$	$\gamma = 1.4$
250	2.203	1.146	2.142	2.104	1.925
	(0.962)	(0.765)	(0.978)	(1.231)	(1.024)
350	2.458	1.107	2.154	2.116	1.823
	(0.784)	(0.452)	(0.450)	(1.114)	(0.978)
450	1.067	1.047	2.045	2.123	1.450
	(0.452)	(0.596)	(0.258)	(1.080)	(0.856)
		$\lambda = 3.2$	$k = 2.5$	$\alpha = 1.5$	$\gamma = 3$
250	1.854	1.256	1.478	1.149	1.853
	(0.927)	(0.451)	(0.301)	(0.856)	(1.420)
350	1.745	1.024	1.201	1.131	1.741
	(0.847)	(0.237)	(0.214)	(0.723)	(1.204)
450	1.680	1.345	1.635	1.085	1.658
	(0.784)	(0.478)	(0.481)	(0.456)	(1.004)

6. Application

Here, we prove the potentiality of the KwEW distribution by means of a real data set using both MLEs and Bayesian approaches.

6.1. The MLEs approach. By using MLEs method, we fit the two-parameter Weibull (Weibull), exponential-Weibull (EW) [5], extended Weibull (ExtW) [20], Marshall-Olkin exponential-Weibull (MO-EW) [22], Kumaraswamy Weibull (Kw-W) [4] and KwEW distributions to the Aarset data [1] on lifetimes of 50 components, which possess a bathtub-shaped failure rate property. The density functions of these models are given below (for $x > 0$):

- The Weibull density function

$$f(x) = \frac{k}{\lambda}\left(\frac{x}{\lambda}\right)^{k-1} \mathrm{e}^{-(\mathrm{x}/\lambda)^{\mathrm{k}}},\ \mathrm{k} > 0,\ \lambda > 0;$$

Table 4. MLEs of the parameters (standard errors in parentheses) for the Aarset data

Distributions	Estimates				
Weibull$(k\,,\,\lambda)$	3.441197	47.05054			
	(0.000248)	(0.036047)			
E-W$(\lambda,\,\beta,\,k)$	0.018620	0.040483	0.373635		
	(0.003771)	(0.031143)	(0.188693)		
ExtW$(\,a\,,\,b\,,\,c)$	0.027836	0.942137	0.020278		
	(0.033196)	(0.285026)	(0.319463)		
Mo-EW$(a\,,b\,,c\,,\alpha)$	0.027083	0.161829	0.328829	3.599999	
	(0.006184)	(0.124196)	(0.143844)	(1.87102)	
Kw-W$(a\,,b\,,c\,,\lambda)$	0.340211	0.145696	1.209999	0.089756	
	(0.201699)	(0.106772)	(0.294355)	(0.079873)	
KwEW$(\lambda\,,\beta\,,k\,,\alpha\,,\gamma)$	0.004366	0.209999	0.116764	3.516432	18.99999
	(0.001879)	(0.175644)	(0.057365)	(1.61287)	(15.3596)

Table 5. Goodness-of-fit statistics for the Aarset data

Distributions	$-\hat{\ell}$	AIC	BIC	A^*	W^*
Weibull$(k,\,\lambda)$	240.98	485.959	489.783	3.53566	0.532984
E-W$(\lambda,\,\beta,\,k)$	239.463	484.927	490.663	2.92873	0.513036
ExtW$(a,\,b,\,c)$	240.957	487.914	493.65	3.5425	0.53549
Mo-EW$(a,\,b,\,c,\,\alpha)$	235.515	479.03	486.678	2.21706	0.34524
Kw-W$(a,\,b,\,c,\,\lambda)$	235.925	479.851	487.499	2.48043	0.424629
KwEW$(\lambda,\,\beta,\,k,\,\alpha,\,\gamma)$	**233.087**	**476.175**	**485.735**	**2.11894**	**0.32768**

- The EW density function

$$f(x) = \left(\lambda + \beta\,k\,x^{k-1}\right)\,\mathrm{e}^{-\lambda\,\mathrm{x}-\beta\,\mathrm{x}^{\mathrm{k}}},\ \lambda, \beta, \mathrm{k} > 0;$$

- The ExtW density function

$$f(x) = a\,(c + b\,x)\,x^{-2+b}\,\mathrm{e}^{-\mathrm{c/x}-\mathrm{ax}^{\mathrm{b}}\mathrm{e}^{-\mathrm{c/x}}},\ \mathrm{a}, \mathrm{b} > 0,\ \mathrm{c} \geq 0;$$

- The MO-EW density function

$$f(x) = \frac{\alpha\left(a + b\,c\,x^{-1+c}\right)\,\mathrm{e}^{-(\mathrm{ax}+\mathrm{bx}^{\mathrm{c}})}}{\left[1 - (1-\alpha)\,\mathrm{e}^{-(\mathrm{ax}+\mathrm{bx}^{\mathrm{c}})}\right]^2},\ \lambda, \beta, k, \alpha > 0;$$

- The Kw-W density function

$$f(x) = a\,b\,c\,\lambda^{\mathrm{c}}\,x^{c-1}\,\mathrm{e}^{-(\mathrm{x}\lambda)^{\mathrm{c}}}\,\left\{1 - \mathrm{e}^{-(\mathrm{x}\lambda)^{\mathrm{c}}}\right\}^{\mathrm{a}-1}\left[1 - \left\{1 - \mathrm{e}^{-(\mathrm{x}\lambda)^{\mathrm{c}}}\right\}^{\mathrm{a}}\right]^{\mathrm{b}-1},$$
$$a, b, c, \lambda > 0.$$

The parameters of the above distributions are estimated by maximizing the log-likelihoods using the *NMaximize* command in the symbolic computational package *Mathematica*. Table 4 lists the MLEs (and the corresponding standard errors in parentheses) of the parameters. Table 5 gives the values of minus the maximized log-likelihood $(-\hat{\ell})$, Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC), Anderson-Darling (A^*) and Cramér-von Mises (W^*) goodness-of-fit statistics for some fitted models. Since the values of these statistics are smaller for the KwEW distribution compared to those values of the Weibull, EW, ExtW, MO-EW and Kw-W distributions, the proposed distribution is a very competitive model for lifetime data analysis. Plots of the fitted KwEW, Weibull, E-W, ExtW,

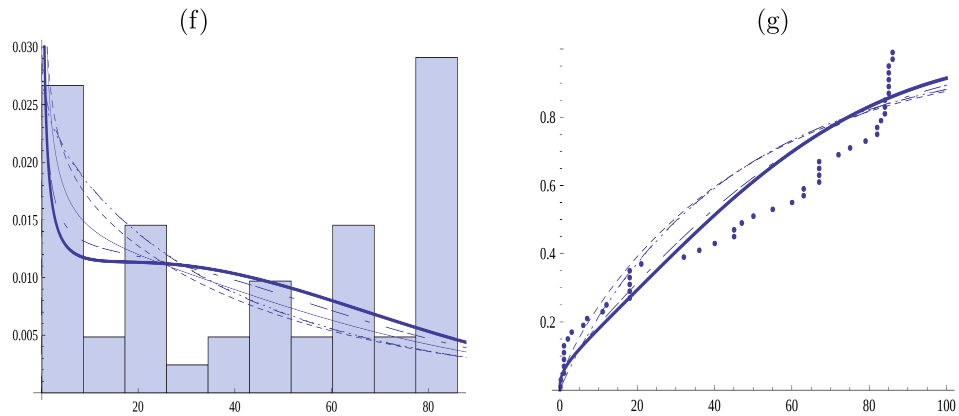

Figure 5. (f) The estimated KwEW density superimposed on the histogram for the Aarset data with other models. (g) The empirical cdf and the estimated cdf's of other models, where Kw-Ew is represented by (Thick line), Kw-W by (Thin line), MO-EW by (Long and short dashed line), ExtW by (Long dashed line), E-W by (dashed line) and Weibull by (Dotted line)

Mo-EW and Kw-W densities and the histogram of the data are displayed in Figure 5(f). In Figure 5(g), we plot the empirical cumulative function and the estimated cdf's for the KwEW and other distributions, which shows a satisfactory fit of the new model.

6.2. Bayesian approach. The following independent priors are considered to perform the Metropolis-Hastings algorithm: $\lambda \sim \Gamma(0.01, 0.01)$, $\beta \sim \Gamma(0.01, 0.01)$, $k \sim \Gamma(0.01, 0.01)$, $\alpha \sim \Gamma(0.01, 0.01)$ and $\gamma \sim \Gamma(0.01, 0.01)$, so that we have vague prior distributions. Considering these prior density functions, we generate two parallel independent runs of the Metropolis-Hastings with size 150,000 for each parameter, disregarding the first 15.000 iterations to eliminate the effect of the initial values and, to avoid correlation problems, we consider a spacing of size 10, obtaining a sample of size 13,500 from each chain. To monitor the convergence of the Metropolis-Hastings, we perform the methods suggested by Cowles and Carlin [7]. To monitor the convergence of the Metropolis-Hastings, we use the between and within sequence information, following the approach developed in Gelman and Rubin [9], to obtain the potential scale reduction, $\widehat{R}$. In all cases, these values were close to one, indicating the convergence of the chain. The approximate posterior marginal density functions for the parameters are presented in Figure 6. In Table 6, we report posterior summaries for the parameters of the new model. We note that the values for the means a posteriori (Table 6) are quite close (as expected) to the MLEs given in Table 5. Here, SD represents the standard deviation from the posterior distributions of the parameters and HPD represents the 95% highest posterior density (HPD) intervals.

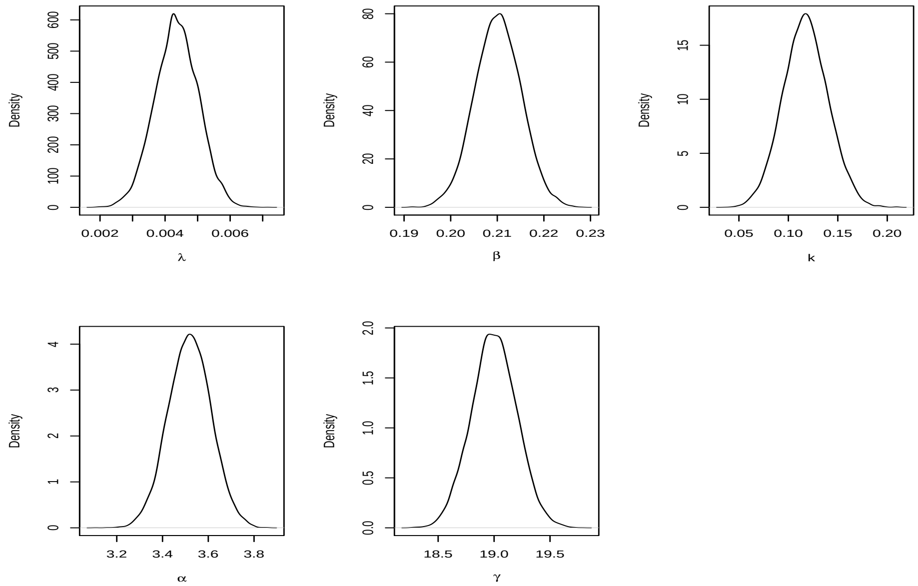

Figure 6. Approximate posterior marginal densities for the parameters from the KwEW model for the Aarset data.

Table 6. Posterior summaries for the parameters from the KwEW model for the Aarset data.

Parameter	Mean	SD	HPD (95%)	$\hat{R}$
λ	0.0044	0.0007	$(0.0031; 0.0057)$	1.0052
β	0.2102	0.0050	$(0.2005; 0.2200)$	1.0002
k	0.1175	0.0227	$(0.0740; 0.1630)$	1.0018
α	3.5188	0.0934	$(3.3338; 3.7012)$	0.9999
γ	19.0003	0.2027	$(18.6049; 19.3974)$	1.0008

7. Bivariate KwEW Distribution

Suppose $U_1 \sim \text{KwEW}(\gamma_1, \alpha, \lambda, \beta, k)$, $U_2 \sim \text{KwEW}(\gamma_2, \alpha, \lambda, \beta, k)$ and $U_3 \sim \text{KwEW}(\gamma_3, \alpha, \lambda, \beta, k)$ are independently distributed. Define $X_1 = max(U_1, U_3)$ and $X_2 = max(U_2, U_3)$. Then the bivariate vector $(X_1, X_2) \sim$ KwEW $(\gamma_1, \gamma_2, \gamma_3, \alpha, \lambda, \beta, k)$.

Now, we construct the joint CDF of X_1 and X_2. Since

$$F(x_1, x_2) = P(X_1 \leq x_1, X_2 \leq x_2)\,,$$

we have

$$\begin{aligned} F(x_1,x_2) &= P(max(U_1,U_3)\le x_1,(max(U_2,U_3)\le x_2) \\ &= P(U_1\le x_1,U_3\le x_1,U_2\le x_2,U_3\le x_2) \\ &= P(U_1\le x_1,U_2\le x_2,U_3\le min(x_1,x_2)\,. \end{aligned}$$

Since $U_i, i=1,2,3$ are independent, one gets

$$\begin{aligned} F(x_1,x_2) &= P(U_1\le x_1,U_2\le x_2,U_3\le min(x_1,x_2) \\ &= F(x_1,\gamma_1,\alpha,\lambda,\beta,k)\,F(x_2,\gamma_2,\alpha,\lambda,\beta,k)\,F(z,\gamma_3,\alpha,\lambda,\beta,k) \\ &= \left[1-\left\{1-\left(1-\mathrm{e}^{-\lambda\,\mathrm{x}_1-\beta\,\mathrm{x}_1^{\mathrm{k}}}\right)^{\alpha}\right\}^{\gamma_1}\right] \\ &\quad \left[1-\left\{1-\left(1-\mathrm{e}^{-\lambda\,\mathrm{x}_2-\beta\,\mathrm{x}_2^{\mathrm{k}}}\right)^{\alpha}\right\}^{\gamma_2}\right] \\ &\quad \times 1-\left\{1-\left(1-\mathrm{e}^{-\lambda\,\mathrm{z}-\beta\,\mathrm{z}^{\mathrm{k}}}\right)^{\alpha}\right\}^{\gamma_3}\,, \end{aligned} \tag{7.1}$$

where $z=min(x_1,x_2)\,.$

Combining (1.5) and (7.1), we obtain the joint cdf of the bivariate KwEW distribution as:

$$F(x_1,x_2)=\begin{cases} \left[1-\left\{1-\left(1-\mathrm{e}^{-\lambda\,\mathrm{x}_1-\beta\,\mathrm{x}_1^{\mathrm{k}}}\right)^{\alpha}\right\}^{\gamma_1+\gamma_3}\right] & \\ \times\left[1-\left\{1-\left(1-\mathrm{e}^{-\lambda\,\mathrm{x}_2-\beta\,\mathrm{x}_2^{\mathrm{k}}}\right)^{\alpha}\right\}^{\gamma_2}\right]\,, & x_1\le x_2 \\ \left[1-\left\{1-\left(1-\mathrm{e}^{-\lambda\,\mathrm{x}_1-\beta\,\mathrm{x}_1^{\mathrm{k}}}\right)^{\alpha}\right\}^{\gamma_1}\right] & \\ \times\left[1-\left\{1-\left(1-\mathrm{e}^{-\lambda\,\mathrm{x}_2-\beta\,\mathrm{x}_2^{\mathrm{k}}}\right)^{\alpha}\right\}^{\gamma_2+\gamma_3}\right]\,, & x_2\le x_1 \\ 1-\left\{1-\left(1-\mathrm{e}^{-\lambda\,\mathrm{x}_1-\beta\,\mathrm{x}_1^{\mathrm{k}}}\right)^{\alpha}\right\}^{\gamma_1+\gamma_2+\gamma_3}\,, & x_1=x_2=x \end{cases}\,. \tag{7.2}$$

The joint pdf of (X_1,X_2) is given by

$$f(x_1,x_2)=\begin{cases} f_1(x_1,x_2)\,, & x_1\le x_2 \\ f_2(x_1,x_2)\,, & x_2\le x_1 \\ f_3(x_1,x_2)\,, & x_1=x_2=x \end{cases}\,.$$

Now, $f_1(x_1,x_2)$ and $f_2(x_1,x_2)$ can easily be obtained by taking second order partial differentiation (i.e $f(x_1,x_2)=\frac{\partial^2 F(x_1,x_2)}{\partial x_1\partial x_2}$) of the bivariate KwEW cdf given in (7.2) and obtain the following forms:

$$
\begin{aligned}
f_1(x_1,x_2) &= \alpha^2\gamma_2(\gamma_1+\gamma_3)\left(\beta(-k)\,x_1^{k-1}-\lambda\right)\left(\beta(-k)\,x_2^{k-1}-\lambda\right)\\
&\quad\times\left(1-\mathrm{e}^{-\lambda \mathrm{x}_1-\beta \mathrm{x}_1^{\mathrm{k}}}\right)^{\alpha-1}\mathrm{e}^{-\lambda(\mathrm{x}_1+\mathrm{x}_2)-\beta(\mathrm{x}_1^{\mathrm{k}}+\mathrm{x}_2^{\mathrm{k}})}\left(1-\mathrm{e}^{-\lambda \mathrm{x}_2-\beta \mathrm{x}_2^{\mathrm{k}}}\right)^{\alpha-1}\\
&\quad\times\left(1-\left(1-\mathrm{e}^{-\lambda \mathrm{x}_2-\beta \mathrm{x}_2^{\mathrm{k}}}\right)^{\alpha}\right)^{\gamma_2-1}\left(1-\left(1-\mathrm{e}^{-\lambda \mathrm{x}_1-\beta \mathrm{x}_1^{\mathrm{k}}}\right)^{\alpha}\right)^{\gamma_1+\gamma_3-1} \qquad (7.3)
\end{aligned}
$$

and

$$
\begin{aligned}
f_2(x_1,x_2) &= \alpha^2\gamma_1(\gamma_2+\gamma_3)\left(\beta(-k)\,x_1^{k-1}-\lambda\right)\left(\beta(-k)\,x_2^{k-1}-\lambda\right)\\
&\quad\times\left(1-\mathrm{e}^{-\lambda \mathrm{x}_1-\beta \mathrm{x}_1^{\mathrm{k}}}\right)^{\alpha-1}\mathrm{e}^{-\lambda(\mathrm{x}_1+\mathrm{x}_2)-\beta(\mathrm{x}_1^{\mathrm{k}}+\mathrm{x}_2^{\mathrm{k}})}\left(1-\mathrm{e}^{-\lambda \mathrm{x}_2-\beta \mathrm{x}_2^{\mathrm{k}}}\right)^{\alpha-1}\\
&\quad\times\left(1-\left(1-\mathrm{e}^{-\lambda \mathrm{x}_2-\beta \mathrm{x}_2^{\mathrm{k}}}\right)^{\alpha}\right)^{\gamma_1-1}\left(1-\left(1-\mathrm{e}^{-\lambda \mathrm{x}_1-\beta \mathrm{x}_1^{\mathrm{k}}}\right)^{\alpha}\right)^{\gamma_2+\gamma_3-1}. \qquad (7.4)
\end{aligned}
$$

But $f_3(x_1,x_2)$ can not be derived in the similar way. For this, we use the following identity

$$
\begin{aligned}
&\int_0^\infty\int_0^{x_2} f_1(x_1,x_2)\,\mathrm{dx}_1\mathrm{dx}_2+\int_0^\infty\int_0^{\mathrm{x}_1}\mathrm{f}_2(\mathrm{x}_1,\mathrm{x}_2)\,\mathrm{dx}_1\mathrm{dx}_2\\
&\quad+\int_0^\infty f_3(x,x)\,\mathrm{dx}=1\\
&\quad= I_1+I_2+\int_0^\infty f_3(x,x)\,\mathrm{dx}=1\,.
\end{aligned}
$$

Let

$$
\begin{aligned}
I_1 &= \alpha\gamma_2\int_0^\infty\left(\beta(-k)\,x_2^{k-1}-\lambda\right)\,\mathrm{e}^{-\lambda\,\mathrm{x}_2-\beta\,\mathrm{x}_2^{\mathrm{k}}}\left(1-\mathrm{e}^{-\lambda\,\mathrm{x}_2-\beta\,\mathrm{x}_2^{\mathrm{k}}}\right)^{\alpha-1}\\
&\quad\times\left(1-\left(1-\mathrm{e}^{-\lambda\,\mathrm{x}_2-\beta\,\mathrm{x}_2^{\mathrm{k}}}\right)^{\alpha}\right)^{\gamma_2-1}\,\mathrm{dx}_2\\
&\quad\times\alpha\,(\gamma_1+\gamma_3)\int_0^{x_2}\left(\beta(-k)\,x_1^{k-1}-\lambda\right)\mathrm{e}^{-\lambda\,\mathrm{x}_1-\beta\,\mathrm{x}_1^{\mathrm{k}}}\left(1-\mathrm{e}^{-\lambda\,\mathrm{x}_1-\beta\,\mathrm{x}_1^{\mathrm{k}}}\right)^{\alpha-1}\\
&\quad\times\left(1-\left(1-\mathrm{e}^{-\lambda\,\mathrm{x}_1-\beta\,\mathrm{x}_1^{\mathrm{k}}}\right)^{\alpha}\right)^{\gamma_1+\gamma_3-1}\,\mathrm{dx}_1\,,
\end{aligned}
$$

then

$$
\begin{aligned}
I_1 &= \alpha\gamma_2\int_0^\infty\left(\beta(-k)\,x_2^{k-1}-\lambda\right)\,\mathrm{e}^{-\lambda\,\mathrm{x}_2-\beta\,\mathrm{x}_2^{\mathrm{k}}}\left(1-\mathrm{e}^{-\lambda\,\mathrm{x}_2-\beta\,\mathrm{x}_2^{\mathrm{k}}}\right)^{\alpha-1}\\
&\quad\times\left(1-\left(1-\mathrm{e}^{-\lambda\,\mathrm{x}_2-\beta\,\mathrm{x}_2^{\mathrm{k}}}\right)^{\alpha}\right)^{\gamma_1+\gamma_2+\gamma_3-1}\,\mathrm{dx}_2\,. \qquad (7.5)
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
I_2 = \alpha\gamma_1 \int_0^\infty & \left(\beta(-k)\, x_1^{k-1} - \lambda\right)\, \mathrm{e}^{-\lambda\, \mathrm{x}_1 - \beta\, \mathrm{x}_1^{\mathrm{k}}} \left(1 - \mathrm{e}^{-\lambda\, \mathrm{x}_1 - \beta\, \mathrm{x}_1^{\mathrm{k}}}\right)^{\alpha-1} \\
& \times \left(1 - \left(1 - \mathrm{e}^{-\lambda\, \mathrm{x}_1 - \beta\, \mathrm{x}_1^{\mathrm{k}}}\right)^{\alpha}\right)^{\gamma_1 - 1} \mathrm{dx}_2 \\
& \times \alpha\,(\gamma_2 + \gamma_3) \int_0^{x_1} \left(\beta(-k)\, x_2^{k-1} - \lambda\right) \mathrm{e}^{-\lambda\, \mathrm{x}_2 - \beta\, \mathrm{x}_2^{\mathrm{k}}} \left(1 - \mathrm{e}^{-\lambda\, \mathrm{x}_2 - \beta\, \mathrm{x}_2^{\mathrm{k}}}\right)^{\alpha-1} \\
& \times \left(1 - \left(1 - \mathrm{e}^{-\lambda\, \mathrm{x}_2 - \beta\, \mathrm{x}_2^{\mathrm{k}}}\right)^{\alpha}\right)^{\gamma_2 + \gamma_3 - 1} \mathrm{dx}_2\,,
\end{aligned}
$$

then

$$
\begin{aligned}
I_2 = \alpha\gamma_1 \int_0^\infty & \left(\beta(-k)\, x^{k-1} - \lambda\right)\, \mathrm{e}^{-\lambda\, \mathrm{x}_1 - \beta\, \mathrm{x}_1^{\mathrm{k}}} \left(1 - \mathrm{e}^{-\lambda\, \mathrm{x}_1 - \beta\, \mathrm{x}_1^{\mathrm{k}}}\right)^{\alpha-1} \\
& \times \left(1 - \left(1 - \mathrm{e}^{-\lambda\, \mathrm{x}_1 - \beta\, \mathrm{x}_1^{\mathrm{k}}}\right)^{\alpha}\right)^{\gamma_1 + \gamma_2 + \gamma_3 - 1} \mathrm{dx}_2\,. \qquad (7.6)
\end{aligned}
$$

From (7.5) and (7.6), one obtains

$$
\begin{aligned}
\int_0^\infty f_3(x,x)\, \mathrm{dx} = \alpha\gamma_3 & \int_0^\infty \left(\beta(-\mathrm{k})\, \mathrm{x}^{\mathrm{k}-1} - \lambda\right)\, \mathrm{e}^{-\lambda\, \mathrm{x} - \beta\, \mathrm{x}^{\mathrm{k}}} \\
& \times \left(1 - \mathrm{e}^{-\lambda\, \mathrm{x} - \beta\, \mathrm{x}^{\mathrm{k}}}\right)^{\alpha-1} \left(1 - \left(1 - \mathrm{e}^{-\lambda\, \mathrm{x} - \beta\, \mathrm{x}^{\mathrm{k}}}\right)^{\alpha}\right)^{\gamma_1 + \gamma_2 + \gamma_3 - 1} \mathrm{dx}\,.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
f_3(x,x) = \alpha\gamma_3 \int_0^\infty & \left(\beta(-k)\, x^{k-1} - \lambda\right)\, \mathrm{e}^{-\lambda\, \mathrm{x} - \beta\, \mathrm{x}^{\mathrm{k}}} \left(1 - \mathrm{e}^{-\lambda\, \mathrm{x} - \beta\, \mathrm{x}^{\mathrm{k}}}\right)^{\alpha-1} \\
& \times \left(1 - \left(1 - \mathrm{e}^{-\lambda\, \mathrm{x} - \beta\, \mathrm{x}^{\mathrm{k}}}\right)^{\alpha}\right)^{\gamma_1 + \gamma_2 + \gamma_3 - 1}\,. \qquad (7.7)
\end{aligned}
$$

8. Conclusions

In the last two decades, several authors have been interested in developing methods for generating distributions with more flexibility in applications and data modeling. There has been a growing interest among statisticians and applied researchers in constructing flexible lifetime models in order to improve the modeling of survival data. In particular, some authors proposed new extensions of the classical Weibull model. In this paper, we introduce a five–parameter distribution obtained by applying the Kumaraswamy generator defined by Cordeiro *et al.* [6] to the exponential-Weibull model given by Cordeiro *et al.* [5]. Interestingly, the proposed model has increasing, upside-down bathtub and bathtub shaped hazard rate functions. We study some of its mathematical properties. We discuss the maximum likelihood method and a Bayesian approach to make inference on the model parameters. In the Bayesian approach, the selection of proper priors is difficult to examine and it is left to the interested readers for further study. Also, the monitoring the rate of convergence of the associated MCMC method will be an important issue to look after. An application proves its flexibility to analysis of real data. We also discuss a bivariate extension of the KwEW distribution. The distributional results developed in this paper can have numerous applications in the physical and biological sciences, reliability theory, hydrology, medicine, meteorology, engineering and survival analysis.

Appendix A. The unified Fox–Wright generalized hypergeometric function

Here,

$$ {}_p\Psi_q^*\Big[\begin{array}{c}(a,A)_p\\(b,B)_q\end{array}\Big|\,z\Big] = \sum_{n=0}^\infty \frac{\prod_{j=1}^p (a_j)_{A_j n}}{\prod_{j=1}^q (b_j)_{B_j n}}\,\frac{z^n}{n!} \tag{8.1}$$

stands for the *unified variant of the Fox–Wright generalized hypergeometric function* with p upper and q lower parameters; $(a,A)_p$ denotes the parameter p–tuple $(a_1,A_1),\cdots,(a_p,A_p)$ and $a_j\in\mathbb{C}$, $b_i\in\mathbb{C}\setminus\mathbb{Z}_0^-$, $A_i,B_j>0$ for all $j=\overline{1,p}, i=\overline{1,q}$. The power series converges for suitably bounded values of $|z|$ when

$$\Delta_{p,q} = 1-\sum_{j=1}^p A_j + \sum_{j=1}^q B_j > 0\,.$$

In the case $\Delta=0$, the convergence holds in the open disc $|z|<\beta=\prod_{j=1}^q B_j^{B_j}\cdot\prod_{j=1}^p A_j^{-A_j}$.

The function ${}_1\Psi_0^*$ is called *confluent*. The convergence condition $\Delta_{1,0}=1-A_1>0$ is of special interest for us.

We point out that the original definition of the Fox–Wright function ${}_p\Psi_q[z]$ (consult formula collection [8] and the monographs [11], [15]) contains gamma functions instead of the generalized Pochhammer symbols used here. However, these two functions differ only up to constant multiplying factor, that is

$$ {}_p\Psi_q\Big[\begin{array}{c}(a,A)_p\\(b,B)_q\end{array}\Big|\,z\Big] = \frac{\prod_{j=1}^p\Gamma(a_j)}{\prod_{j=1}^q\Gamma(b_j)}\,{}_p\Psi_q^*\Big[\begin{array}{c}(a,A)_p\\(b,B)_q\end{array}\Big|\,z\Big]\,.$$

The unification's motivation is clear - for $A_1=\cdots=A_p=B_1=\cdots=B_q=1$, the fucntion ${}_p\Psi_q^*[z]$ reduces exactly to the well-known generalized hypergeometric function ${}_pF_q[z]$.

Appendix B. Meijer G–function

The symbol $G_{p,q}^{m,n}(\cdot|\cdot)$ denotes Meijer's $G-$function [24] defined in terms of the Mellin–Barnes integral as

$$ G_{p,q}^{m,n}\Big(z\Big|\begin{array}{c}a_1,\cdots,a_p\\b_1,\cdots,b_q\end{array}\Big) = \frac{1}{2\pi\mathrm{i}}\oint_{\mathfrak{C}}\frac{\prod_{j=1}^m\Gamma(b_j-s)\prod_{j=1}^n\Gamma(1-a_j+s)\,z^s}{\prod_{j=m+1}^q\Gamma(1-b_j+s)\prod_{j=n+1}^p\Gamma(a_j-s)}\mathrm{ds}, \tag{8.2}$$

where $0\le m\le q$, $0\le n\le p$ and the poles a_j,b_j are such that no pole of $\Gamma(b_j-s), j=\overline{1,m}$ coincides with any pole of $\Gamma(1-a_j+s), j=\overline{1,n}$; i.e. $a_k-b_j\notin\mathbb{N}$, while $z\neq0$. $\mathfrak{C}$ is a suitable integration contour which startes at $-\mathrm{i}\infty$ and goes to $\mathrm{i}\infty$ separating the poles of $\Gamma(b_j-s), j=\overline{1,m}$ which lie to the right of the contour, from all poles of $\Gamma(1-a_j+s), j=\overline{1,n}$, which lie to the left of $\mathfrak{C}$. The integral converges if $\delta=m+n-\frac12(p+q)>0$ and $|\arg(z)|<\delta\pi$, see [14, p. 143] and [24].

The G function's Mathematica code reads

$$\texttt{MeijerG[}\{\{a_1,...,a_n\},\{a_{n+1},...,a_p\}\},\{\{b_1,...,b_m\},\{b_{m+1},...,b_q\}\},z\texttt{]}.$$

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